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ON EXTREMA OF FUNCTIONALS

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Introduction. The present paper is dealing with study of extrema of functionals. One simple generalization of Vajnberg's result on existence of minimum of non-linear functional f is given and the condition for uniqueness of minimum is established. These conditions concern the second differential of f. Another theorem, where the sufficient conditions (concerning gradient of the functional in question) for existence and uniqueness of extremum of f are given, is presented. Furthermore, several simple conditions for weak convergence of minimizing sequence are given and strong convergence is investigated, too.

Assuming existence of a unique minimum of the functional in question, a simple condition concerning the second differential of f is sufficient for the strong convergence of minimizing sequence. Given a sequence $\psi_n(x) = \Phi(x) - f_n(x)$ of functionals, where Φ is non-linear, f_n are linear (we are working in reflexive Banach spaces) and letting $\psi_n(x_n) = \min\limits_{x \in E} \psi_n(x)$ (this minimum existing), $(n = 0, 1, 2, \dots)$ the implication $f_n \to f_0 \Rightarrow x_n \rightharpoonup x_0$ holds under certain conditions.

Terminology and notations used in this paper. Real Banach space is denoted by E (or E_x, E_y etc.) - E^* is

the space of all linear and bounded functionals on E; the symbol $[E_x \to E_y]$ denotes the set of all linear and bounded mappings of E_x to E_y.
Let F be an operator from E_x to E_y. We shall denote by $DF(x,h)$ linear Gateaux' differential of operator F in the point x, i.e.

$$DF(x,h) = \lim_{t\to 0} \frac{F(x+th)-F(x)}{t}, \quad h \in E_x \text{, where}$$

$DF(x,h)$ is bounded and linear in variable h. If f is a functional on E having a linear Gateaux' differential on the set $M \subset E$, then

(1) $$Df(x,h) = F(x)h,$$

where $F(x) \in [E \to E_1]$, x being fixed, $x \in M$. The operator F defined by the equation (1) is called gradient of the functional f and we shall write

$$F(x) = \operatorname{grad} f(x).$$

Operator F defined on E to E^* is called potential on the set $M \subset E$, if there is such a functional f that the equality

$$\operatorname{grad} f(x) = F(x)$$

holds for all $x \in M$.

Remark 1. If the operator F defined on E to E^* is potential on $M \subset E$, then there exists only one functional f, for which $f(x_0) = f_0$ (x_0 being a fixed point in M) and $F(x) = \operatorname{grad} f(x)$ on M; the functional f is expressed by:

$$(2) \qquad f(x) = f_0 + \int_0^1 F(x_0 + t(x - x_0))(x - x_0)\,dt$$

under certain conditions (see [1],§ 5) which are fulfilled whenever this relation is used. Weak convergence is denoted by $\xrightarrow{w}$

Remark 2 ([2], § 3). Banach space has a weakly compact sphere if and only if it is reflexive.

Lemma 1 ([1],§ 9). Given a Banach space E with a weakly compact sphere and given a bounded weakly closed set $\sigma \subset E$ and a lower-semicontinuous functional on E, then f is bounded from below on σ and there exists $\min\limits_{x \in \sigma} f(x)$.

Lemma 2. Let E be a Banach space with a weakly compact sphere; let f be lower-semicontinuous functional on E, $x_0 \in E$ and suppose that there is a $K > 0$ such that $r > K$ implies

$$\inf_{\|x\| = r} f(x) \geqq f(x_0) .$$

Then there exists an absolute minimum of $f(x)$, i.e. $\min\limits_{x \in E} f(x)$.

Proof. Let $r_0 = \max(K, \|x_0\|)$; $D_r = \{x ; \|x\| \le r\}$. There exists $\min\limits_{x \in D_r} f(x)$ according to Lemma 1. Now it is trivial to show that

$$\min_{x \in E} f(x) = \min_{x \in D_{r_0}} f(x) .$$

Definition. A point x_0 is a critical point of the functional f if

$$\operatorname{grad} f(x_0) = \theta , \quad (\|\theta\| = 0).$$

Theorem 1. Let E be a Banach space with a weakly compact sphere. Assume that: 1) The functional f has

Gateaux' differential of the first and the second orders on E and the inequality

(3) $$D^2 f(x, h, h) \geq \gamma(\|h\|) \cdot \|h\|$$

holds for all $h \in E$, where $\gamma(t)$ is a continuous, real-valued function on $\langle 0, +\infty)$, non-negative such that

(4) $$\lim_{R \to \infty} \frac{1}{R} \int_0^R \gamma(t) dt = \infty .$$

2) $D^2 f(tx, h, h)$ is continuous for $t \in \langle 0, 1 \rangle$.

Then there exists $\min_{x \in E} f(x)$. Furthermore, if $\gamma(t) > 0$ for $t > 0$, then there exists only one extremal point.

__Proof__. The first assumption implies the lower-semicontinuity of functional f in any sphere in E. According to Lemmas 1 and 2 it is sufficient to show that there exists a number $R_0 > 0$ such that for $R > R_0$ the inequality

$$\inf_{\|x\|=R} f(x) \geq f(x_0)$$

holds (x_0 is a point in the sphere $\{x; \|x\| \leq R_0\}$). Let $F(x) = \operatorname{grad} f(x)$. Then, according to (2), we can write

$$F(x)h = F(\theta)h + \int_0^1 DF(tx, x) h \, dt ;$$

particularly, for $h = x$ we have

$$F(x)x = F(\theta)x + \int_0^1 DF(tx, x)x \, dt \geq F(\theta)x + \|x\| \cdot \gamma(\|x\|).$$

Consequently, the relation

$$f(x) = f(\theta) + \int_0^1 F(tx)tx \frac{dt}{t}$$

implies the estimate

$$f(x) \geq f(\theta) + \int_0^1 [F(\theta)tx + \|tx\| \cdot \gamma(\|tx\|)] \frac{dt}{t} =$$
$$= f(\theta) + F(\theta)x + R \cdot \int_0^1 \gamma(tR)\,dt$$

on the sphere $\|x\| = R$, or $f(x) \geq f(\theta) + R \cdot (-\|F(\theta)\| + \int_0^1 \gamma(tR)dt)$. But $\int_0^1 \gamma(tR)\,dt = \frac{1}{R} \cdot \int_0^R \gamma(t)\,dt$, so that for a given $K > 0$ there exists a number R_0 such that for $R > R_0$ the inequality $f(x) \geq f(\theta) + K$ holds on the sphere $\|x\| = R$, i.e. $\inf_{\|x\| = R} f(x) > f(\theta)$.

The second part of theorem is trivial. If both x_1 and x_2 are critical points and $x_1 - x_2 \neq 0$, we have

$$0 = Df(x_2, h) - Df(x_1, h) = D^2 f(x_1 + \tau(x_2 - x_1), h, x_2 - x_1)$$

for all $h \in E$; especially for $h = x_2 - x_1$ we have a contradiction.

Remark 3 ([1], § 9). If x_0 is an extremal point of f on the open set $\omega \subset E$ and there exists $Df(x_0, h)$, then the point x_0 is critical.

Theorem 2. Let E be a Banach space with the weakly compact sphere; F potential operator on E to E^*; $x_0 \in E$ and let $F(x_0 + t(x - x_0))(x - x_0)$ be continuous for $t \in \langle 0, 1 \rangle$. Assume that there exists a measurable function $\lambda_{x_0}(s)$, defined on $\langle 0, \infty)$ such that

a) $\frac{\lambda_{x_0}(s)}{s}$ is bounded on any finite interval;

b) there exists R_0 such that $\int_0^{R_0} \frac{\lambda_{x_0}(s)}{s}\,ds > 0$;

c) $F(x)(x - x_0) \geq \lambda_{x_0}(\|x - x_0\|)$;

d) $x_n \xrightarrow{w} x \Longrightarrow F(x)(x - x_0) \leq \underline{\lim}\, F(x_n)(x_n - x_0)$.

Let us designate by $f(x)$ the functional for which

$F(x) = \operatorname{grad} f(x)$.

I Then there exists a local minimum of the functional f and accordingly a critical point.

II Furthermore, if

e) $\int_0^R \frac{\lambda_{x_0}(s)}{s}\, ds > 0$ for $R \geq R_0$ then there exists an absolute minimum of f.

III Furthermore, if $\int_0^R \frac{\lambda_{x_0}(s)}{s}\, ds > 0$ for $R > 0$,

then the absolute minimum is unique.

IV If, for arbitrary points $x_1, x_2 \in E$; $x_1 \neq x_2$ $(F(x_1) - F(x_2))(x_1 - x_2) > 0$ then f has at most one critical point.

Proof. We shall prove that f is weakly lower-semicontinuous on E. The first assertion then follows from Lemma 1 and the fact that $f(x) > f(x_0)$ for x; $\|x - x_0\| = R$. (According to Lemma 1 there exists $\min_{\|x - x_0\| \leq R} f(x)$ and as a result of the relation $f(x) > f(x_0)$ on $\|x - x_0\| = R$ there exists a critical point.)

Let $x_n, \tilde{x} \in E$; $x_n \xrightarrow{w} \tilde{x}$. The inequality

$$F(x_0 + t(x - x_0))(x - x_0) \geq \frac{\lambda_{x_0}(t \cdot \|x - x_0\|)}{t \cdot \|x - x_0\|} \cdot \|x - x_0\|$$

holds on the assumption (c) (t is positive). Because of boundedness of $\|x_n - x_0\|$ ($\|x_n - x_0\|$ is bounded owing to weak convergence of $\{x_n\}$) we have according to (a)

(5) $F(x_0 + t(x_n - x_0))(x_n - x_0) > -M$; $M > 0$, $t \in \langle 0, 1 \rangle$; $n = 1, 2, \ldots$.

Now,

$$(6)\begin{cases} \int_0^1 F(x_0+t(\tilde{x}-x_0))(\tilde{x}-x_0)\,dt \overset{(d)}{\leq} \int_0^1 \underline{\lim}\, F(x_0+t(x_n-x_0))(x_n-x_0)\,dt \leq \\ \qquad \leq \underline{\lim} \int_0^1 F(x_0+t(x_n-x_0))(x_n-x_0)\,dt\ , \end{cases}$$

where the last inequality follows from Fatou's lemma which can be used according to (5). Now, using the relation

$$f(x) = f(x_0) + \int_0^1 F(x_0+t(x-x_0))(x-x_0)\,dt$$

and applying the inequality (6), we obtain the desired result.

II The second statement trivially follows from Lemma 2 and the assumption (e) using the fact

$$f(x) = f(x_0) + \int_0^1 F(x_0+t(x-x_0))(x-x_0)\,dt\ .$$

III Let $f(x_1) = f(x_2)$ be minimum of $f(x)$; $x_1 \neq x_2$.
Then we have

$$f(x_2) = f(x_1) + \int_0^1 F(x_1+t(x_2-x_1))(x_2-x_1)\,dt$$

so that the following relation must hold

$$0 = \int_0^1 F(x_1+t(x_2-x_1))(x_2-x_1)\,dt \geq \int_0^1 \lambda\cdot(t\cdot|x_2-x_1|)\frac{dt}{t} = \int_0^R \frac{\lambda(s)}{s}\,ds > 0,$$

which is a contradiction.

IV Let $x_1 \neq x_2$; $x_1, x_2 \in E$;

$$\operatorname{grad} f(x_1) = F(x_1) = 0;\quad \operatorname{grad} f(x_2) = F(x_2) = 0 .$$

Then we have $0 = (F(x_1) - F(x_2))(x_1 - x_2) > 0$;
i.e. a contradiction.

<u>Theorem 3</u>. Let E be a space with a weakly compact sphere, $f(x)$ weakly lower-semicontinuous functional on E, x_0 point of the local minimum of f such that there exists $r > 0$ such that for $\{x;\ 0 < \|x-x_0\| < r\}$ the relation $f(x) > f(x_0)$ holds. Let $\|x_n - x_0\| \leq r$,

$f(x_n) \to f(x_0)$. Then $x_n \xrightarrow{w} x_0$.

Proof. Let us suppose the contrary. The sequence $\{x_n\}$ is bounded so that there is a subsequence $\{x_{n_k}\}$; $x_{n_k} \xrightarrow{w} \tilde{x} \neq x_0$. Then we have

$$f(\tilde{x}) \leq \underline{\lim}\, f(x_{n_k}) = f(x_0) \Longrightarrow f(\tilde{x}) = f(x_0),$$

which is a contradiction.

Theorem 4. Let E and f be defined just as in Theorem 3. Let $\lim\limits_{\|x\| \to \infty} f(x) = \infty$ and let there be a unique minimum of f; let us denote it $f(x_0)$. Then the implication $f(x_n) \to f(x_0) \Longrightarrow x_n \xrightarrow{w} x_0$ holds.

Proof. It is clear that $f(x) > f(x_0)$ for $x \in E - \{x_0\}$. Let $\{x_n\}$ be such a sequence that $f(x_n) \to f(x_0)$. Either $\{x_n\}$ is bounded and the assertion follows from Theorem 3 or $\|x_n\| \to \infty$ but in this case the assumption $f(x_n) \to f(x_0)$ does not hold.

Theorem 5. Let E be a Banach space with a weakly compact sphere; $f(x)$ functional on E which satisfies all the conditions of Theorem 1 so that there is $\min\limits_{x \in E} f(x) = f(x_0) = d$. Let $\{x_n\}$ be minimizing sequence i.e. $f(x_n) \to f(x_0)$. Let there be a number $c > 0$ and a point $t_0 \in (0, \infty)$ such that for $t \geq t_0$ the inequality $\gamma(t) > c$ holds. Then $x_n \to x_0$.

Proof. Let

$$g(x, y) = \frac{1}{2} f(x) + \frac{1}{2} f(y) - f\left(\frac{x+y}{2}\right).$$

We shall arrange the expression on the right-hand side using formula (2) and Fubini's theorem:

$$g(x,y)=\frac{1}{2}\cdot[f(x)-f(\frac{x+y}{2})]+\frac{1}{2}\cdot[f(y)-f(\frac{x+y}{2})]=$$

$$=\frac{1}{2}\cdot\int_0^1 Df(\frac{x+y}{2}+t\cdot\frac{x-y}{2},\frac{x-y}{2})dt+\frac{1}{2}\cdot\int_0^1 Df(\frac{x+y}{2}+t\cdot\frac{y-x}{2},\frac{y-x}{2})dt=$$

$$=\frac{1}{4}\cdot\int_0^1[Df(\frac{x+y}{2}+t\cdot\frac{x-y}{2},x-y)-Df(\frac{x+y}{2}+t\cdot\frac{y-x}{2},x-y)]dt=$$

$$=\frac{1}{4}\cdot\int_0^1 dt\int_0^1 D^2f(\frac{x+y}{2}+t\cdot\frac{y-x}{2}+s\cdot t\cdot(x-y),x-y,t\cdot(x-y))ds=$$

$$=\frac{1}{4}\cdot\int_0^1 tdt\int_0^1 D^2f(\frac{x+y}{2}+t\cdot\frac{y-x}{2}+s\cdot t\cdot(x-y),x-y,x-y)ds.$$

Using the first assumption in Theorem 1 we obtain

$$g(x,y)\geq\frac{1}{4}\cdot\int_0^1\gamma(\|x-y\|)\cdot\|x-y\|tdt=\frac{1}{8}\gamma(\|x-y\|)\cdot\|x-y\|.$$

Further, $\varepsilon_1>0$ being arbitrary, there exists n_0 such that $f(x_n)\leq d+\varepsilon_1$ for all $n\geq n_0$. Then the following relation holds:

$$g(x_n,x_0)\leq\frac{d+\varepsilon_1}{2}+\frac{d}{2}-d<\varepsilon_1.$$

Choosing $\varepsilon=\frac{\varepsilon_1}{8}$ we have proved that for arbitrary $\varepsilon>0$ there exists $n_0>0$ such that for $n\geq n_0$ the relation

(7) $$\gamma(\|x_n-x_0\|)\cdot\|x_n-x_0\|<\varepsilon$$

holds.

Now, the minimizing sequence $\{x_n\}$ is bounded by the last assumption of the theorem in question (one can prove it easily by contradiction); let $\|x_n-x_0\|\leq K<\infty$. If we can choose a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that for some $\varepsilon_0>0$ the relation $\|x_{n_k}-x_0\|>\varepsilon_0$ holds, then

$$\lim\sup \gamma(\|x_n - x_0\|)\cdot\|x_n - x_0\| \geq \min_{t\in\langle \varepsilon_0 K\rangle}\gamma(t)\cdot\varepsilon_0\,(>0);$$

this is in contradiction with (7).

Lemma 3. Let E be a Banach space with a weakly compact sphere; $\phi(x)$ is a non-linear functional on E. Let $\psi_f(x) = \phi(x) - f(x)$ for an arbitrary linear functional f on E. Given a positive number K_1 let $\psi_f(x)$ satisfy the conditions of Theorem 1 for those f for which $\|f\| \leq K_1$. Let us denote $\min_{x\in E}\psi_f(x)$ by $\psi_f(x_f)$. Then there is a positive number K_2 (depending on K_1) such that $\|x_f\| \leq K_2$.

Proof. In the first part of the proof of Theorem 1 we obtained the estimate

$$\psi_f(x) \geq \psi_f(\theta) + R\cdot(-\|F_f(\theta)\| + \int_0^1 \gamma(tR)\,dt), \tag{8}$$

where $F_f(x) = \operatorname{grad}\psi_f(x)$.

Here we have $F_f(x) = \operatorname{grad}\phi(x) - f = \Phi(x) - f$.

From (8) it follows

$$\phi(x) \geq \phi(\theta) + R\cdot(-\|\Phi(\theta)\| - 2K_1 + \int_0^1 \gamma(tR)\,dt);$$

according to this inequality there exists $R_0 > 0$ such that for $R > R_0$ the relation $\phi(x) > \phi(\theta)$ holds. Now it can be shown clearly that for arbitrary $K_2 > R_0$ there is $\|x_f\| \leq K_2$. Actually, if $\|f\| \leq K_1$, we obtain from (8)

$$\psi_f(x) - K_1 > \psi_f(\theta) - \|\Phi(\theta)\| - K_1 + \int_0^1 \gamma(tR)\,dt$$

and $-\|\Phi(\theta)\| - K_1 \geq -\|\Phi(\theta)\| - 2K_1$, so that the inequality $\psi_f(x) > \psi_f(\theta)$ holds on the sphere $\|x\| = R \geq R_1$ (where R_1 is a number, $R_1 \leq R_0$) and the point of $\min\psi_f(x)$ cannot be contained outside of sphere

$\|x\| = K_2$.

Remark 4. Roughly speaking, if the functionals f are in a fixed sphere then there is a fixed sphere which contains all the points of $\min \psi_f(x)$ (under certain conditions).

Theorem 6. Let E be a Banach space with a weakly compact sphere, let Φ be a non-linear functional on E. Let f_i $(i = 0, 1, 2, \dots)$ be linear functionals on E, $f_n \to f_0$ (in E^*) $(n = 1, 2, \dots)$. Let us write $\psi_i(x) = \Phi(x) - f_i(x)$. Let $\psi_i(x)$ satisfy the conditions of Theorem 1. Let $\psi_i(x_i) = \min_{x \in E} \psi_i(x)$. Then $x_n \to x_0$ in E.

Proof. x_i is an extremal point of functional $\psi_i(x)$ so that $\operatorname{grad} \psi_i(x_i) = 0$, i.e.

$$0 = \operatorname{grad} \psi_n(x_n) = \operatorname{grad} \Phi(x_n) - f_n \quad (n = 0, 1, 2, \dots).$$

From this fact it follows

$$\|\operatorname{grad} \Phi(x_n) - \operatorname{grad} \Phi(x_0)\| - \|f_n - f_0\| \leq 0 ,$$

and further

$$(9) \quad \|\operatorname{grad} \Phi(x_n) - \operatorname{grad} \Phi(x_0)\| \leq \|f_n - f_0\| \xrightarrow[(n \to \infty)]{} 0 .$$

It is $\operatorname{grad} \Phi(x_i) h = D\Phi(x_i, h)$ for $h \in E$. Let $h_i = x_i - x_0$. Because of $f_n \to f_0$ there is a positive number K_1 such that $\|f_i\| \leq K_1$ and, according to Lemma 3, there is a number $K_2 > 0$ such that $\|x_i\| \leq K_2$, so that $\|h_i\| \leq K$. Now, according to Remark 1, we have

$$D\Phi(x_n, h) - D\Phi(x_0, h) = \int_0^1 D^2\Phi(x_0 + t(x_n - x_0), h, x_n - x_0)\,dt$$

and if $h_n = x_n - x_0$ we obtain

$$D\Phi(x_n, h_n) - D\Phi(x_0, h_n) \geq \gamma(\|h_n\|) \cdot \|h_n\| .$$

Let ε be an arbitrary positive number. Now, for $\varepsilon_1 = \frac{\varepsilon}{K}$ there is $n_0 > 0$ such that for $n \geq n_0$ the following relation holds (according to (9)):

$$\gamma(\|h_n\|)\cdot\|h_n\| \leq \|grad\,\phi(x_n) - grad\,\phi(x_0)\|\cdot\|h_n\| < \varepsilon_1\cdot K = \varepsilon$$

so that we have proved:

for arbitrary $\varepsilon > 0$ there exists $n_0 > 0$ such that for $n \geq n_0$ the following relation holds: $\gamma(\|x_n - x_0\|)\cdot\|x_n - x_0\| < \varepsilon$. Now, as in Theorem 5, we obtain $x_n \longrightarrow x_0$.

Remark. After the paper was submitted the authors became aware that Theorem 1 is stated in "M.M. Vajnberg: O minimume vypuklych funkcionalov, UMN 20(1965),121,No.1, 239-240" without proof.

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