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ON THE NEUMANN PROBLEM IN POTENTIAL THEORY

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We shall deal with the Fredholm method for solving the second boundary value problem in potential theory imposing no à priori restrictions on the boundary of the set considered. Let us first briefly recall the classical situation assuming that G is an open set in R^m with a sufficiently smooth boundary B. We are looking for a harmonic function h in G whose normal derivative has prescribed limits at points of B. If we assume h to be represented as a newtonian potential of a single layer distributed over B then the problem reduces to an integral equation of the second kind

$$\varphi(x) + \int_B \varphi(y) K(x,y) \, d H_{m-1}(y) = f(x)$$

for the unknown density φ of the single layer (H_{m-1} stands for $(m-1)$-dimensional Hausdorff measure) and we have the Fredholm theory at our disposal. This method of treating the Neumann problem is well known and widely used in a number of different situations. Let us also note that this method does not require boundedness of the Dirichlet integral of the harmonic function h. One of its disadvantages consists in very strong restrictions on B which are usually connected with the existence and behavior of normal derivatives of single layer potentials. It seems therefore to be of interest to know what restrictions on the boundary are essentially connected with the method and which of them are superfluous. We shall describe here some results in this direction obtained by methods of geometric measure theory. In the classical formulation of the Neumann problem some à priori restrictions on the boundary are inevitable. At least, there must be some normal if we wish to speak of a normal derivative. These restrictions may be avoided if we characterize the normal derivative $\frac{\partial h}{\partial n}$ weakly introducing the functional

$$(1) \qquad \int_B \varphi \frac{\partial h}{\partial n} \, d H_{m-1}$$

over the class $\mathcal{D}$ of all infinitely differentiable functions φ with compact support in R^m. Employing the Gauss-Green formula one may transform (1) into the integral

$$(2) \qquad \int_G \operatorname{grad} h(x) \cdot \operatorname{grad} \varphi(x) \, dx$$

involving no restrictions on B at all. From now on we assume that G is an arbitrary open set with a compact boundary B. Noting that (2) is meaningful whenever $|\operatorname{grad} h|$ is summable over every bounded portion of G we are led to adopt the following definition (compare, e.g., Constantinescu-Cornea: Ideale Ränder Riemannscher Flächen):

Definition 1. If h is a harmonic function in G such that $|\operatorname{grad} h|$ is summable on every bounded open subset of G, we define the distribution Nh over $\mathcal{D}$ by

$$\langle \varphi, Nh \rangle = \int_G \operatorname{grad} h(x) \cdot \operatorname{grad} \varphi(x) \, dx, \quad \varphi \in \mathcal{D}.$$

For reasons that are clear from the above remarks the distribution Nh is termed the generalized normal derivative of h. (Similar functionals suitable for characterizing the boundary values in connection with the first boundary value problem were introduced by L.C. Young.) It is easily seen that Nh has support contained in B. Indeed, if $\varphi \in \mathcal{D}$ vanishes near B, then there is a bounded open set P with a smooth boundary C such that $G \cap$ support $\varphi \subset P \subset P \cup C \subset G$ and we have

$$\langle \varphi, Nh \rangle = \int_P \operatorname{grad} h(x) \cdot \operatorname{grad} \varphi(x)\, dx = \int_C \varphi \frac{\partial h}{\partial n}\, dH_{m-1} = 0 .$$

Let us now consider the Banach space $C^*(B)$ of all signed Borel measures with support in B; total variation is taken as a norm in $C^*(B)$. With every $\mu \in C^*(B)$ we associate the corresponding newtonian potential $\mathcal{U}\mu = p * \mu$, where $p(x) = \dfrac{|x|^{2-m}}{2-m}$ or $p(x) = \log \dfrac{1}{|x|}$ according as $m > 2$ or $m = 2$. Since the gradient of $h = \mathcal{U}\mu$ is summable on bounded portions of G, the distribution $Nh = N\mathcal{U}\mu$ introduced above is available for every $\mu \in C^*(B)$. In general, $N\mathcal{U}\mu$ need not be a measure (in the sense usual in distribution theory). We thus arrive naturally at the following question:

__Problem 1__. What must be the shape of G in order that $N\mathcal{U}\mu$ be a measure for every $\mu \in C^*(B)$?

We know that support $N\mathcal{U}\mu \subset B$, so that $N\mathcal{U}\mu \in C^*(B)$ whenever $N\mathcal{U}\mu$ is a measure.

Before investigating the above mentioned problem we start with the following simpler question:

Problem 2. Fix $y \in B$ and denote by δ_y the unit point mass (= Dirac measure) concentrated at y. What necessary and sufficient restrictions are to be imposed on G in order that $N\mathcal{U}\delta_y \in C^*(B)$?

In order to be able to formulate the answer in a geometric fashion we introduce the concept of a hit of a half-line on G. Let us agree to denote by $\Omega_r(y)$ the open ball of center y and radius $r > 0$ and let Γ stand for the unit sphere $\{\theta : |\theta| = 1\}$ in R^m. A point $x \in H = \{y + t\theta : t > 1\}$ will be termed a hit of the half-line H on G provided $\Omega_r(x) \cap H \cap G \neq \emptyset$ and $\Omega_r(x) \cap (H - G)$ has a positive linear measure for every $r > 0$. The number (possibly zero or infinite) of all the hits of $\{y + t\theta : t > 0\}$ on G will be denoted by $v^G(y, \theta)$. For fixed G and y, $v^G(y, \theta)$ is a Baire function of the variable θ on Γ and we are justified to form the integral

$$v^G(y) = \int_\Gamma v^G(y, \theta)\, d H_{m-1}(\theta) .$$

With this notation we are now in position to formulate the following answer to the question raised in Problem 2 :

Proposition 1. Let $y \in B$. Then $N\mathcal{U}\delta_y \in C^*(B)$ if and only if

$$v^G(y) < \infty . \tag{3}$$

Proof of this proposition may be obtained by techniques developed in connection with investigations of functions whose partial derivatives are measures (E. De Giorgi, H. Federer, W.H. Fleming, K. Krickeberg, J. Mařík, Chr. Y. Pauc).

Remark. (3) implies that G has a well defined m-density at y, which will be denoted by

$$d_G(y) = \lim_{r \downarrow 0} \frac{\text{volume } (\Omega_r(y) \cap G)}{\text{volume } \Omega_r(y)} .$$

Using the above proposition it is not difficult to derive the following theorem which settles the Problem 1 :

Theorem 1. $N \mathcal{U} \mu \in C^*(B)$ for every $\mu \in C^*(B)$ if and only if

$$\sup_{y \in B} v^G(y) < \infty . \tag{4}$$

If (4) holds then $\mu \longrightarrow N \mathcal{U} \mu$ is a bounded operator on $C^*(B)$.

Before proceeding further we shall describe some consequences regarding B which are implied by conditions like (3) and (4). First of all, we have the following

Proposition 2. (4) implies

$$\sup_{y \in R^m} v^G(y) < \infty .$$

Let us now recall the notion of the exterior normal introduced by H. Federer. Given $y \in R^m$ and $\theta \in \Gamma$ we denote by $S(y, \theta)$ the half-space $\{x : (x-y) \cdot \theta < 0\}$. Following H. Federer we term $\theta \in \Gamma$ the exterior normal of G at y provided the symmetric difference of G and $S(y, \theta)$ has m-density zero at y. It is easily seen that, for every y, there is at most one exterior normal of G at y in this sense. The set $\hat{B}$ of all y at which the exterior normal is available is termed the reduced boundary of G. Clearly, $\hat{B} \subset B$. For $y \in \hat{B}$ we shall denote by $n^G(y) = n(y)$ the exterior normal of G at y. Besides that, we agree to put $n^G(y) = n(y) = 0$

(= zero vector) provided $y \notin \hat{B}$. The following proposition is a consequence of known results on sets with finite perimeter due to E. De Giorgi and H. Federer.

Proposition 3. If $v^G(y_j) < \infty$ for a $(m+1)$-tuple of points $\{y_j : 1 \leq j \leq m+1\}$ in general position (i.e., not situated on a single hyperplane), then $H_{m-1}(\hat{B}) < \infty$ and

$$v^G(x) \leq H_{m-1}(\hat{B}) \cdot [\text{distance } (x, B)]^{1-m}$$

for any $x \notin B$.

Combining this result with proposition 2 we see that

$H_{m-1}(\hat{B}) < \infty$ whenever (4) holds.

Let us now impose (4) on G and investigate more closely the operator

(5) $$\mu \rightarrow N \mathcal{U} \mu .$$

First we state the following

Proposition 4. For any fixed $y \in B$, $\dfrac{n(x) \cdot (x-y)}{|x-y|^m}$ is a summable (H_{m-1}) function of the variable x on B and

$$v^G(y) = \int_B \frac{|n(x) \cdot (x-y)|}{|x-y|^m} \, d H_{m-1}(x) .$$

Denoting by $C(B)$ the Banach space of all real-valued continuous functions on B with the usual norm, we are thus justified to associate with every $f \in C(B)$ the integral

$$Wf(y) = \int_B f(x) \frac{n(x) \cdot (x-y)}{|x-y|^m} \, d H_{m-1}(x)$$

corresponding to the classical double layer potential.

__Theorem 2__. Let $A = H_{m-1}(\Gamma)$. Then, for every $f \in C(B)$,

$$\overline{W} f(y) = A(d_G(y) - \frac{1}{2}) f(y) - Wf(y)$$

is a continuous function of the variable y on B and the operator (5) is adjoint to the operator

$$\frac{1}{2} A I + \overline{W}$$

acting on $C(B)$; here, as usual, we denote by I the identity operator.

Now we shall be concerned with the Neumann problem in the following formulation: Given $\nu \in C^*(B)$ determine a $\mu \in C^*(B)$ with $N \mathcal{U} \mu = \nu$.

Theorem 2 shows that this problem reduces to solving the equation

$$(6) \qquad (\frac{1}{2} A I + \overline{W})^* \mu = \nu .$$

In order to be able to apply the Riesz-Schauder theory to the equation (6) we are naturally led to investigate the Fredholm radius of the operator $\overline{W}$.

Letting T range over all compact operators acting on $C(B)$ we put

$$\omega \overline{W} = \inf_T \| \overline{W} - T \| ,$$

so that the Fredholm radius of $\overline{W}$ equals the reciprocal of $\omega \overline{W}$. Let us also introduce the following

__Definition 2__. For fixed $y \in R^m$, $\theta \in \Gamma$ and $r > 0$ denote by $v_r^G(y, \theta)$ the number (possibly zero or infinite) of those hits of the half-line $\{y + t\theta : t > 0\}$ on G that are contained in $\Omega_r(y)$. Then $v_r^G(y, \theta)$

is a Baire function of the variable θ and we are justified to define

$$v^G(y) = \int_\Gamma v_r^G(y, \theta)\, d\, H_{m-1}(\theta).$$

This notation enables us to formulate the following theorem describing relations between the analytic quantity $\omega \overline{W}$ and the geometric structure of B.

<u>Theorem 3</u>. Let B_I stand for the set of all isolated points of B and put $E = B - B_I$ or $E = B$ according as B_I is finite or not. Then

$$\omega \overline{W} = \lim_{r \downarrow 0} \sup_{y \in E} \{A\, |d_G(y) - \tfrac{1}{2}| + v_r^G(y)\}.$$

It is interesting to observe that the smaller is $\omega \overline{W}$, the nicer must be B. If $\omega \overline{W} \geq \frac{1}{2} A$, then $H_{m-1}(B) = \infty$ is possible. If $\omega \overline{W} < \frac{1}{2} A$, then $H_{m-1}(B) < \infty$ and, moreover, there is a closed set $F \subset B$ with $H_{m-1}(F) = 0$ such that every point in $B - F$ has a neighborhood in B which is a non-parametric lipschitzian surface. If $\omega \overline{W} < \frac{1}{4} A$, then every point in E has a neighborhood in B which is a non-parametric lipschitzian surface; hence it follows, in particular, that G has only a finite number of components. Finally, $\omega \overline{W} = 0$ (which means that $\overline{W}$ is a compact operator) implies that E is a surface of class C^1.

Proofs of the last assertions rely on investigations concerning regularity of sets with finite perimeter due to E. De Giorgi and M. Miranda.

Let us now return to the adjoint equations

$$(7) \qquad (\tfrac{1}{2} A I + \overline{W})^* \mu = \nu \quad (\text{over } C^*(B)),$$

(8) $\left(\frac{1}{2} AI + \overline{W}\right) f = g$ (over $C(B)$).

If $\omega \overline{W}$ is sufficiently small, then $\overline{W}$ is close to a compact operator and the Fredholm alternative applies to (7),(8). Besides that, G has only a finite number of components. If $B_1, \dots, B_q$ are the boundaries of the bounded components of G and χ_j designates the characteristic function of B_j on B , then $\{\chi_1, \dots, \chi_q\}$ is a basis in the space of all solutions χ of the equation

$$\left(\frac{1}{2} AI + \overline{W}\right)\chi = 0$$

and we obtain the following theorem on the Neumann problem:

In order that $\nu\ (\in C^*(B))$ belong to the range of the operator $N\mathcal{U}$ it is necessary and sufficient that $\nu(B_j) = 0,\ 1 \leqq j \leqq q$.

Detailed proofs of the above results together with further related investigations and corresponding references may be found in the author's paper "The Fredholm method in potential theory" (supported by the National Science Foundation, United States of America) which will appear in the Transactions of the American Mathematical Society.

Remark. The above text is an abstract of a lecture prepared originally for the conference Equadiff held in Bratislava (September 1-7,1966). Since the talk was scheduled for September 5 when the author was not able to participate in the conference, the lecture could not be delivered.

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