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A NOTE ON GENERALIZED SOUSLIN'S PROBLEM

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A natural generalization of the well-known Souslin's hypothesis can be stated as follows:

(S_μ): there exists no linear order L with the properties

(1) L has no first and no last element,

(2) L is continuous; i.e. all cuts in L are Dedekind ones,

(3) any set of non-overlapping intervals on L is of power less than $\aleph_\mu$,

(4) there is no dense subset of L of power less than $\aleph_\mu$.

Obviously, (S_1) is the original Souslin's hypothesis. A linear order with properties (1) - (4) will be called generalized Souslin's order of type μ.

It is easy to prove (S_μ) for such μ that $\aleph_\mu$ is a singular cardinal ([3]). In [4], an extent of the class $\boldsymbol{C}_0$ is examined. Results of Hanf [1] imply that (S_μ) holds for each μ with $\aleph_\mu \notin \boldsymbol{C}_0$ (compare (3) and P_1). Analoguous results were obtained in [3], too.

In [2], T. Jech demonstrated the non-provability of the Souslin's hypothesis (S_1) from the axioms A-E of Gödel-Bernays set theory. The aim of this remark is to establish the non-provability of the generalized Souslin's hypothesis (S_μ) for μ such that $\mu = \nu + 1$ and $\aleph_\nu$ is a regular cardinal. The problem of validity of (S_μ) e.g. for $\mu = \omega + 1$ or for the first inaccessible cardinal remains fully open.

The construction of the model ∇ in which generalized Souslin's order of type μ exists will follow step by step that of Jech. We consider ramified graphs on $\omega_\nu \times \omega_\mu$. The necessary and sufficient condition for the existence of generalized Souslin's order of type μ is the existence of a ramified graph of power $\aleph_\mu$ which has no chains or antichains of power $\aleph_\mu$ (this equivalence is provable by an easy modification of [5] even for all regular $\aleph_\mu$).

The definition of the regular graph from [2] must be modified. A ramified graph r is regular iff

(v) there is $\alpha \leq \omega_\mu$ (the length of the graph) such that $\mathcal{D}(r) = \omega_\nu \times \alpha$,

(vi) if $x \in h_\beta$, $\beta < \gamma < \alpha$, then there are $y_1, y_2 \in h_\gamma$, $y_1 \neq y_2$, with $\langle x\ y_1 \rangle \in r$, $\langle x\ y_2 \rangle \in r$,

(vii) for each chain $s \subseteq r$ the length of which is less than α and is cofinal with $\aleph_\gamma$ for some $\gamma < \nu$ there exists a chain $s' \subseteq r$ with $s' \supseteq s$, $s' \neq s$.

Again, c is the set of all regular ramified graphs of lengths ω_μ , b is the set of all regular ramified graphs with lengths less than ω_μ , $u_r = \{ q \in c ; q \supseteq r \}$ for each $r \in b$.

Lemma 4.6. in [2] can be now proved for $\alpha < \omega_\mu$ using a transfinite sequence of type $\leq \omega_\nu$ going to α . The important lemma 4.7 is obvious if α is not limit. For α limit, distinguish two cases:

a) α is cofinal with $\aleph_\gamma$ for some $\gamma < \nu$. Then the cardinality of the set of all chains of lengths α in r

is $\aleph_\nu^{\aleph_\gamma} = \aleph_\nu$ (we may suppose the generalized continuum hypothesis in the set theory; nevertheless, the proof would fail in this point for $\aleph_\nu$ singular even under this assumption). Hence, we can enumerate all chains of lengths α by ordinal numbers from ω_ν : f_0 , f_1 ,..., f_β ,... and extend f_β by adding the point $\langle \beta, \alpha \rangle$ in α -th row.

b) α is cofinal with no $\aleph_\gamma$ for $\gamma < \nu$. We enumerate points of $\omega_\nu \times \alpha$ by ordinal numbers from ω_ν : e_0, $e_1, \dots, e_\beta, \dots$. For each $\beta \in \omega_\nu$ choose a chain s_β of length α going through e_β and extend this chain by adding the point $\langle \beta, \alpha \rangle$ from α -th row. Obviously, we obtain a regular ramified graph of length $\alpha + 1$ containing r.

This analogy of lemma 4.8 is obvious: If $\{r_\beta ; \beta \in \gamma\}$ is a transfirnite sequence of elements of b with $\gamma \in \omega_\mu$ and $r_{\beta_1} \subseteq r_{\beta_2}$ for $\beta_1 < \beta_2$, then $r = \bigcup \{ r_\beta ; \beta \in \gamma \}$ belongs to b .

Define a topology t on the set c as in 4.11, take for B the Boolean algebra of all open regular sets of $\langle c, t \rangle$ and define the function $f \in \mathcal{C}(B)$ by $f(\langle x, y \rangle) = V\{ r ; \langle x, y \rangle \in r \}$ for $x, y \in \omega_\nu \times \omega_\mu$.

Cardinals of the model $\nabla(B, x)$ are absolute, since $\sigma(B) = \aleph_\mu$ and $\mu(B) = \aleph_{\mu+1}$. It remains to prove that in the model $\nabla(B, x)$ f is a ramified graph of power $\aleph_\mu$ which has no chains or antichains of power $\aleph_\mu$. It can be done fully analogously to [2], only in the proof of lemma 4.18 the point $\alpha < \omega_\mu$ with $g(\alpha) = \alpha$ should be chosen not to be cofinal with any $\aleph_\gamma$ for $\gamma < \nu$.

R e f e r e n c e s

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