Petr Štěpánek; Petr Vopěnka

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DECOMPOSITION OF METRIC SPACES INTO NOWHERE DENSE SETS

Petr ŠTĚPÁNEK, Petr VOPĚNKA, Praha

The method which will be described in this paper is based on the results achieved by means of models of the set theory, especially of the ∇ -models (see [2]). The fundamental parameters of a ∇ -model are a complete Boolean algebra B and a ultrafilter z on B and special properties of the model $\nabla(B, \varkappa)$ are determined by the choice of a suitable algebra B . Conversely, some propositions of the set theory have some consequences in the theory of Boolean algebras.

We shall use some concepts introduced in [2] and we shall apply the propositions concerning complete Boolean algebras on topological spaces. It enables us to prove a theorem about decomposition of certain types of metric spaces and uniform spaces into an increasing sequence of nowhere dense sets. We shall show that the solution of an analogous general problem depends on continuum hypothesis.

§ 1. Preliminaries

<u>1.1 Definition</u>. Let a be a system of sets. We define

$$Ex(a) \equiv (x)(y)[(x \in a \mathbin{\&} y \in a \mathbin{\&} x \neq y) \rightarrow x \cap y = 0]$$

<u>1.2 Definition</u>. Let $\langle P, \tau \rangle$ be a topological space, $\sigma \subset P$. We define $Reg\, \sigma = Int(\bar{\sigma})$. We say that σ is a regular open set, if $Reg\, \sigma = \sigma$.

1.3 Lemma. Let $\langle P, \tau \rangle$ be a topological space, $\mathcal{B} \subset \tau$ system of open sets such that the following conditions are satisfied:

(1) $\mathrm{E}\varkappa(\mathcal{B})$,

(2) $\overline{\bigcup_{u \in \mathcal{B}} u} = P$.

Then the system $Reg\,\mathcal{B} = \{Reg\,u;\ u \in \mathcal{B}\}$ satisfies also conditions (1),(2).

1.4 Lemma. Let $\langle P, \tau \rangle$ be a topological space and system $\{A_\lambda ;\ \lambda \in I\}$ satisfy conditions (1),(2) in lemma 1.3. Let, for every $\lambda \in I$, $F_\lambda \subset A_\lambda$ be a closed and nowhere dense set in the subspace $\langle A_\lambda, \tau \rangle$. If we put $F = P - \bigcup_\lambda A_\lambda$ then $T = F \cup \bigcup_\lambda F_\lambda$ is closed and a nowhere dense set in $\langle P, \tau \rangle$.

Proof. It can be easily seen that T is a closed set. We shall prove that the set P - T is dense. Let $0 \neq \sigma \subset P$ be an open set. By (2), there is $\lambda_0 \in I$ such that $\sigma \cap A_{\lambda_0} = \sigma_{\lambda_0} \neq 0$. It holds evidently,

$$(P - T) \cap \sigma_{\lambda_0} = [P - (F \cup \bigcup_\lambda F_\lambda)] \cap \sigma_{\lambda_0} = \sigma_{\lambda_0} \cap (P - F_{\lambda_0}) \neq 0 .$$

Every open set $\sigma \neq 0$ has the non-empty intersection with the set P - T .

It follows that P - T is dense.

1.5 Proposition. Let $\langle P, \tau \rangle$ be a topological space. The set $Reg\,\tau = \{Reg\,u;\ u \in \tau\}$ forms a complete Boolean algebra (compare [1]).

§ 2. The Suslin number

2.1 Definition. Let $\langle P, \tau \rangle$ be a topological space.

We define

$$\mu\langle P,\tau\rangle = \min(\alpha;\ card(\alpha)\ \&\ \neg\exists a\,(Ex(a)\ \&\ a\subset\tau\ \&\ card\,a=\alpha))$$

we say that $\langle P, \tau \rangle$ is a saturated topological space if the following condition is satisfied: $0 \neq u \in \tau \longrightarrow$
$\longrightarrow \mu\langle u, \tau\rangle = \mu\langle P, \tau\rangle$.

2.2 Lemma. Let $\langle P, \tau \rangle$ be a topological space and let $b_0 \subset \tau$ satisfy the following conditions:

(a) $\neg\,(0 \in b_0)$,

(b) $(v)(v \in \tau\ \&\ v \neq 0)\,\exists u\,[u \in b_0\ \&\ u \subset v]$.

Then there exists $b_1 \subset b_0$ such that (1) $Ex\,(b_1)$,

(2) $\overline{\bigcup_{u \in b_1} u} = P$.

Proof. Let A be a choice class. For every ordinal number α , we define the sets a_α as follows: $a_0 = A' b_0$

$a_\alpha = A' G_\alpha$, where $G_\alpha = b_0 \cap \tau \restriction Int\,(P - \bigcup_{\beta<\alpha} a_\beta)$.

Obviously, there is $a_\alpha = 0$ for $\alpha \geq \mu\langle P, \tau\rangle$. Put $b_1 = \{a_\alpha\ ;\ \alpha < \mu\langle P, \tau\rangle$. It can be easily verified that b_1 satisfies conditions (1) and (2).

2.3 Lemma. Let $\langle P, \tau \rangle$ be a topological space. Then there exists a system $b \subset \tau$ such that

(1) $Ex\,(b)$,

(2) $\overline{\bigcup_{u \in b} u} = P$,

(3) $(u)[u \in b \longrightarrow \langle u, \tau\rangle$ is saturated].

Proof. Let $b_0 \subset \tau$ be the system of all non-empty saturated sets in $\langle P, \tau \rangle$. As the condition

$(u)(v)[(u, v \in \tau \;\&\; u \subset v) \rightarrow \mu\langle u, \tau\rangle \leq \mu\langle v, \tau\rangle]$

is satisfied, it is obvious that some set $v \in \mathcal{B}_0$ is contained in each set $u \in \tau \;\&\; u \neq 0$ (at least an isolated point). Now the assertion follows immediately from 2.2.

<u>2.4 Theorem (see also [3])</u>. Let $\langle P, \tau \rangle$ be a topological space. Then $\mu\langle P, \tau\rangle$ is a regular cardinal number.

<u>Proof</u>. First, let $\langle P, \tau \rangle$ be a saturated space and let $\mu\langle P, \tau\rangle = \aleph_\alpha$. Let $\{d_\gamma\}_{\gamma \in \omega_\beta}$ be an increasing sequence of cardinal numbers confinal with such that $d_\gamma \in \aleph_\alpha$ for every $\gamma \in \omega_\beta$ and $\omega_\beta \in \omega_\alpha$. Obviously, there exists a system $\mathcal{B}_0 = \{u_\gamma\}_{\gamma \in \omega_\beta}$ such that the following condition is satisfied: $\mathcal{B}_0 \subset \tau \;\&\; \neg(0 \in \mathcal{B}_0) \;\&$ $\& \; E x(\mathcal{B}_0)$. For every u_γ, the equality $\mu\langle u_\gamma, \tau\rangle = \aleph_\alpha$ holds. Thus, there exists a system $\mathcal{B}_\gamma \subset \tau$ such that $E x(\mathcal{B}_\gamma)$ and $card(\mathcal{B}_\gamma) = d_\gamma$. Put $\mathcal{B} = \bigcup_{\gamma \in \omega_\beta} \mathcal{B}_\gamma$. It holds $E x(\mathcal{B})$, $\mathcal{B} \subset \tau$ and $card\, \mathcal{B} = \aleph_\alpha$ and that is a contradiction. It follows that $\aleph_\alpha$ is a regular cardinal number.

Further, let $\langle P, \tau \rangle$ be an arbitrary topological space. According to 2.3, there exists a system $\mathcal{B}_1 \subset \tau$ satisfying (1),(2),(3).

Put $card\, \mathcal{B}_1 = \aleph_\beta < \mu\langle P, \tau\rangle$ and $\aleph_\alpha = \sup_{u \in \mathcal{B}_1} \mu\langle u, \tau\rangle$.

If $\aleph_\beta \geq \aleph_\alpha$ then $\mu\langle P, \tau\rangle = \aleph_{\beta+1}$, i.e. a regulr cardinal.

If $\aleph_\alpha > \aleph_\beta$ and $\aleph_\alpha = \mu\langle u, \tau\rangle$ for some $u \in \mathcal{B}_1$, then $\mu\langle P, \tau\rangle = \mu\langle u, \tau\rangle$ is a regular cardinal as it has been already proved.

Let us suppose that the sequence $\{\mu\langle u, \tau\rangle\}$ has not the greatest element. Let us construct the increasing sequence of cardinal numbers for $\lambda \in \omega_\beta$ such that $\aleph_{\mu_\lambda} = \mu \langle u_\lambda, \tau \rangle$ and that $\aleph_\alpha = \sum_{\lambda \in \omega_\beta} \aleph_{\mu_\lambda}$. To every $\aleph_{\mu_\lambda}$ we can construct a system $\mathcal{b}_\lambda \subset \tau$ in the set $u_{\lambda+1}$ such that $E\,x\,(\mathcal{b}_\lambda)$ and $card\,(\mathcal{b}_\lambda) = \aleph_{\mu_\lambda}$. Put $\mathcal{b} = \bigcup_\lambda \mathcal{b}_\lambda$. Evidently, $card\,\mathcal{b} = \aleph_\alpha$. It holds $\mu \langle P, \tau \rangle > \aleph_\alpha$. If there were $\mu \langle P, \tau \rangle > \aleph_{\alpha+1}$, there would have to exist a system $\mathcal{b} \subset \tau$ such that $card\,\mathcal{b} = \aleph_{\alpha+1}$ and $E\,x\,(\mathcal{b})$ - a contradiction. Thus $\mu \langle P, \tau \rangle = \aleph_{\alpha+1}$, i.e. $\mu \langle P, \tau \rangle$ is a regular cardinal.

2.5 Lemma. Let $\langle P, \tau \rangle$ be a regular topological space. Then $\mu \langle P, \tau \rangle \neq \aleph_0$.

Proof. Let $x_0 \in P$ be a non-isolated point. Let $x_1 \neq x_0$. Then there exists a closed neighbourhood V_1 of the point x_0 such that $x_1 \in P - V_1 = U_1$. Further, there exists $x_2 \neq x_0$, $x_2 \in Int\,(V_1)$ and a closed neighbourhood V_2 of the point x_0 such that $x_2 \in P - V_2$. Put $U_2 = Int\,(V_1) \cap (P - V_2)$. In this way we can construct the sets $U_n = Int\,(V_{n-1}) \cap (P - V_n)$ for every $n < \omega_0$. Denote $\mathcal{U} = \{ U_n ; n < \omega_0 \}$. Obviously, $\mathcal{U} \subset \tau$, $E\,x\,(\mathcal{U})$ and $card\,\mathcal{U} = \aleph_0$ - a contradiction. Thus the points of the space $\langle P, \tau \rangle$ are isolated. If $card\,P < \aleph_0$, then $\mu \langle P, \tau \rangle = \aleph_0$ does not hold. If $card\,P = \aleph_0$, then $\mu \langle P, \tau \rangle = \aleph_1$ - a contradiction.

Remark. The statement of lemma 2.5 holds for every topological space. As we shall deal with the uniform space, we do not prove it in full generality.

2.6 Definition. Let $\langle P, \tau \rangle$ be a topological space. We define $\mathfrak{X}\langle P, \tau \rangle = \min(\mathrm{card}\, b)$, where b is a base for the topology τ.

2.7 Proposition. Let $\langle P, \tau \rangle$ be a metric space. Then $\mu\langle P, \tau \rangle$ is an isolated cardinal number. If $\mathfrak{X}\langle P, \tau \rangle = \aleph_\alpha$, then $\mu\langle P, \tau \rangle = \aleph_{\alpha+1}$.

Proof. Let ρ be a metric, generating the topology τ. Let $\mu\langle P, \tau \rangle > \aleph_0$. For every $n \in \omega_0$, let $A^n = \{A^n_\lambda, \lambda \in I_n\}$ be a system of non-empty sets such that (a) $E \times (A^n)$,

(b) $A^n \subset \tau$,

(c) $\lambda \in I_n \rightarrow (x)(y)[x, y \in A^n_\lambda \rightarrow \rho(x,y) < \frac{1}{n}]$,

(d) $\overline{\bigcup_{\lambda \in I_n} A^n_\lambda} = P$.

For every $n < \omega_0$ and $\lambda \in I_n$ let $x^n_\lambda \in A^n_\lambda$ be a chosen point. Put

$$T = \{x^n_\lambda ;\ n < \omega_0, \lambda \in I_n\}, \beta_n = \mathrm{card}(I_n), \beta = \sum_{n<\omega_0} \beta_n .$$

Obviously, $\mathrm{card}\, T = \beta < \mu\langle P, \tau \rangle$. We shall show that T is a dense set in P. Let $0 \neq \sigma \subset P$ be an arbitrary open set. Then there exists $x_0 \in \sigma$, $n < \omega_0$ such that $U_{\frac{1}{n}}(x_0) \subset \sigma$, where $x \in U_{\frac{1}{n}}(x_0) \equiv \rho(x, x_0) < \frac{1}{n}$. According to (d) there exists $\lambda_0 \in I_{4n}$ such that $A^{4n}_{\lambda_0} \cap U_{\frac{1}{n}}(x_0) \neq 0$. It can be easily seen that $A^{4n}_{\lambda_0} \subset U_{\frac{1}{n}}(x_0)$. Thus $x^{4n}_{\lambda_0} \in \sigma$. According to the well known theorem on metric spaces the following inequality holds:

$$\aleph_\alpha = \mathcal{X}\langle P, \tau \rangle \leq card\ T \leq \beta < \mu \langle P, \tau \rangle .$$

If there were $\mu \langle P, \tau \rangle > \aleph_{\alpha+1}$, then a system $\mathcal{b} \subset \tau$ would exist such that $card\ \mathcal{b} = \aleph_{\alpha+1}$, $E \times (\mathcal{b})$. It is impossible and thus $\mu \langle P, \tau \rangle = \aleph_{\alpha+1}$.

<u>Remark</u>. If $\langle P, \tau \rangle$ is a separable metric space then $\mu \langle P, \tau \rangle = \aleph_1$. If $\langle P, \tau \rangle$ is a non-separable metric space, then $\mu \langle P, \tau \rangle = \aleph_{\alpha+1} \geq \aleph_2$.

§ 3. The (α, β)-system on the Boolean algebra

<u>3.1 Definition</u>. Let B be a complete Boolean algebra, z an ultrafilter on B . We say that a system $A_{\alpha\beta} = \{w(\gamma, \delta);\ \gamma \in \omega_\alpha, \delta \in \omega_\beta\}$ is (α, β)-system on B with respect to z if the following conditions are satisfied: There is a $u \in z$ such that

(1) $(\delta \in \omega_\beta \ \&\ \gamma_1 \neq \gamma_2) \rightarrow w(\gamma_1, \delta) \wedge w(\gamma_2, \delta) = 0$,

(1') $(\gamma \in \omega_\alpha \ \&\ \delta_1 \neq \delta_2) \rightarrow w(\gamma, \delta_1) \wedge w(\gamma, \delta_2) = 0$,

(2) $\bigvee_{\delta \in \omega_\beta} w(\gamma, \delta) = u$ for every $\gamma \in \omega_\alpha$,

(2') $\bigvee_{\gamma \in \omega_\alpha} w(\gamma, \delta) = u$ for every $\delta \in \omega_\beta$.

We say that $A_{\alpha\beta}$ is a (α, β)-system on B if $A_{\alpha\beta}$ satisfies these conditions with respect to all ultrafilters z on B .

<u>Remark</u>. The notion of the (α, β)-system on the Boolean algebra with respect to an ultrafilter was introduced in [2]. From the definition it is not clear why we speak about

an ultrafilter z instead of an element $u \in z$ whose existence is required. It can be easily seen that the above defined (α, β)-system has the properties (2),(2') with respect to all ultrafilters if and only if $u = 1$, i.e. u is the greatest element of the Boolean algebra B. (α, β)-system on B with respect to an ultrafilter z has the important property, which was proved in [2]:

If there exists an (α, β)-system on the Boolean algebra B with respect to an ultrafilter z and γ, δ are arbitrary ordinal numbers such that $\alpha \leq \gamma \leq \delta \leq \beta$ then there exists an (γ, δ)-system on B with respect to z, as well.

The elements $u, v \in z$ which correspond to the both systems according to the definition can be different. We shall prove the similar assertion for (α, β)-systems, i.e. we shall prove that if there exists an (α, β)-system on B with respect to $u = 1$ then for an arbitrary pair of ordinal numbers γ, δ such that $\alpha \leq \gamma \leq \delta \leq \beta$ there exists an (γ, δ)-system with respect to $u = 1$.

__3.2 Lemma__. Let B be a complete Boolean algebra of all regular open sets of the space $\langle P, \tau \rangle$. Let us suppose that there exists an (α, β)-system on B. Then for every pair of ordinal numbers γ, δ such that $\alpha \leq \gamma \leq \delta \leq \beta$ there exists a (γ, δ)-system on B.

__Proof__. Let $0 \neq \sigma \in \tau$. Let z be an ultrafilter generated by $\mathrm{Reg}\ \sigma$. On B, there exists an (α, β)-system with respect to z. According to (2) there exist $u \in \mathcal{z}$ and a (γ, σ)-system $A_{\gamma\sigma}$ such that $\bigvee_{\mu \in \omega_\sigma} w(\lambda, \mu) = u$ & $\bigvee_{\lambda \in \omega_\gamma} w(\lambda, \mu) = u$.

Put $\sigma_1 = u \cap \sigma \neq 0$, $A^1_{\gamma\sigma} = \{w(\lambda, \mu) \cap \sigma_1 ; \lambda \in \omega_\gamma, \mu \in \omega_\sigma\}$.

Obviously, $A^1_{\gamma\sigma} \subset \tau$ and the sets $F^{\sigma_1}_\lambda = \sigma_1 - \bigcup_{\mu \in \omega_\gamma} w(\lambda, \mu)$ for $\lambda \in \omega_\gamma$, $H^{\sigma_1}_\mu = \sigma_1 - \bigcup_{\lambda \in \omega_\gamma} w(\lambda, \mu)$ for $\mu \in \omega_\sigma$ are closed and nowhere dense in σ_1. We have shown that in every non-empty open set σ there exist $\sigma_1 \neq 0$ and $A^1_{\gamma\sigma}$ having the properties of the (γ, σ)-system. (i)

According to 2.2 there exists a system $\ell_1 \subset \tau$ such that $E \asymp (\ell_1)$, $F = P - \bigcup_{u \in \ell_1} u$ is a closed nowhere dense set and every $u \in \ell_1$ satisfies (i). For every $u \in \ell_1$, let $A^u_{\gamma\sigma} = \{w^u(\lambda, \mu); \lambda \in \omega_\gamma, \mu \in \omega_\sigma\}$ be a corresponding (γ, σ)-system. Put $A_{\gamma\sigma} = \{w(\lambda, \mu);$ $\lambda \in \omega_\gamma$, $\mu \in \omega_\sigma\}$, where $w(\lambda, \mu) = \bigcup_{u \in \ell_1} w^u(\lambda, \mu)$ for every $\lambda \in \omega_\gamma$, $\mu \in \omega_\sigma$. (1),(1′) can be easily verified. According to the definition of $A_{\gamma\sigma}$ it can be immediately seen that

$$P - \bigcup_{\lambda \in \omega_\gamma} w(\lambda, \mu) = F \cup \bigcup_{u \in \ell_1} F^u_\mu \quad \text{and} \quad P - \bigcup_{\mu \in \omega_\gamma} w(\lambda, \mu) = F \cup \bigcup_{u \in \ell_1} H^u_\lambda .$$

According to 1.4 these sets are closed and nowhere dense for every $\lambda \in \omega_\gamma$, $\mu \in \omega_\sigma$. According to 1.3 $\mathrm{Reg}\, A_{\gamma\sigma} = \{\mathrm{Reg}\, w(\lambda, \mu)\}$ is the (γ, σ)-system satisfying the conditions of this lemma.

<u>Remark</u>. In the general theory of ∇-models the existence of an (α, β)-system on a complete Boolean algebra

B with respect to the ultrafilter z is equivalent to the existence of a one-to-one mapping of ordinal number ω_{α}^{*} onto ω_{β}^{*} in model $\nabla(B, z)$. From this point of view, lemma 3.2 is only a translation of the well known Cantor-Bernstein theorem.

<u>3.3 Proposition</u>. Let B be an algebra of all regular open sets on $\langle P, \tau \rangle$. Let there exist an (α, β)-system on B, where ω_{β} is a regular ordinal and $\alpha < \beta$. Then there exists a system $\{F_{\lambda}\}_{\lambda \in \omega_{\beta}}$ such that the following conditions are satisfied:

(1) $\lambda \in \omega_{\beta} \longrightarrow F_{\lambda}$ is a closed nowhere dense set,

(2) $\lambda_1 < \lambda_2 \longrightarrow F_{\lambda_1} \subset F_{\lambda_2}$,

(3) $P = \bigcup_{\lambda \in \omega_{\beta}} F_{\lambda}$.

<u>Proof</u>. Let $A_{\alpha\beta} = \{w(\lambda, \mu); \lambda \in \omega_{\alpha}, \mu \in \omega_{\beta}\}$ be an (α, β)-system on B. Put $C_{\sigma} = \bigcup_{\gamma \in \omega_{\alpha}} w(\gamma, \sigma)$. For arbitrary $x \in P$, define $\tau(x) = card\{\xi; x \in C_{\xi}\}$. If there were $\tau(x) > \aleph_{\alpha}$ then there would exist ordinal numbers $\xi_1 \neq \xi_2$, $\gamma \in \omega_{\alpha}$ such that $x \in w(\gamma, \xi_1) \cap w(\gamma, \xi_2)$ - a contradiction. Thus, $\tau(x) \leq \aleph_{\alpha}$. Put $F_{\lambda} = P - \bigcup_{\lambda < \xi} C_{\xi}$ for $\lambda < \omega_{\beta}$. A system $\{F_{\lambda}\}_{\lambda \in \omega_{\beta}}$ is an increasing sequence of closed and nowhere dense sets. Let $x \in P$ be an arbitrary point. Let $\alpha(x) = \sup_{x \in C_{\xi}} \xi$. There is $\alpha(x) < \omega_{\beta}$ because ω_{β} is a regular ordinal. Thus, $x \in \bigcup_{\lambda < \omega_{\beta}} F_{\lambda}$.

<u>3.4 Lemma</u>. Let $\langle P, \tau \rangle$ be a saturated metric space. Let

$\mu \langle P, \tau \rangle = \aleph_{\alpha+1}$. Let B be a complete Boolean algebra of all regular open sets in P. Then there exists a $(0, \alpha)$-system on B.

Proof. Let $(A_{\lambda_1})_{\lambda_1 \in \omega_\alpha} \subset \tau$ be a system such that the following conditions are satisfied:

(a) $A_{\lambda_1} \neq 0$ for every $\lambda_1 \in \omega_\alpha$,

(b) $Ex((A_{\lambda_1})_{\lambda_1 \in \omega_\alpha})$,

(c) $\bigcup_{\lambda_1} A_{\lambda_1}$ is dense in P,

(d) $(x)(y)[x, y \in A_{\lambda_1} \rightarrow \rho(x,y) < 1]$ for every $\lambda_1 \in \omega_\alpha$;

For every $\lambda_1 \in \omega_\alpha$ let $(A_{\lambda_1 \lambda_2})_{\lambda_2 \in (\omega_\alpha - \{\lambda_1\})}$ be a system such that $A_{\lambda_1\lambda_2} \neq 0$, $A_{\lambda_1\lambda_2} \in \tau$, $Ex((A_{\lambda_1\lambda_2})_{\lambda_2 \in \omega_\alpha - \{\lambda_1\}}))$ and $\bigcup_{\lambda_2} A_{\lambda_1 \lambda_2}$ is dense in A_{λ_2}. Analogously for every one-to-one sequence $\lambda_1, \dots, \lambda_n \in \omega_\alpha$, let $(A_{\lambda_1 \dots \lambda_{n+1}})_{\lambda_{n+1} \in (\omega_\alpha - \{\lambda_1, \dots \lambda_n\})}$ be a disjoint system of open sets such that $(x)(y)[x, y \in A_{\lambda_1 \dots \lambda_{n+1}} \rightarrow \rho(x,y) < \frac{1}{n+1}]$ and $\bigcup_{\lambda_{n+1}} A_{\lambda_1 \dots \lambda_{n+1}}$ is dense in $A_{\lambda_1 \dots \lambda_n}$. Put $w(n, \sigma) = \bigcup_{\lambda_n = \sigma} A_{\lambda_1 \dots \lambda_n}$ for $n < \omega_0$, $\sigma \in \omega_\alpha$.

Consider the following implications:

(1) $n_1 \neq n_2 \rightarrow w(n_1, \sigma) \cap w(n_2, \sigma) = 0$,

(1′) $\sigma_1 \neq \sigma_2 \rightarrow w(n, \sigma_1) \cap w(n, \sigma_2) = 0$,

(2) $n < \omega_0 \rightarrow \bigcup_{\sigma} w(n, \sigma)$ is dense in P,

(2′) $\sigma \in \omega_\alpha \rightarrow \bigcup_{n} w(n, \sigma)$ is dense in P.

It can be shown easily that (1),(1′),(2) are true. We shall show that (2') is also true. Let $0 \neq \sigma \in \tau$. Thus there exist $x_0 \in \sigma$ and $n < \omega_0$ such that $\mathcal{U}_{\frac{1}{n}}(x_0) = \{x ; \rho(x, x_0) < \frac{1}{n}\} \subset \sigma$. If there were not $A_{\lambda_1 \dots \lambda_{4n}} \subset \mathcal{U}_{\frac{1}{n}}(x_0)$ for any set $A_{\lambda_1 \dots \lambda_{4n}}$ then $\mathcal{U}_{\frac{1}{4n}}(x_0) \cap A_{\lambda_1 \dots \lambda_{4n}} = 0$ for all $A_{\lambda_1 \dots \lambda_{4n}}$ and that is impossible. Let $A_{\lambda_1 \dots \lambda_{4n}} \subset \sigma$. If $\delta \neq \lambda_i$ for every $i = 1, 2, \dots, 4n$, then $A_{\lambda_1 \dots \lambda_{4n}\delta} \subset \sigma$. Thus $\bigcup_{n<\omega_0} w(n, \delta) \cap \sigma \neq 0$ for every $\sigma \neq 0$, $\sigma \in \tau$, i.e. $\bigcup_n w(n, \delta)$ is dense in $\langle P, \tau \rangle$. According to 1.3 the system $\{\operatorname{Reg} w(n, \delta) ; n < \omega_0, \delta \in \omega_\alpha\}$ is a $(0, \alpha)$-system on B.

3.5 Definition. We say that the topological space $\langle P, \tau \rangle$ is nowhere separable if this condition is satisfied:

$$0 \neq \sigma \in \tau \longrightarrow \langle \sigma, \tau \rangle \text{ is not separable.}$$

3.6 Theorem. Let $\langle P, \tau \rangle$ be a nowhere separable metric space. Then there exists a system $\{F_\lambda\}_{\lambda \in \omega_1}$ such that

(1) $\lambda \in \omega_1 \longrightarrow F_\lambda$ is a closed nowhere dense set,

(2) $\lambda_1 < \lambda_2 \longrightarrow F_{\lambda_1} \subset F_{\lambda_2}$,

(3) $P = \bigcup_\lambda F_\lambda$.

Proof. According to 2.3, there exists $\mathcal{b} \subset \tau$ such that $E \times (\mathcal{b})$ and $\overline{\bigcup_{u \in \mathcal{b}} u} = P$, for every $u \in \mathcal{b}$ the subspace $\langle u, \tau \rangle$ is saturated. According to 2.7 the inequality $\mu \langle u, \tau \rangle = \aleph_{\alpha+1} > \aleph_1$ holds for every $u \in \mathcal{b}$. By 3.4, there exists a $(0, \alpha)$-system. By 3.2 there exists also a $(0, 1)$-system

$A_{01}^{u} = \{w_u(n, \sigma); n < \omega_0, \sigma \in \omega_1\}$. Put $w(\gamma, \sigma) = \bigcup_{u \in \mathcal{b}} w_u(\gamma, \sigma)$

for every $\gamma < \omega_0$, $\sigma \in \omega_1$. Similarly as in the proof of 3.2, we shall verify that the system $\{\mathrm{Reg}\, w(\gamma, \sigma);\ \gamma < \omega_0, \sigma \in \omega_1\}$ is a $(0, 1)$-system on the Boolean algebra of all regular open sets in $\langle P, \tau \rangle$. The statement follows immediately from 3.3.

3.7 Lemma. A linear topological space $\langle P, \tau \rangle$ is saturated.

Proof. Let $U \subset P$ be a non-empty open saturated set. Then there exist $x_0 \in P$ and an open neighbourhood V of zero in $\langle P, \tau \rangle$, such that $x_0 + V \subset U$. Thus V is also saturated set and $P = \bigcup_{n < \omega_0} n\, V$. Obviously $\mu \langle V, \tau \rangle \leq \mu \langle P, \tau \rangle$. If there were $\mu \langle P, \tau \rangle > \mu \langle V, \tau \rangle$ then there would exist n_0 such that $\mu \langle n_0 V, \tau \rangle = \mu \langle P, \tau \rangle$ – contradiction, because $\mu \langle n_0 V, \tau \rangle = \mu \langle V, \tau \rangle$.

3.8 Corollary. Let $\langle P, \tau \rangle$ be a metric linear space which is not separable. Then there exists a system $\{F_u\}_{u \in \omega_1}$ with properties (1),(2) and (3) from 3.6.

Remark. According to the above propositions, a metric nowhere separable space is the union of an increasing sequence of $\aleph_1$ nowhere dense sets. A separable metric space consists of $2^{\aleph_0}$ points. If continuum hypothesis $2^{\aleph_0} = \aleph_1$ holds, then a separable metric space without isolated points is the union of $\aleph_1$ nowhere dense sets. According to 2.3

there exists the decomposition $\mathcal{b}$ of an arbitrary metric space $\langle P, \tau \rangle$ into saturated sets. Some subspaces $\langle u, \tau \rangle$, where $u \in \mathcal{b}$, are separable, the other ones are nowhere separable. If continuum hypothesis holds the separable subspaces $\langle u, \tau \rangle$ without isolated points are the union of $\aleph_1$ nowhere dense sets and the other ones are even the union of an increasing sequence of $\aleph_1$ nowhere dense sets. According to 1.4 it can be easily seen that the whole space $\langle P, \tau \rangle$ without isolated points is the union of the system (non necessary monotony) of $\aleph_1$ nowhere dense sets. It can be easily seen that the separable metric space without isolated points is not the union of increasing sequence of $\aleph_1$ closed nowhere dense sets. Let T be a countable dense set in P . If there were $P = \bigcup_{\lambda \in \omega_1} F_\lambda$ then there would exist an index λ_0 such that $T \subset F_{\lambda_0}$ - contradiction.

If $\langle P, \tau \rangle$ is in addition a complete metric space, then according to the well known Baire's theorem $\langle P, \tau \rangle$ is not the union of $\aleph_0$ nowhere dense sets. In this case $\langle P, \tau \rangle$ is the union exactly of $\aleph_1$ nowhere dense sets .

If the continuum hypothesis does not hold, then the situation is more complicated. In paper [4] there is constructed a model of the set theory in which the continuum hypothesis does not hold and the interval $I = \langle 0, 1 \rangle$ is not the union of less than $2^{\aleph_0}$ nowhere dense sets. In an analogous way as above it can be proved that a metric space without the isolated points is always the union of $2^{\aleph_0}$ nowhere dense sets.

The described method can be extended also to certain uniform spaces.

3.9 Definition. Let $\langle P, \tau \rangle$ be a uniform space. We define $\omega(\tau) = \min(\operatorname{card}(\mathcal{U}))$, where $\mathcal{U}$ is a base of the filter of neighbourhoods of the diagonal $\Delta \subset P \times P$, which generates the structure of the uniformity τ.

3.10 Proposition.[x)] Let $\langle P, \tau \rangle$ be a saturated uniform space, B a complete Boolean algebra of all regular open sets in P. Let $\aleph_\alpha = \omega(\tau) < \mu\langle P, \tau \rangle = \aleph_\beta$. Then for every γ, δ such that $\alpha \le \gamma \le \delta < \beta$, there exists a (γ, δ)-system in B.

Proof. Let $\{V_\xi ; \xi \in \omega_\alpha\}$ be a fundamental system of neighbourhoods of Δ. Let V be a neighbourhood of the point $x_0 \in P$. We say that V is of order $\xi \in \omega_\alpha$ if $V \times V \subset V_\xi$ is satisfied. We say that a system of open sets $A_\delta = \{u_\lambda ; \lambda \in I_\delta\} \subset \tau$ is of order δ in an open set A if $Ex(A_\delta)$, $\overline{\bigcup_{\lambda \in I_\delta} u_\lambda} = A$ holds and for every $\lambda \in I_\delta$ there exists x_λ such that u_λ is a neighbourhood of order δ of the point x_λ.
Put
$$\aleph_{\gamma_\delta} = \min_{A_\delta}(\operatorname{card}(A_\delta)), \quad \aleph_{\gamma*} = \sum_{\delta \in \omega_\alpha} \aleph_{\gamma_\delta} < \mu\langle P, \tau \rangle .$$

For every γ such that $\alpha \le \gamma$ & $\gamma^* \le \gamma < \beta$ there exist systems A_γ of all orders in P such that $\operatorname{card} A_\gamma = \aleph_\gamma$. Using the transfinite induction we define the sets $A_{\lambda_1 \ldots \lambda_\varepsilon}$. If ε is an isolated ordinal number $\varepsilon = \varepsilon_0 + 1$, then $\{A_{\lambda_1 \ldots \lambda_\varepsilon}\}$ $\lambda_\varepsilon \neq \lambda_\nu$

x) see p. 517

for $\nu < \varepsilon$, $\lambda_\varepsilon \in \omega_\gamma\}$ will denote a system of order ε in the set $A_{\lambda_1 \dots \lambda_{\varepsilon_0}}$.

If ε is a limit ordinal number and $\alpha = \{\lambda_\nu\}_{\nu \in \varepsilon}$ is a one-to-one sequence of ordinal numbers $\lambda_\nu \in \omega_\gamma$, put $A_\alpha = \operatorname{Int} \bigcap_{\nu \in \varepsilon} A_{\lambda_1 \dots \lambda_\nu}$. If $A_\alpha \neq 0$, then $\{A_{\lambda_1 \dots \lambda_\varepsilon};\ \lambda_\varepsilon \neq \lambda_\nu$ for $\nu \in \varepsilon$, $\lambda_\nu \in \omega_\gamma\}$ denotes a system of order ε for the set A_α.

For $\varepsilon \in \omega_\alpha$, $\mu \in \omega_\alpha$ put $w(\varepsilon, \mu) = \bigcup_{\lambda_\varepsilon = \mu} A_{\lambda_1 \dots \lambda_\varepsilon}$. It can be easily seen that

(1) $\varepsilon_1 \neq \varepsilon_2 \rightarrow w(\varepsilon_1, \mu) \cap w(\varepsilon_2, \mu) = 0$,

(1') $\mu_1 \neq \mu_2 \rightarrow w(\varepsilon, \mu_1) \cap w(\varepsilon, \mu_2) = 0$,

(2) $\bigcup_\mu w(\varepsilon, \mu)$ is dense for every $\varepsilon \in \omega_\alpha$

hold.

It remains to verify (2'), i.e. that $\bigcup_\varepsilon w(\varepsilon, \mu)$ is dense in P for every $\mu \in \omega_\gamma$. Let $0 \neq \sigma \subset \tau$. Then there exist $\sigma_0 \in \omega_\alpha$ and $x_0 \in P$ such that the neighbourhood U defined by $U \times \{x_0\} = V_{\sigma_0} \cap \{P \times \{x_0\}\}$ satisfies $U \subset \sigma$. Let $\sigma_1 \in \omega_\alpha$ be such an ordinal number that $\overset{3}{V}_{\sigma_1} \subset V_{\sigma_0}$. Let us take a neighbourhood U_1 of the point x_0 of order σ_1. Obviously, there exists a set $A_{\lambda_1 \dots \lambda_{\sigma_1}}$ such that $A_{\lambda_1 \dots \lambda_{\sigma_1}} \cap U_1 \neq 0$. Then $A_{\lambda_1 \dots \lambda_{\sigma_1}} \subset U \subset \sigma$. Let $x \in A_{\lambda_1 \dots \lambda_{\sigma_1}}$. Then $\langle x_0\, x \rangle \in \overset{3}{V}_{\sigma_1}$ thus $x \in U \subset \sigma$. If for any

$\nu \neq \sigma_1$ there is not $\lambda_\nu = \mu$ then $A_{\lambda_1 \dots \lambda_{\sigma_1} \mu} \subset \sigma$.
Thus $\sigma \cap \bigcup_{\varepsilon} w(\varepsilon, \mu) \neq 0$. We have constructed an (α, γ)-system on the algebra B, $A_{\alpha\gamma} = \{ Reg\, w(\varepsilon, \mu) ; \varepsilon \in \omega_\alpha, \mu \in \omega_\gamma \}$.
The second part of the proposition follows from 3.2.

3.11 Proposition. Let $\langle P, \tau \rangle$ be a uniform space, $\omega(\tau) = \aleph_\alpha$ and let the following condition be satisfied:

$$0 \neq \sigma \in \tau \longrightarrow \mu \langle \sigma, \tau \rangle > \aleph_{\alpha+1} .$$

Then there exists a system $\{ F_\lambda \}_{\lambda \in \omega_{\alpha+1}}$ such that:

(1) $\lambda \in \omega_{\alpha+1} \longrightarrow F_\lambda$ is a closed nowhere dense set,

(2) $\lambda_1 < \lambda_2 \longrightarrow F_{\lambda_1} \subset F_{\lambda_2}$,

(3) $\bigcup_{\lambda} F_\lambda = P$.

Proof. Let ℓ be the decomposition into saturated sets such that $E \times (\ell)$ and $\overline{\bigcup_{u \in \ell} u} = P$. According to 3.10, in every set there exists an $(\alpha, \alpha+1)$-system. Similarly as in the proof of 3.2 it is possible to construct an $(\alpha, \alpha+1)$-system on the algebra B of all regular open sets. Now the statement follows from 3.3.

3.12 Corollary. Let $\langle P, \tau \rangle$ be a linear topological space such that $\omega(0) = \aleph_\alpha$ and $\mu \langle P, \tau \rangle > \aleph_{\alpha+1}$, where $\omega(0)$ is the least cardinality of the fundamental system of neighbourhoods of zero in P. Then there exists a system $\{F_\lambda\}_{\lambda \in \omega_{\alpha+1}}$ with properties (1),(2) and (3) from 3.12.

Proof. Linear topological space is uniform and saturated. Evidently $\omega(0) = \omega(\tau)$. The statement follows from 3.10 and 3.11.

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