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A REMARK ON THE THEORY OF LATTICE POINTS IN ELLIPSOIDS

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1. Let r be an integer ≥ 2. Let $Q(u_j) = \sum_{j,k=1}^{r} a_{kj} u_k u_j$ be a positive definite quadratic form whose discriminant is denoted by D and let $\overline{Q}$ be the form conjugated with Q. Let M_j, b_j and α_j be real numbers, $M_j > 0$ $(j = 1, 2, \ldots, r)$. The numbers m and n (supplied with indices if convenient) are always integers. The letter c denotes (various) positive constants depending at most from Q, M_j, b_j and α_j $(j = 1, 2, \ldots, r)$. (Positive constants may depend e.g. on ε, then we write $c(\varepsilon)$.)

For a natural p, let μ_p be the p-dimensional Lebesgue measure. By an integral we mean the (absolutely convergent) Lebesgue integral; we put

$$\int_{a-i\infty}^{a+i\infty} f(s)\,ds = i\int_{-\infty}^{\infty} f(a+it)\,dt$$

for $a \in E_1$ (provided the integral on the left side exists). The symbols O and Ω are used with regard to the limiting process for $x \to +\infty$ and the constants involved are of the "type c". $A \ll B$ means $|A| \leq cB$. If $A \ll B$ and $B \ll A$ we write $A \asymp B$.

The present remark is devoted to the study of certain properties of the function

(1) $$A(x) = A(x; \alpha_j) = \sum e^{2\pi i \sum_{j=1}^{r} \alpha_j u_j} ,$$

where the summation runs over all systems $u_1, u_2, \dots, u_k$ of real numbers, satisfying $Q(u_j) \le x$ and $u_j \equiv b_j \pmod{M_j}$ for $j = 1, 2, \dots k$. Put

$$V(x) = V(x; \alpha_j) = \frac{\pi^{\frac{k}{2}} x^{\frac{k}{2}} e^{2\pi i \sum_{j=1}^{k} \alpha_j b_j}}{\sqrt{D} \prod_{j=1}^{k} M_j \, \Gamma(\frac{k}{2}+1)} \delta$$

($\delta = 1$ if all numbers $\alpha_1 M_1, \alpha_2 M_2, \dots, \alpha_k M_k$ are integers, $\delta = 0$ otherwise). Landau proved ([2] pp.54 and 74)

(2) $$P(x) = P(x; \alpha_j) = A(x; \alpha_j) - V(x; \alpha_j) = O(x^{\frac{k}{2} - \frac{k}{k+1}})$$

and, if $A(x) \not\equiv 0$,

(3) $$P(x) = \Omega(x^{\frac{k-1}{4}}).$$

Clearly, without loss of generality, we are able to assume $0 \le b_j < M_j$ and $0 \le \alpha_j < \frac{1}{M_j}$ $(j = 1, 2, \dots, k)$.

2. Denoting $\mathfrak{M} = \langle 0, \frac{1}{M_1}) \times \langle 0, \frac{1}{M_2}) \times \dots \times \langle 0, \frac{1}{M_k})$, let us examine the function

(4) $$\int_{\mathfrak{M}} |A(x)|^{2p} d\alpha_1 d\alpha_2 \dots d\alpha_k = \int_{\mathfrak{M}} |P(x)|^{2p} d\alpha_1 d\alpha_2 \dots d\alpha_k,$$

where p is a natural number ($\delta = 1$ in $\mathfrak{M}$ only for $\alpha_j = 0$, $j = 1, 2, \dots, k$).

<u>Lemma 1</u>.

(5) $$\int_{\mathfrak{M}} |P(x)|^{2p} d\alpha_1 d\alpha_2 \dots d\alpha_k = \frac{1}{\prod_{j=1}^{k} M_j} \Sigma 1,$$

where the summation runs over all systems

(6) $$n_{1k}, n_{2k}, \dots, n_{kk}, \quad m_{1k}, m_{2k}, \dots, m_{kk} \quad (k = 1, 2, \dots, p)$$

satisfying

(7) $$Q(n_{jk} M_j + b_j) \le x, \quad Q(m_{jk} M_j + b_j) \le x \quad (k = 1, 2, \dots, p)$$

and $\sum_{k=1}^{p} n_{jk} = \sum_{k=1}^{p} m_{jk} \qquad (j = 1, 2, \ldots, n).$

(8)

Proof: Clearly,

$$\int_{\mathfrak{M}} |A(x)|^{2p} d\alpha_1 d\alpha_2 \ldots d\alpha_n = \int_{\mathfrak{M}} A^p(x) \overline{A^p(x)} d\alpha_1 d\alpha_2 \ldots d\alpha_n =$$

(9)

$$= \sum \int_{\mathfrak{M}} e^{2\pi i \sum_{j=1}^{n} \alpha_j M_j \sum_{k=1}^{p} (m_{jk} - n_{jk})} d\alpha_1 d\alpha_2 \ldots d\alpha_n .$$

(the summation runs over all systems (6) satisfying (7)). Since for an integer m and $M > 0$ it holds that

$$\int_0^{\frac{1}{M}} e^{2\pi i \alpha M m} d\alpha = \begin{cases} 0 & \text{for } m \neq 0 \\ \frac{1}{M} & \text{for } m = 0 \end{cases}$$

we can infer immediately from (9) the assertion of the lemma.

Lemma 2. Let n and p be natural numbers, and let $\mathfrak{A}$ be a measurable set with $\mu_n(\mathfrak{A}) < +\infty$. Let f be a measurable function on $\mathfrak{A}$.* Then

(10) $$\sqrt[p]{\int_{\mathfrak{A}} |f(t)|^p dt} \leq \underset{t \in \mathfrak{A}}{\text{vrai sup}} |f(t)| \sqrt[p]{\mu_n(\mathfrak{A})}\ ^{1)}$$

and

(11) $$\lim_{p \to \infty} \sqrt[p]{\int_{\mathfrak{A}} |f(t)|^p dt} = \underset{t \in \mathfrak{A}}{\text{vrai sup}} |f(t)| .$$

Proof: We may assume $\mu_n(\mathfrak{A}) > 0$. Put

$$T = \underset{t \in \mathfrak{A}}{\text{vrai sup}} |f(t)| .$$

If $T = +\infty$ (10) holds. Let $T < T' < +\infty$. We have

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1) $\underset{t \in \mathfrak{A}}{\text{vrai sup}} |f(t)| = \underset{\mathfrak{A}}{\text{vrai sup}} |f(t)| = \inf_{\substack{\mathfrak{B} \subset \mathfrak{A} \\ \mu_n(\mathfrak{B}) = 0}} \sup_{t \in \mathfrak{A} - \mathfrak{B}} |f(t)| .$

$$\sup_{t \in \mathfrak{A}-\mathfrak{L}} |f(t)| \leq T'$$

for a suitable subset $\mathfrak{L}$ of $\mathfrak{A}$ such that $\mu_n(\mathfrak{L}) = 0$ and thus

$$\sqrt[p]{\int_{\mathfrak{A}} |f(t)|^p dt} \leq T' \sqrt[p]{\mu_n(\mathfrak{A})} .$$

As T' is arbitrary, (10) follows. Let $T > 0$ (otherwise (10) implies (11)) and $0 < T' < T$. Putting

$$\mathfrak{L} = \{ t \in \mathfrak{A} ; \ |f(t)| \geq T' \}$$

we have necessarily $\mu_n(\mathfrak{L}) > 0$ and further

$$\sqrt[p]{\int_{\mathfrak{M}} |f(t)|^p dt} \geq T' \sqrt[p]{\mu_n(\mathfrak{L})} .$$

From this and (10) using the limit for $p \to +\infty$ and having in view that T' is arbitrary ($T' < T$), we obtain (11).

Theorem 1.

$$\operatorname{vrai\,sup}_{(\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathfrak{M}} | P(x; \alpha_j) | \asymp x^{\frac{n}{2}} .$$

Proof: Let p be a natural number. Denoting $S(x, p)$ the right hand side in (5) we infer from lemmas 1 and 2 ($A(x)$ is continuous in $\mathfrak{M}$ and thus measurable)

$$\operatorname{vrai\,sup}_{\mathfrak{M}} |A(x)| = \operatorname{vrai\,sup}_{\mathfrak{M}} |P(x)| = \lim_{p \to +\infty} \sqrt[2p]{\int_{\mathfrak{M}} |P(x)|^{2p} d\alpha_1 d\alpha_2 \dots d\alpha_n}$$

i.e.

(12) $$\operatorname{vrai\,sup}_{\mathfrak{M}} |P(x)| = \lim_{p \to +\infty} \sqrt[2p]{S(x, p)} .$$

Putting

$$B(x) = A(x; 0, 0, \dots, 0)$$

it is

$$S(x, p) \ll B^{2p}(x) .$$

But, by (2)

$$B(x) \asymp x^{\frac{n}{2}} .$$

Therefore

$$(13) \qquad \sqrt[2p]{S(x,p)} \ll x^{\frac{n}{2}} .$$

Note that

$$Q(u_j) \asymp \max_{j=1,2,\dots,n} |u_j|^2$$

holds for $(u_1, u_2, \dots, u_n) \in E_n$ and hence

$$S(x,p) \gg \sum 1 ,$$

where the summation runs over all systems (6) satisfying (8) and

$$|m_{jk} M_j + b_j| \ll \sqrt{x} , \quad |n_{jk} M_j + b_j| \ll \sqrt{x}$$

$(k = 1, 2, \dots, p, j = 1, 2, \dots, n)$. This implies

$$S(x,p) \gg \sum 1 ,$$

where the summation runs over all systems (6) satisfying (8) and

$$|m_{jk}| \ll \sqrt{x}, \quad |n_{jk}| \ll \sqrt{x}$$

$(k = 1, 2, \dots, p, j = 1, 2, \dots, n)$.

Put

$$R(u) = \sum 1$$

for $u > 0$, where the summation runs over all $m_1, m_2, \dots$ $\dots, m_{2p-1}$ for which

$$|m_j| \le u \quad (j = 1, 2, \dots, 2p-1)$$

and

$$\Big| \sum_{j=1}^{2p-1} m_j \Big| \le u .$$

We shall examine the function $R(u)$. Clearly, we may consider only natural u. Putting

$$\vartheta(\alpha) = \sum_{m=-u}^{u} e^{2\pi i \alpha m}$$

for $\alpha \in \langle 0, 1 \rangle$, i.e.

$$\vartheta(\alpha) = \begin{cases} \dfrac{\sin(2u+1)\pi\alpha}{\sin \pi\alpha} & \text{for } \alpha \in (0,1) \\ 2u+1 & \text{for } \alpha = 0, 1 \end{cases},$$

it is easily seen that

$$R(u) = \sum_{m=-u}^{u} \int_0^1 \vartheta^{2p-1}(\alpha)\, e^{-2\pi i \alpha m}\, d\alpha = \int_0^1 \vartheta^{2p-1}(\alpha)\, \overline{\vartheta(\alpha)}\, d\alpha$$

and thus

$$R(u) = \int_0^1 \left(\frac{\sin(2u+1)\pi\alpha}{\sin \pi\alpha} \right)^{2p} d\alpha .$$

But , by lemma 2, we have

$$\lim_{p \to +\infty} \sqrt[2p]{R(u)} = \operatorname*{vrai\,sup}_{\alpha \in (0,1)} \left| \frac{\sin(2u+1)\pi\alpha}{\sin \pi\alpha} \right| = 2u+1 .$$

Finally, we obtain the relation

$$\lim_{p \to +\infty} \sqrt[2p]{S(x,p)} \gg \left(\lim_{p \to +\infty} \sqrt[2p]{R(c\sqrt{x})} \right)^{k} \gg x^{\frac{k}{2}} ,$$

proving together with (13) and (12) the theorem.

Comparing the result with (2) we can see a remarkable non-uniformity of this estimation on $P(x)$. If we confine ourselves to the case of a_{jk}, M_j, b_j $(j,k = 1, 2, \dots, k)$ being integers, the comparison becomes still more surprising. In [3] the following theorem is stated under these assumptions and for $k > 5$:

There exists a set $\mathcal{L} \subset \mathfrak{M}$, $\mu_k(\mathcal{L}) = 0$ such that

$$P(x; \alpha_j) = O(x^{\frac{k}{4}+\varepsilon})$$

for $(\alpha_1, \alpha_2, \dots, \alpha_k) \in \mathfrak{M} - \mathcal{L}$ and for every $\varepsilon > 0$ (the constants in O-relation are of type $c(\varepsilon)$).

Nevertheless, by the theorem 1 we have

$$\sup_{(\alpha_1,\alpha_2,\dots,\alpha_k)\in \mathfrak{M}-\mathfrak{L}} |P(x;\alpha_j)| >> x^{\frac{k}{2}} .$$

We emphasize, for next purposes, an imporant consequence of lemma 1 (see [3] theorem 4):

<u>Theorem 2</u>.

$$\int_{\mathfrak{M}} |P(x)|^2 d\alpha_1\, d\alpha_2 \dots d\alpha_k = \frac{A(x;0,0,\dots,0)}{M_1 M_2 \dots M_k} . \tag{14}$$

3. Kendall [1] examined similar expressions with regard to $b_1, b_2, \dots, b_k$ under assumption $\alpha_1 = \alpha_2 = \dots = \alpha_k = 0$, $M_1 = M_2 = \dots = M_k$, especially for the case of a circle. In the following we shall prove his result in general form and in a different way on the basis of the identity of landau.

Let $0 \leq \lambda_1 < \lambda_2 < \dots$ $(0 < \lambda'_1 < \lambda'_2 < \dots)$ be the sequence of all numbers of the form $Q(m_j M_j + b_j)$ $(\bar{Q}(\frac{m_j}{M_j} - \alpha_j) > 0)$ for all integers $m_1, m_2, \dots, m_k$. Let

$$a_n = \sum e^{2\pi i \sum_{j=1}^{k} \alpha_j (m_j M_j + b_j)}$$

$$\left(a'_n = \sum e^{2\pi i \sum_{j=1}^{k} \frac{b_j}{M_j} m_j} ,\ a''_n = \sum 1 \right) ,$$

where the summation runs over all systems $m_1, m_2, \dots, m_k$ satisfying $Q(m_j M_j + b_j) = \lambda_n$ $(\bar{Q}(\frac{m_j}{M_j} - \alpha_j) = \lambda'_n)$. Denote

$$\mathfrak{N} = \langle 0, M_1) \times \langle 0, M_2) \times \dots \times \langle 0, M_k) .$$

<u>Theorem 3</u>.

$$\int_{\mathfrak{N}} |P(x)|^2 db_1\, db_2 \dots db_k = \frac{x^{\frac{k}{2}}}{D \prod_{j=1}^{k} M_j} \sum_{n=1}^{\infty} a''_n \frac{J^2_{\frac{k}{2}}(2\pi\sqrt{\lambda'_n x})}{\lambda'^{\frac{k}{2}}_n} . \tag{15}$$ 2)

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2) J_ν is the Bessel function.

Proof: Put

$$(16) \qquad A_\varphi(x;\alpha_j) = \frac{1}{\Gamma(\varphi+1)} \sum_{\lambda_n \le x} a_n (x-\lambda_n)^\varphi$$

$$(17) \quad V_\varphi(x;\alpha_j) = \frac{\pi^{\frac{k}{2}} x^{\frac{k}{2}+\varphi} e^{2\pi i \sum_{j=1}^{k} \alpha_j b_j}}{\sqrt{D} \prod_{j=1}^{k} M_j \, \Gamma(\frac{k}{2}+\varphi+1)} \delta, \quad P_\varphi(x;\alpha_j) = A_\varphi(x;\alpha_j) - V_\varphi(x;\alpha_j)$$

for $\varphi \in E_1, \varphi \ge 0$ (It is $A_0 = A$, $V_0 = V$, $P_0 = P$, $\int_0^x A_\varphi(y)dy = A_{\varphi+1}(x)$ etc.) Because

$$\theta(s) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n s} = \frac{\pi^{\frac{k}{2}}}{\sqrt{D} \prod_{j=1}^{k} M_j s^{\frac{k}{2}}} \left(e^{2\pi i \sum_{j=1}^{k} \alpha_j b_j} \delta + \sum_{n=1}^{\infty} a'_n e^{-\frac{\pi^2 \lambda'_n}{s}} \right)$$

for s complex such that $Re\, s > 0$ and, clearly,

$$A_\varphi(x) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \frac{e^{xs}\theta(s)}{s^{\varphi+1}} \, ds$$

$(a \in E_1, a > 0, \rho > 0)$

we obtain

$$(18) \qquad P_\varphi(x) = \frac{x^{\frac{k}{4}+\frac{\rho}{2}}}{\sqrt{D} \prod_{j=1}^{k} M_j \, \pi^\rho} \sum_{n=1}^{\infty} a'_n \, \frac{J_{\frac{k}{2}+\rho}(2\pi\sqrt{\lambda'_n x})}{\lambda_n'^{\frac{k}{4}+\frac{\rho}{2}}}$$

for $\rho > \frac{k}{2}$ (similarly as Landau [2] pp.258-264 for integer $\rho > \frac{k}{2}$).

The series on the right hand side is moreover absolutely and uniformly convergent for $(b_1, b_2, \dots, b_k) \in \mathcal{H}$ (still for $\rho > \frac{k}{2}$). It implies

$$(19) \int_{\mathcal{H}} P_\varphi(x;\alpha_j) P_\varphi(x;-\alpha_j) db_1 db_2 \dots db_k = \frac{x^{\frac{k}{2}+\rho}}{D \prod_{j=1}^{k} M_j \pi^{2\rho}} \sum_{n=1}^{\infty} a''_n \frac{J^2_{\frac{k}{2}+\rho}(2\pi\sqrt{\lambda'_n x})}{\lambda_n'^{\frac{k}{2}+\rho}},$$

because of

$$\int_{\mathcal{H}} a'_n \overline{a'_m} \, db_1 db_2 \dots db_k = \begin{cases} a''_n \prod_{j=1}^{k} M_j & \text{for } n = m \\ 0 & \text{for } n \ne m \end{cases}$$

and the interchangeability of summation and integration.

The right hand side of (19) is a continuous function of the variable ρ for $\rho \geq 0$ (in the point 0 we mean the one-sided continuity) and holomorphic with respect to in the half-plane $Re\,\rho > 0$. We shall investigate the left side. Let $\mathcal{L}$ be the set of all $(b_1, b_2, \dots, b_n) \in \mathcal{R}$ such that $Q(m_j M_j + b_j) = x$ for suitable $m_1, m_2, \dots, m_n$. Clearly $\mu_n(\mathcal{L}) = 0$. From (16) and (17) it follows that the function $P_\rho(x; \alpha_j)$ and conclusively the integrand in (19) is continuous in the domain $\rho \geq 0$ and holomorphic with respect to ρ in the half-plane $Re\,\rho > 0$ for $(b_1, b_2, \dots, b_n) \in \mathcal{R} - \mathcal{L}$. Using the Lebesgue theorem we thus obtain that the function $(\rho_1 = Re\,\rho,\ \rho_2 = Im\,\rho)$

$$F(\rho) = F(\rho_1 + i\rho_2) = \int_{\mathcal{R}} P_\rho(x; \alpha_j)\, P_\rho(x_1 - \alpha_j)\, db_1\, db_2 \dots db_n$$

is continuous for $\rho \geq 0$ and her derivatives

$$\frac{\partial F(\rho)}{\partial \rho_1} = \int_{\mathcal{R}} \frac{\partial P_\rho(x; \alpha_j) P_\rho(x_1 - \alpha_j)}{\partial \rho_1} db_1\, db_2 \dots db_n,$$

$$\frac{\partial F(\rho)}{\partial \rho_2} = \int_{\mathcal{R}} \frac{\partial P_\rho(x; \alpha_j) P_\rho(x_1 - \alpha_j)}{\partial \rho_2} db_1\, db_2 \dots db_n$$

are continuous for $\rho_1 > 0$. Since the Cauchy-Riemann conditions hold (almost everywhere in $\mathcal{R}$)for the integrand in (19) they hold for the function F and thus the left side in (19) is holomorphic with respect to ρ in the half-plane $Re\,\rho > 0$. Because (19) is proved for $\rho \in E_1,\ \rho > \frac{n}{2}$ we can, using the theorem of uniqueness, state that (19) holds in the half-plane $Re\,\rho > 0$ and we obtain (15) usin the limiting process $\rho \to 0_+$.

4. Applying (2) to (14) and the asymptotic properties of Bessel functions to (15) we are ready to state

Lemma 3.

(20)
$$\int_{\mathfrak{M}} |P(x)|^2 \, d\alpha_1 \, d\alpha_2 \ldots d\alpha_k \asymp x^{\frac{k}{2}}$$

(21)
$$\int_{\mathfrak{N}} |P(x)|^2 \, db_1 \, db_2 \ldots db_k = O(x^{\frac{k}{2}-\frac{1}{2}}), \quad \int_{\mathfrak{N}} |P(x)|^2 \, db_1 \, db_2 \ldots db_k = \Omega(x^{\frac{k}{2}-\frac{1}{2}})$$

Analogously as Kendall did in [1], we can now derive from (20) and (21) a series results using the well- nown lemma (see [4],p.345):

Lemma 4. Let $\mathfrak{A} \subset E_k$ be a measurable set and let $f_n(t)$ $(n = 1, 2, \ldots)$ be a function measurable on $\mathfrak{A}$. Let the series

$$\sum_{n=1}^{\infty} \int_{\mathfrak{A}} |f_n(t)|^2 \, d\epsilon$$

be convergent. Then there exists a set $\mathfrak{L} \subset \mathfrak{A}$, $\mu_k(\mathfrak{L}) = 0$ such that

$$\lim_{n \to +\infty} f_n(t) = 0$$

for any $t \in \mathfrak{A} - \mathfrak{L}$.

If we put in this lemma $\mathfrak{A} = \mathfrak{M}$ or $\mathfrak{A} = \mathfrak{N}$,

$$f_n(\alpha_1, \alpha_2, \ldots, \alpha_k) = P(x_n) \, x_n^{-\frac{k}{4}} \lambda^{-1}(x_n)$$

or

$$f_n(b_1, b_2, \ldots, b_k) = P(x_n) \, x_n^{-\frac{k}{4}+\frac{1}{4}} \lambda^{-1}(x_n)$$

respectively, where $\lambda(x)$ is a positive function of x and $\{x_n\}$ is an increasing sequence of positive real numbers,

$$\lim_{n \to +\infty} x_n = +\infty, \quad \sum_{n=1}^{\infty} \frac{1}{\lambda^2(x_n)} < +\infty$$

we can obtain from (20) and (21) by suitable choice of $\lambda(x)$ and $\{x_n\}$ the following results:

Theorem 4. a) If an increasing sequence $\{x_n\}$ of positive real numbers has a zero exponent of convergence then

$$(22) \qquad P(x_n) = O(x_n^{\frac{k}{4}+\varepsilon}) \qquad (P(x_n) = O(x_n^{\frac{k-1}{4}+\varepsilon}), \text{ resp.})$$

almost everywhere in $\mathfrak{M}$ ($\mathfrak{N}$ resp.) for $n \to +\infty$ and for any $\varepsilon > 0$ (the constants in O -relations are $c(\varepsilon, \{x_n\})$.

b) If $\lambda(x)$ is an arbitrary increasing positive function, $\lim_{x \to +\infty} \lambda(x) = +\infty$, it is

$$\liminf_{x \to +\infty} \frac{|P(x)|}{x^{k/4}\lambda(x)} = 0 \qquad (\liminf_{x \to \infty} \frac{|P(x)|}{x^{k/4 - 1/4}\lambda(x)} = 0, \text{ resp.})$$

almost everywhere in $\mathfrak{M}$ ($\mathfrak{N}$ resp.).

c) If the exponent of convergence of the sequence λ_2, $\lambda_3, \dots$ is 2γ it is

$$(23) \qquad P(x) = O(x^{\frac{k}{4}+\gamma+\varepsilon})$$

almost everywhere in $\mathfrak{M}$ for any $\varepsilon > 0$ (the constants in O -relation are $c(\varepsilon)$).

Proof: The assertions a) and b) are obvious. Let $\varepsilon > 0$ be an arbitrary real number. Putting $x_n = \lambda_n$ and $\lambda(x) = x^{\gamma+\varepsilon}$ we obtain

$$P(\lambda_n) = O(\lambda_n^{\frac{k}{4}+\gamma+\varepsilon})$$

almost everywhere in $\mathfrak{M}$ (for $n \to +\infty$). Because $\sigma = 1$ in $\mathfrak{M}$ only for $\alpha_j = 0$ ($j = 1, 2, \dots, k$) we have also

$$P(\lambda_n) = A(\lambda_n) = O(\lambda_n^{\frac{k}{4}+\gamma+\varepsilon})$$

almost everywhere in $\mathfrak{M}$ (for $n \to +\infty$)and thus ($A(x) = P(x) = A(\lambda_n)$ for $\lambda_n \le x < \lambda_{n+1}$, $\sigma = 0$)

$$P(x) = O(x^{\frac{r}{4} + \gamma + \varepsilon})$$

almost everywhere in $\mathfrak{M}$,i.e. (23).

A comparison of the two results (22) indicates that possible definitiveness of the exponent in (3) may be due to the nature of the system $\ell_1, \ell_2, \dots, \ell_r$.

R e f e r e n c e s

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