Jaroslav Blažek; Milan Koman

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ON AN EXTREMAL PROBLEM CONCERNING GRAPHS

Jaroslav BLAŽEK, Milan KOMAN, Praha

(Preliminary communication)

In this paper, a generalization of a problem proposed by P. Erdős (see e.g. [1, p.87]) and of a problem proposed by P. Turán (see e.g. [2]) is studied. This generalization may be formulated as follows (see also [3]): Let G be a finite graph without loops and multiple edges, the complementary graph of which consists of k components (of connecticity), each having the form of a complete graph $\langle n_i \rangle$, $i = 1, 2, \dots, k$. The problem is to find the minimal number of intersection points of edges for all immersions x) of G into the Euclidean plane E_2. This number will be denoted by $p_k(n_1, n_2, \dots, n_k)$.

1. Upper estimate of $p_k(n_1, n_2, \dots, n_k)$.

a) In a particular case (the problem of P. Erdős), for $n_1 = n_2 = \dots = n_k = 1$, the following upper bound has been proved (see [4] and [3]):

$$(1)\quad p_k(1, 1, \dots, 1) \leqq \frac{1}{4}\left[\frac{k}{2}\right]\left[\frac{k-1}{2}\right]\left[\frac{k-2}{2}\right]\left[\frac{k-3}{2}\right].$$

b) In another particular case (the problem of P. Turán), for $k = 2$, K. Zarankiewicz proved in his paper [2]

x) The term "immersion" is used in the same sense as in [1].

(2) $p_2(n_1,n_2) \leq [\frac{n_1}{2}][\frac{n_1-1}{2}][\frac{n_2}{2}][\frac{n_2-1}{2}] = K(n_1,n_2)$.

c) For $k = 3$, by using a generalization of Zarankiewicz's construction from [2], it can be proved that

$$p_3(n_1,n_2,n_3) \leq K(n_1,n_2+n_3)+K(n_2,n_1+n_3)+K(n_3,n_1+n_2)- \\ - K(n_1,n_2)-K(n_1,n_3)-K(n_2,n_3),$$

where $K(a,b)$ is the symbol defined in (2).

d) In general, for $k \geq 4$ we may suppose that in the sequence $n_1, n_2, \dots, n_k$ all odd integers are preceded by all even integers. We shall use the following notations:

$\overline{m} = [\frac{m+1}{2}]$, $\underline{m} = [\frac{m}{2}]$ (for any integer m);

$a_1 = \overline{n}_1,\ a_2 = \underline{n}_2,\ a_3 = \overline{n}_3,\ a_4 = \underline{n}_4, \dots$;

$b_1 = \underline{n}_1,\ b_2 = \overline{n}_2,\ b_3 = \underline{n}_3,\ b_4 = \overline{n}_4, \dots$;

$$N_i = \sum_{\substack{j=1 \\ j \neq i}}^{k} n_j \qquad (i = 1, 2, \dots, k).$$

Then it is possible, by using a generalization of the construction B from [3], to prove this upper estimate:

$$p_k(n_1,n_2,\dots,n_k) \leq \sum_{i=1}^{k} K(n_i,N_i) - \sum_{\substack{i,j=1 \\ i<j}}^{k} K(n_i,n_j) + \\ + L(n_1,n_2,\dots,n_k) + \varepsilon M(a_i,b_i),$$

where

$$L(n_1,n_2,\dots,n_k) = \sum_{\substack{r,s,t,u=1 \\ r<s<t<u}}^{k} (a_r a_s a_t a_u + a_r a_s b_t b_u + \\ + a_r b_s b_t a_u + b_r a_s a_t b_u + b_r b_s a_t a_u + b_r b_s b_t b_u)$$

and where $\varepsilon = 1$ if in the number of odd integers in the sequence $n_1, n_2, \ldots, n_k$ is odd, and $\varepsilon = 0$ otherwise; $M(a_i, b_i)$ is a function of degree 2 in $a_1, \ldots, a_k, b_1, \ldots, b_k$.

2. Lower estimate of $p_k(n_1, n_2, \ldots, n_k)$. It seems to us that all upper bounds mentioned in part 1 do not differ essentially from the number $p_k(n_1, n_2, \ldots, n_k)$. But the establishment of a precise enough lower bound seems to be rather difficult.

In case $n_1 = n_2 = \cdots = n_k = 1$ is proved in [4] and [3]

(3) $$k\, p_{k-1}(1, 1, \ldots, 1) \leqq (k-4)\, p_k(1, 1, \ldots, 1)$$

and
$$\frac{3}{280} k(k-1)(k-2)(k-3) \leqq p_k(1, 1, \ldots, 1).$$

For $k = 2$, in [2] the proof of the inequality

(4) $$K(n_1, n_2) \leqq p_2(n_1, n_2)$$

is not correct because of an incorrect application of Lemma 2 (see [2],p.139). We do not know (if $\min(n_1, n_2) \geqq 5$) any proof of (4). We can only prove the following inequality analogous to (3):

(5) $$n_1 p_2(n_1 - 1, n_2) \leqq (n_1 - 2)\, p_2(n_1, n_2).$$

In general, we can prove

$$\sum_{i=1}^{k} n_i p_k(n_1, \ldots, n_{i-1}, n_i - 1, n_{i+1}, \ldots, n_k) \leqq$$
$$\leqq (n_1 + n_2 + \cdots + n_k - 4)\, p_k(n_1, n_2, \ldots, n_k)$$

which is a generalization of (3) and (5).

R e f e r e n c e s

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