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ON THE DIFFERENTIABILITY OF MAPPINGS IN FUNCTIONAL SPACES

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This remark deals with the differentiability of mappings in functional spaces. We establish some conditions for the existence of Fréchet differentials for the mappings acting in reflexive Banach spaces (Theorem 2,3).

Moreover, the connection between the Gateaux and Fréchet differentials is derived and also some basic properties of bounded differentials are established. In last section, using the arguments similar to those of M.M. Vajnberg [1,chapt.I] we give some sufficient conditions for the boundedness and continuity of the Gateaux differentials.

1. First of all we introduce some well-known notation and definitions. Let X,Y be linear normed spaces, $(X \to Y)$ the space of all linear continuous mappings of X into Y. Throughout this paper by a word "space" there is meant a real space. We shall use the symbols " $\longrightarrow$ " and " $\xrightarrow{W}$ " to denote the strong and weak convergence in X(Y), respectively. A mapping $F: X \to Y$ of X into Y is said to be strongly (weakly), [demi-] continuous at $x_0 \in X$ if $x_n \xrightarrow{W} x_0$, $[x_n \longrightarrow x_0]$ implies $F(x_n) \to F(x_0)$, $[(F(x_n) \xrightarrow{W} F(x_0))]$, respectively. A mapping $F: X \to Y$ is called compact on a set $M \subset X$ if for every bounded subset $N \subset M$, the set F(N) is compact

in Y. Let us recall that $F: X \to Y$ is said to be completely compact on a bounded set $M \subset X$ if F is uniformly continuous and compact on M. For an another equivalent definition cf. [1,chapt.I]. The following result is due to M.M. Vajnberg [1,chapt.I]: If X is a reflexive Banach space and $F: X \to X$ is strongly continuous on $D_R = \{ x \in X ; \| x \| \leqq R \}$, then F is completely compact on D_R.

By $VF(x_0, h)$ (by $DF(x_0, h)$) we denote the Gâteaux (a linear Gâteaux) differential of a mapping $F: X \to Y$ at $x_0 \in X$, respectively. By $dF(x_0, h)$ we shall understand the Fréchet differential of F at $x_0 \in X$, $(h \in X)$, cf.[1,chapt.I].

The concept of bounded differentials was proposed by G.A. Suchomlinov [2]. His definition is as follows: We shall say that a mapping $F: X \to Y$ has at $x_0 \in X$ a bounded differential $dVF(x_0, h)$ if for any given $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that if $|t| < \delta$, then

$$\| \frac{1}{t} [F(x_0 + th) - F(x_0)] - dVF(x_0, h) \| < \varepsilon$$

uniformly with respect to $h \in X$, $\| h \| = 1$ and $dVF(x_0, h)$ is bounded on the unit sphere $\| h \| = 1$.

Suchomlinov [2] proved the following assertion: If $F: X \to$ $\to X$ is a mapping of Banach space X into itself having a bounded uniform differential at $x_0 \in X$, then $dVF(x_0, h) =$ $= dF(x_0, h)$. The result of Ivanov [3] is as follows:

<u>Theorem 1</u> (Ivanov [3]). Let X be a finite dimensional Banach space, $f: X \to E_1$ a real functional on X. If there exists the Gâteaux differential $Vf(x_0, h)$ and f is Lipschitzian in a neighbourhood $\mathcal{U}(x_0)$ of $x_0 \in X$:

(1) $|f(x+h)-f(x)| \leqq M\|h\|, \ x \in \mathcal{U}(x_0), x+h \in \mathcal{U}(x_0)$

(M = const), then f possesses a bounded differential $dVf(x_0, h)$ at $x_0 \in X$.

Recall that (1) implies

$$|Vf(x_0, h) - Vf(x_0, h_1)| \leqq M \|h - h_1\|$$

for every $h, h_1 \in X$. The Ivanov's theorem gives immediately the following consequence: Under the condition of Theorem 1, let there exist $Df(x_0, h)$ at $x_0 \in X$. Then f possesses the Fréchet differential $df(x_0, h)$ at $x_0 \in X$.

Let us remark that Theorem 1 does not hold for any Banach space. The Vajnberg's result [1,4] states: If there exists the Gateaux derivative $F'(x)$ in a neighbourhood $\mathcal{U}(x_0)$ of $x_0 \in X$ and is continuous (in norm topology of $(X \to X)$) in x_0, then $F: X \to Y$ possesses the Fréchet derivative at $x_0 \in X$.

For another result cf. Theorem 2 [4,chapt.VIII,§ 3]. The proof of mentioned theorem depends essentially on uniform continuity (in norm topology of $(X \to X)$) of Gâteaux derivative $F'(x)$ in some neighbourhood $\|x - x_0\| < r$ of x_0. The above results were generalized by G. Marinescu [9,th.2,3]. But these assertions also depend on continuity (under the direction h) of Gâteaux derivative $F'(x)$ in the norm topology of $(X \to X)$.

Recall that there is a completely another situation in complex Banach spaces, cf.[6,chapt.IV,7].

2. We shall say that the Gâteaux differential $VF(x_0, h)$, $x_0 \in X$ is strongly (or weakly) continuous in (x_0, h), $h \in X$ (h is an arbitrary element of X) jointly if $x_n \xrightarrow{w} x_0$,

$h_n \xrightarrow{w} h$ imply $VF(x_n, h_n) \to VF(x_0, h)$ (or $VF(x_n, h_n) \xrightarrow{w} VF(x_0, h)$).

Now we shall prove the following

Theorem 2. Let X be a reflexive Banach space, $F: X \to X$ a mapping of X into itself. Suppose that F possesses the Gâteaux differential $VF(x, h)$ in a convex neighbourhood $\mathcal{U}(x_0)$ of $x_0 \in X$. If $VF(x_0, h)$ is strongly continuous in (x_0, h), $h \in X$ jointly, then F possesses the Fréchet derivative $F'(x_0)$ at x_0 and $VF(x_0, h) = dF(x_0, h) = F'(x_0)h$.

Proof. Let ε be an arbitrary positive number, h a fixed (but arbitrary) element of X. Then there exists a constant $\delta_1(\varepsilon) > 0$ such that if $|t| < \delta_1(\varepsilon)$, then

(2) $$\left\| \frac{1}{t} \omega(x_0, th) \right\| < \varepsilon ,$$

where $\omega(x_0, th) = F(x_0 + th) - F(x_0) - VF(x_0, th)$.
To prove our theorem, we need to show that the numbers $\delta_1(\varepsilon)$ have a positive lower bound $\delta(\varepsilon)$ for $h \in X$, $\|h\| = 1$ and that the inequality (2) is valid for these h. Suppose contrary, there exists a positive number ε_0 with the following property: for every n ($n = 1,2,\ldots$) there exist $h_n \in X$ ($\|h_n\| = 1$) and t_n such that $|t_n| < \frac{1}{n}$ and

(3) $$\left\| \frac{1}{t_n} \omega(x_0, t_n h_n) \right\| > \varepsilon_0 .$$

Since X is reflexive space and $\|h_n\| = 1$, passing to a subsequence $\{h_{n_k}\}$ we have that $h_{n_k} \xrightarrow{w} h_0 \in X$.

Since ε_0 and $h_0 \in X$ are given, there exists a positive number $\delta_2(\varepsilon_0)$ such that if $|t| < \delta_2(\varepsilon_0)$, then

(4) $$\left\| \frac{1}{t}\, \omega(x_0, t h_0) \right\| < \frac{\varepsilon_0}{3} .$$

Since $\{h_{n_k}\}$ is a subsequence of $\{h_n\}$ and if $\frac{1}{n_k} < \delta_2(\varepsilon_0)$, then there exists t_{n_k} such that $|t_{n_k}| \leqq \frac{1}{n_k} < \delta_2(\varepsilon_0)$ and

(5) $$\left\| \frac{1}{t_{n_k}}\, \omega(x_0, t_{n_k} h_{n_k}) \right\| > \varepsilon_0 .$$

Let us note that $x_0 + t_{n_k} h_{n_k} \in \mathcal{U}(x_0)$, $x_0 + t_{n_k} h_0 \in \mathcal{U}(x_0)$ for sufficiently large k. But by assumption

(6) $$\begin{aligned} F(x_0 + t_{n_k} h_{n_k}) - F(x_0) &= VF(x_0, t_{n_k} h_{n_k}) + \omega(x_0, t_{n_k} h_{n_k}), \\ F(x_0 + t_{n_k} h_0) - F(x_0) &= VF(x_0, t_{n_k} h_0) + \omega(x_0, t_{n_k} h_0). \end{aligned}$$

Now let $e \in X^*$ be any linear continuous functional on X such that $\| e \| = 1$. By the mean-value theorem

(7) $$\begin{aligned} (F(x_0 + t_{n_k} h_{n_k}) - F(x_0), e) &= (VF(x_0 + \alpha_k t_{n_k} h_{n_k}, t_{n_k} h_{n_k}), e), \\ (F(x_0 + t_{n_k} h_0) - F(x_0), e) &= (VF(x_0 + \beta_k t_{n_k} h_0, t_{n_k} h_0), e), \end{aligned}$$

where (x, e) denotes the value of e at the point $x \in X$, $0 < \alpha_k < 1$, $0 < \beta_k < 1$ and $\alpha_k = \alpha_k(e)$, $\beta_k = \beta_k(e)$. Adding and substracting $(VF(x_0, t_{n_k} h_0), e)$ and according to (6),(7)

$$\begin{aligned} (\omega(x_0, t_{n_k} h_{n_k}) - \omega(x_0, t_{n_k} h_0), e) &= (VF(x_0 + \alpha_k t_{n_k} h_{n_k}, t_{n_k} h_{n_k}), e) - \\ &- (VF(x_0, t_{n_k} h_{n_k}), e) + (VF(x_0, t_{n_k} h_0), e) - \\ &- (VF(x_0 + \beta_k t_{n_k} h_0, t_{n_k} h_0), e) + (VF(x_0, t_{n_k} h_0) - \\ &- VF(x_0, t_{n_k} h_0), e) . \end{aligned}$$

From Hahn-Banach theorem it follows the existence of $e_0 \in X^*$ such that $\| e_0 \| = 1$ and

$$|(\omega(x_0, t_{n_k} h_{n_k}) - \omega(x_0, t_{n_k} h_0), e_0)| =$$

$$= \| \omega(x_0, t_{n_k} h_{n_k}) - \omega(x_0, t_{n_k} h_0) \|$$

Hence

$$\| \frac{1}{t_{n_k}} [\omega(x_0, t_{n_k} h_{n_k}) - \omega(x_0, t_{n_k} h_0)] \| \leqq$$

$$\leqq \| VF(x_0 + \alpha_k t_{n_k} h_{n_k}, h_{n_k}) - VF(x_0, h_0) \| +$$

$$+ \| VF(x_0, h_0) - VF(x_0, h_{n_k}) \| + \| VF(x_0, h_0) -$$

$$- VF(x_0 + \beta_k t_{n_k} h_0, h_0) \| .$$

Since $t_{n_k} \longrightarrow 0$, $h_{n_k} \xrightarrow{W} h_0$, we have that $x_0 + \alpha_k t_{n_k} h_{n_k} \xrightarrow{W} x_0$ and $x_0 + \beta_k t_{n_k} h_0 \longrightarrow x_0$. Hence $x_0 + \beta_k t_{n_k} h_0 \xrightarrow{W} x_0$ and $VF(x_0 + \alpha_k t_{n_k} h_{n_k}, h_{n_k}) \to VF(x_0, h_0)$, $VF(x_0, h_{n_k}) \to VF(x_0, h_0)$, $VF(x_0 + \beta_k t_{n_k} h_0, h_0) \longrightarrow VF(x_0, h_0)$.

Thus, there exists an integer $k_1(\varepsilon_0)$ such that

$$\text{(8)} \qquad \| \frac{1}{t_{n_k}} [\omega(x_0, t_{n_k} h_{n_k}) - \omega(x_0, t_{n_k} h_0)] \| < \frac{2\varepsilon_0}{3}$$

for every k, $k \geqq k_1(\varepsilon_0)$. On the other hand

$$\text{(9)} \quad \| \frac{1}{t_{n_k}} \omega(x_0, t_{n_k} h_{n_k}) \| \leqq \| \frac{1}{t_{n_k}} \omega(x_0, t_{n_k} h_0) \| +$$

$$+ \| \frac{1}{t_{n_k}} [\omega(x_0, t_{n_k} h_{n_k}) - \omega(x_0, t_{n_k} h_0)] \|$$

In view of (9), (8) and (4)

$$\text{(10)} \qquad \| \frac{1}{t_{n_k}} \omega(x_0, t_{n_k} h_{n_k}) \| < \varepsilon_0$$

for every $k \geqq k_1$ and some t_{n_k}, $|t_{n_k}| \leqq \frac{1}{n_k} < \delta_x(\varepsilon_0)$.

But (10) contradicts (3),(5). According to Vajnberg's theorem [1,§ 3,th.3.1] $VF(x_0,h) = DF(x_0,h) = F'(x_0)h$, where $F'(x_0)$ denotes the Gâteaux derivative of F at x_0. Thus $F'(x_0)$ is the Fréchet derivative of F at x_0 and this completes the proof.

Corollary 1. Let X be finite-dimensional Banach space, $F: X \to X$ a mapping of X into itself. Suppose that F possesses the Gâteaux differential $VF(x,h)$ in a convex neighbourhood $\mathcal{U}(x_0)$ of $x_0 \in X$. If $VF(x,h)$ is continuous in (x_0,h), $h \in X$ jointly, then F possesses the Fréchet derivative $F'(x_0)$ at x_0 and $VF(x_0,h) = dF(x_0,h) = F'(x_0)h$.

Remark. Let us note that Corollary 1 does not hold for more general Banach spaces even if we impose on F more restrictive conditions. A. Alexiewicz and W. Orlicz [8] proved that there exists an operation $F(x)$ from a separable Banach space c_0 to itself, satisfying the condition of Lipschitz, having everywhere the Gâteaux differential continuous in x and h jointly and being nowhere Fréchet-differentiable. An another example was proposed by M.M. Vajnberg. Let $h(u) = g(u(x),x)$ be an operator of Nemyckij, where N-function $g(u,x)$ satisfies the conditions of theorem 20.2 [1,chapt.VI,§ 20]. Then $h(u)$ is Gâteaux-differentiable in L_2, $Dh(u,v)$ is continuous in u, v jointly and $h(u)$ satisfies the Lipschitz condition in L_2. But $h(u)$ is nowhere Fréchet-differentiable in L_2 [4,§ 5,p.91-92]. Hence these examples show that the strong continuity of $VF(x,h)$ in (x_0,h), $h \in X$ cannot be replaced in theorem 2 by continuity of $VF(x,h)$ in (x_0,h) $h \in X$ even if we impose on F the Lipschitz condition.

Theorem 3. Let X be a reflexive Banach space, $F: X \to X$ a mapping of X into X. Suppose that F possesses the Gâteaux differential $VF(x, h)$ in a convex neighbourhood $\mathcal{U}(x_0)$ of $x_0 \in X$. If $VF(x_0, h)$ is ~~weakly~~ strongly continuous in (x_0, h), $h \in X$ jointly, then F possesses the Fréchet derivative at x_0 and $VF(x_0, h) = dF(x_0, h) = F'(x_0)h$.

Proof. Is similar to that of Theorem 2.

3. Unless otherwise explicitly stated, X, Y are linear normed spaces. The concept of a bounded differential can be introduced equivalently as follows:

Definition 1. We shall say that a mapping $F: X \to Y$ possesses at $x_0 \in X$ a bounded differential $dVF(x_0, h)$ if

$$F(x_0 + h) - F(x_0) = dVF(x_0, h) + \omega(x_0, h),$$

where $\lim\limits_{\|h\| \to 0} \frac{\|\omega(x_0, h)\|}{\|h\|} = 0$, $dVF(x_0, \alpha h) = \alpha\, dVF(x_0, h)$

for any real α and $dVF(x_0, h)$ is continuous at $h = \emptyset$. Let us note that the continuity of $dVF(x_0, h)$ at $h = 0$ implies the boundedness of $dVF(x_0, h)$ in some neighbourhood of $h = 0$.

Since $dVF(x_0, h)$ is homogeneous in h, $dVF(x_0, h)$ is bounded on any closed ball D_R $(\|x\| \leq R) \subset X$. Instead of the continuity of $dVF(x_0, h)$ at $h = 0$, one may require that $dVF(x_0, h)$ is bounded in some neighbourhood of h = 0, or that $dVF(x_0, h)$ is bounded on some sphere $\|x\| = R > 0$.

Theorem 4. Suppose that $F: X \to Y$ and that F possesses a bounded differential $dVF(x_0, h)$ at $x_0 \in X$.

If F is strongly continuous, uniformly continuous, continuous, weakly continuous, demicontinuous, compact in some neighbourhood $\mathcal{U}(x_0)$ of x_0, then $dVF(x_0, h)$ considered as the mapping in h from X into Y is strongly continuous, uniformly continuous, continuous, weakly continuous, demicontinuous, compact, respectively.

Proof. For instance we shall assume that F is weakly continuous in $\mathcal{U}(x_0)$ of x_0. For any given $h_0 \in X$ let $\{h_n\} \in X$ be a sequence such that $h_n \xrightarrow{W} h_0$. We need to show that $dVF(x_0, h_n) \xrightarrow{W} dVF(x_0, h_0)$ in Y. If this assertion were not true, we could find a positive number ε_0, a linear functional $e_0 \in Y^*$, $\|e_0\| = 1$ and a subsequence $\{h_{n_k}\}$ such that

$$(11)\quad |(dVF(x_0, h_{n_k}) - dVF(x_0, h_0), e_0)| \geqq \varepsilon_0 ,$$

where (y, e_0) denotes the value of e_0 at the point $y \in Y$. Choose a positive number t such that $x_0 + th_0 \in \mathcal{U}(x_0)$, $x_0 + th_n \in \mathcal{U}(x_0)$ for every n $(n = 1,2,\dots)$.

We have

$$(12)\quad |(F(x_0 + th_{n_k}) - F(x_0 + th_0), e_0)| \geqq$$
$$\geqq t|(dVF(x_0, h_{n_k}) - dVF(x_0, h_0), e_0)| - |(\omega(x_0, th_{n_k}), e_0)| -$$
$$- |(\omega(x_0, th_0), e_0)| \geqq t|(dVF(x_0, h_{n_k}) -$$
$$- dVF(x_0, h_0), e_0)| - \|\omega(x_0, th_{n_k})\| -$$
$$- \|\omega(x_0, th_0)\| .$$

Since $h_n \xrightarrow{W} h_0$, $\|h_n\| \leqq C$ for every n. Hence, by our assumption there exists a positive number t_0.

$(0 < t_0 < t)$ and an integer k_0 such that for every $k \geq k_0$

(13) $\|\omega(x_0, t_0 h_{n_k})\| < \frac{\varepsilon_0}{3} t_0$, $\|\omega(x_0, t_0 h_0)\| < \frac{\varepsilon_0}{3} t_0$.

In view of (11),(12),(13)

(14) $|(F(x_0+t_0 h_{n_k}) - F(x_0+t_0 h_0), e_0)| > \frac{\varepsilon_0}{3} t_0 \quad (k \geqq k_0)$.

Since $x_0 + t_0 h_{n_k} \in \mathcal{U}(x_0)$, $x_0 + t_0 h_0 \in \mathcal{U}(x_0)$, $x_0 + t_0 h_{n_k} \xrightarrow{w} x_0 + t_0 h_0$ and F is weakly continuous on $\mathcal{U}(x_0)$, $F(x_0 + t_0 h_{n_k}) \xrightarrow{w} F(x_0 + t_0 h_0)$, which contradicts with (14). Therefore $VF(x_0, h)$ is weakly continuous mapping in $h \in X$. This completes the proof.

<u>Corollary 2</u>. Let $F : X \to Y$ be a mapping of X into Y. Suppose that there exists a bounded differential $dVF(x_0, h)$. If F is completely continuous (or completely compact) in some neighbourhood $\mathcal{U}(x_0)$ of x_0, then $dVF(x_0, h)$ is completely continuous (or completely compact) in $h \in X$.

<u>Corollary 3</u>. Let X be a reflexive Banach space, $F: X \to X$ a mapping of X into X having the property that F is strongly continuous in some neighbourhood $\mathcal{U}(x_0)$ of $x_0 \in X$. Suppose that F possesses the bounded differential at $x_0 \in X$. Then $dVF(x_0, h)$ is completely compact in any closed ball $D_R = \{h \in X;\ \|h\| \leqq R\}$.

4. We introduce the following

<u>Definition 2</u>. A mapping $F : X \to Y$ is called locally weakly uniformly differentiable in $D_R = (x \in X;\ \|x\| < R)$

if for every $\varepsilon > 0$ and $x_0 \in D_R$ there exist two positive constants $\sigma(\varepsilon, x_0)$, $\eta(\varepsilon, x_0)$ such that, if $|t| < \sigma(\varepsilon, x_0)$, then

$$\left\| \frac{1}{t} \omega(x, th) \right\| < \varepsilon$$

holds for every $x \in D(x_0, \eta) \cap D_R$, where

$$\omega(x, th) = F(x+th) - F(x) - VF(x, th),$$

$D(x_0, \eta) = \{x \in X : \|x - x_0\| < \eta\}$ and h is an arbitrary (but fixed) element of X .

A Gâteaux differential $VF(x_0, h)$ is said to be continuous at $x_0 \in X$ if $x_n \to x_0$ implies $VF(x_n, h) \to$ $\to VF(x_0, h)$. Using the arguments similar to those of [1, chapt.I], it is easy to prove the following

<u>Theorem 5</u>. Let $F : X \to Y$ be a mapping having in D_R $(\|x\| < R)$ a continuous Gâteaux differential $VF(x, h)$. Then F is locally weakly uniformly differentiable in D_R .

<u>Theorem 6</u>. Suppose that $F : X \to Y$ is continuous in some neighbourhood $\mathcal{U}(x_0)$ of $x_0 \in X$. If F is locally weakly uniformly differentiable on $\mathcal{U}(x_0)$, then $VF(x_0, h)$ is continuous at $x_0 \in X$.

<u>Proof</u>. For any given (but fixed) $h \in X$, let $\{x_n\} \in X$ be a sequence such that $x_n \to x_0$. Then there exists an integer n_0 such that for every $n \geq n_0$ $x_n \in \mathcal{U}(x_0)$. Since F is locally uniformly differentiable on $\mathcal{U}(x_0)$,

$$F(x_0 + th) - F(x_0) = VF(x_0, th) + \omega(x_0, th),$$

$$F(x_n + th) - F(x_n) = VF(x_n, th) + \omega(x_n, th), \quad n \geq n_0 .$$

Taking $t > 0$ sufficiently small, we have that

$x_0 + th \in \mathcal{U}(x_0)$, $x_n + th \in \mathcal{U}(x_0)$ for every n ($n = 1,2,\ldots$) . For any given $\varepsilon > 0$ there exist two positive constants t_0 $(0 < t_0 < t)$, η (ε, x_0) such that for every $x \in D(x_0, \eta) \cap \mathcal{U}(x_0)$ there is $\| \frac{1}{t_0} \omega(x, t_0 h) \| < \frac{\varepsilon}{4}$. Since $x_n \to x_0$, $x_n \in D(x_0, \eta) \cap \cap \mathcal{U}(x_0)$ for every $n \geq n_1$. Hence

$$(15) \quad \| \frac{1}{t_0} \omega(x_0, t_0 h) \| < \frac{\varepsilon}{4}, \quad \| \frac{1}{t_0} \omega(x_n, t_0 h) \| < \frac{\varepsilon}{4} \quad .$$

$(n \geq n_1)$. Since F is continuous on $\mathcal{U}(x_0)$, for a given $\frac{\varepsilon}{4} t_0$ there exists an integer n_2 such that for every $n \geq n_2$

$$(16) \quad \| F(x_n + t_0 h) - F(x_0 + t_0 h) \| < \frac{\varepsilon}{4} t_0, \quad \| F(x_n) - F(x_0) \| < \frac{\varepsilon}{4} t_0 \quad .$$

By (15) and (16)

$$\| VF(x_n, h) - VF(x_0, h) \| < \| \frac{1}{t_0} \omega(x_0, t_0 h) \| + \| \frac{1}{t_0} \omega(x_n, t_0 h) \| + \frac{\varepsilon}{2} < \varepsilon$$

for every $n \geq n_3$, where $n_3 = max(n_1, n_2, n_3)$.

This concludes the proof.

A mapping $F : X \to Y$ is said to be weakly Lipschitzian [1,chapt.I] at $x_0 \in X$ if for every $h \in X$, $\|h\| = 1$ there exists $\delta(h) > 0$ such that if $|t| < \delta(h)$, then

$$\| F(x_0 + th) - F(x_0) \| \leq C \| th \| ,$$

where a positive constant C does not depend on h .

<u>Theorem 7</u>. Let $F : X \to Y$ be a mapping locally weakly uniformly differentiable on some neighbourhood $\mathcal{U}(x_0)$ of $x_0 \in X$. Suppose that F is continuous in $\mathcal{U}(x_0)$ and weakly Lipschitzian at $x_0 \in X$. Then $VF(x_0, h) = DF(x_0, h)$

and $DF(x_0, h) = F'(x_0)h$, where $F'(x_0)$ denotes the Gateaux derivative of F at x_0.

Proof. According to theorem 6, $VF(x_0, h)$ is continuous at $x_0 \in X$. From the continuity of $VF(x_0, h)$ at $x_0 \in X$ and the existence of $VF(x, h)$ in some neighbourhood of x_0 it follows that $VF(x_0, h) = DF(x_0, h)$. Since F is weakly Lipschitzian at x_0, $DF(x_0, h)$ is bounded. This concludes the proof.

It is easy to prove the following

Theorem 8. Suppose that a mapping $F: X \to Y$ possesses the Gâteaux differential $VF(x_0, h)$ at $x_0 \in X$. Then $VF(x_0, h)$ is continuous at $h = 0$ under an arbitrary direction $u \in X$ (one may suppose that $\|u\| = 1$) if and only if F is continuous at x_0 under the direction u.

Recall that a mapping $F: X \to Y$ is called continuous at $x_0 \in X$ under an arbitrary direction u ($\|u\| = 1$) if $\lim_{t \to \infty} \|F(x_0 + tu) - F(x_0)\| = 0$.

Definition 3. A mapping $F: X \to Y$ is said to be weakly uniformly differentiable in $D_R = \{x \in X : \|x\| < R\}$ if for any given $\varepsilon > 0$ there exists a positive number $\delta(\varepsilon)$ such that if $|t| < \delta(\varepsilon)$, then $\|\frac{1}{t}\omega(x, th)\| < \varepsilon$ for every $x \in D_R$, where $\omega(x, th) = F(x + th) - F(x) - VF(x, th)$ and h is an arbitrary (but fixed) element of X.

Definition 4. Suppose that a mapping $F: X \to Y$ is Gâteaux-differentiable in an open ball $D_{R+\alpha}$ ($\alpha > 0$). We shall say that F possesses an uniformly continuous differential $VF(x, h)$ under the direction $h \in X$ in D_R ($\|x\| < R$) if for any given $\varepsilon > 0$ there exists a number $\delta(\varepsilon, h) > 0$ such that if

$|t| < \delta(\varepsilon, h)$, then

$$\| VF(x + th, h) - VF(x, h) \| < \varepsilon$$

for every $x \in D_R$.

Theorem 9. Let $F : X \to Y$ be a mapping having an uniformly continuous differential $VF(x, h)$ in D_R under the direction $h \in X$. Then F is weakly uniformly differentiable in D_R . Conversely, if F is uniformly continuous in $D_{R+\alpha}$ $(\alpha > 0)$ and F is weakly uniformly differentiable in D_R , then $VF(x, h)$ is uniformly continous in D_R with respect to x .

Remark. Some further results concerning the Gâteaux, Fréchet and bounded differentials will be published later.

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