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A CONTRIBUTION TO THE SUCCESSIVE OVER-RELAXATION METHOD

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1. Introduction. Given a system of linear algebraic equations

$$Ax = k \tag{1}$$

arising from a finite difference treatment of elliptic partial differential equations it is often recommended to use the relaxation method for finding its solution. Successive approximations are calculated according to the following linear recurrence formula

$$x_{n+1} = \mathcal{L}_\omega x_n + b \qquad (n = 0, 1, 2, \dots) \tag{2}$$

(notation in accordance with [1]) where b, x_n are vectors and the matrix $\mathcal{L}_\omega$ is obtained from A according to formula (7) of section 2, and depends on a real parameter $\omega \in (0, 2)$. The convergence of the iteration process obviously depends on the value of the spectral radius $\rho(\mathcal{L}_\omega)$ of $\mathcal{L}_\omega$. If A fulfils conditions (4) of section 2, then there exists a unique ω_b in the interval $(0, 2)$ for which $\rho(\mathcal{L}_\omega)$ attains its minimum. Fig. 1 shows the dependence of $\rho(\mathcal{L}_\omega)$ on ω. The left derivative with respect to ω at ω_b is $-\infty$. For $\omega > \omega_b$ the spectral radius $\rho(\mathcal{L}_\omega) = \omega - 1$.

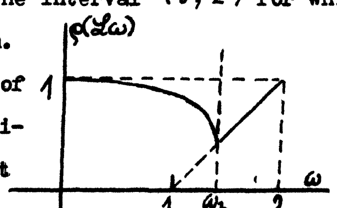

Fig. 1

Therefore it is often recommended to choose $\omega > \omega_b$. Because the actual error $\| x - x_n \| = \| \mathcal{L}_\omega^n (x - x_0) \|$ is not known in carrying out a numerical calculation (the exact solution has been denoted by x), we shall proceed in the following manner: choose a smal real $\varepsilon > 0$ and an initial approximation x_0 and construct a sequence of vectors x_n until $\| x_{n+1} - x_n \| < \varepsilon$. The vector x_n is then taken as the approximate solution. In this case $\| x_{n+1} - x_n \| = \| \mathcal{L}_\omega^n (x_1 - x_0) \|$.

Let us choose an initial vector y and observe the behavior of the number of iterations necessary to achieve $\| \mathcal{L}_\omega^n \ y \| < \varepsilon$, when varying ω.

2. Some basic properties of the successive overrelaxation operator.

Let there be given a matrix of the form

(3) $$A = D(I - U - L);$$

here A is an $n \times n$ matrix, I is the unit matrix, D is a diagonal matrix and L and U are strictly lower and upper triangular matrixes respectively. Let A have the following properties:

a) A is irreducible;

(4) b) A is diagonally dominant with positive diagonal terms;

c) A is consistently ordered and has the property (A) as defined in [2].

The matrix

(5) $$B = L + U$$

is then weakly cyclic of index 2. Let B have the following properties:

a) all eigenvalues of B are real (this is true e.g. for A symmetric)

(6) b) Its positive eigenvalues $\mu_1, \mu_2, \dots, \mu_s$ (counted with regard to their multiplicity) satisfy $1 > \mu_1 > \mu_2 \geq \mu_3 \geq \dots \geq \mu_s > 0$.

The matrix $\mathcal{L}_\omega$ is constructed as follows:

(7) $$\mathcal{L}_\omega = (I - \omega L)^{-1} (\omega U + (1 - \omega) I),$$

and has the following eigenvalues:

(8) $$\begin{cases} \lambda_{2i-1}(\omega) = \left\{ \dfrac{\omega \mu_i + \sqrt{\omega^2 \mu_i^2 - 4(\omega - 1)}}{2} \right\}^2, \\ \lambda_{2i}(\omega) = \left\{ \dfrac{\omega \mu_i - \sqrt{\omega^2 \mu_i^2 - 4(\omega - 1)}}{2} \right\}^2, \\ \qquad i = 1, \dots, s \\ \lambda_{2s+1}(\omega) = \lambda_{2s+2}(\omega) = \dots = \lambda_n(\omega) = 1 - \omega . \end{cases}$$

Let

(9) $$\omega_b = \frac{2}{1 + \sqrt{1 - \mu_1^2}} .$$

Then

(10) $$\lambda_1(\omega) > |\lambda_i(\omega)|, \quad i = 2, \dots, n \quad \text{for } \omega \in (0, \omega_b),$$
$$|\lambda_i(\omega)| = \omega - 1, \quad i = 1, \dots, n \quad \text{for } \omega \in \langle \omega_b, 2).$$

From (4) and (6) it follows that $\lambda_1(\omega)$ is a simple eigenvalue for $\omega \in (0, \omega_b) \cup (\omega_b, 2)$.

Now choose a norm in an unitary n-dimensional space V, and denote the unit eigenvector corresponding to the common eigenvalue $\lambda_1(\omega)$ of $\mathcal{L}_\omega$, and $\mathcal{L}_\omega^*$ by x_n and x_n' respectively. Let $y \in V$ be an arbitrary vector and denote by η_ω a number with the following property:

(11) $$y - \eta_\omega x_\omega \in \mathop{\mathcal{E}}_{u \in V}((u, x_\omega') = 0) = V_\omega \ .$$

Let

(12) $$Q_k(\omega) = \left\| \left(\frac{1}{\lambda_1(\omega)} \mathcal{L}_\omega \right)^k y \right\| \ .$$

For every $\omega \in (0, \omega_b)$ this sequence has the limit

(13) $$\lim_{k \to \infty} Q_k(\omega) = |\eta_\omega| \ .$$

We shall prove that this convergence is uniform with respect to ω in every segment $\langle \alpha, \beta \rangle \subset (0, \omega_b)$.

Let

(14) $$z_\omega = y - \eta_\omega x_\omega \ .$$

Denote by M_ω the operator induced by the operator $\frac{1}{\lambda_1(\omega)} \mathcal{L}_\omega$ on the invariant subspace V_ω. The matrix $\mathcal{L}_\omega$ depends continuously on ω and, as stated above, $\lambda_1(\omega)$ is its simple eigenvalue. Therefore $\lambda_1(\omega)$ is a continuous function of ω, and the same is true for the vectors x_ω and x_ω' and the scalar η_ω. Moreover $\| M_\omega^k \|$, i.e.

$$\sup_{\substack{u \in V_\omega \\ \|u\| = 1}} \left\| \left(\frac{1}{\lambda_1(\omega)} \mathcal{L}_\omega \right)^k u \right\|$$

depends continuously on ω. Choose a fixed $\omega_0 \in (0, \omega_b)$. The operator M_{ω_0} has all eigenvalues less then 1. Therefore there exists an integer q such that $\| M_{\omega_0}^q \| < C < 1$, where C is a positive constant. The continuous dependence of $\| M_\omega^q \|$ on ω implies the existence of a $\vartheta > 0$

such that $\| M_\omega^q \| < C$ holds even for all $\omega \in \langle \omega_0 - \vartheta,$ $\omega_0 + \vartheta \rangle$.

Now construct the following sequences:

$$\begin{array}{ll} & \| x_\omega \|, \ \| M_\omega^q x_\omega \|, \ \| M_\omega^{2q} x_\omega \|, \dots \\ (15) & \| M_\omega x_\omega \|, \ \| M_\omega^{q+1} x_\omega \|, \ \| M_\omega^{2q+1} x_\omega \|, \dots \\ & \dots\dots\dots\dots \\ & \| M^{q-1} x_\omega \|, \ \| M_\omega^{2q-1} x \|, \ \| M_\omega^{3q-1} x_\omega \|, \dots \end{array}$$

These sequences of continuous functions of ω converge monotonically to 0 on $\langle \omega_0 - \vartheta, \ \omega_0 + \eta \rangle$, since $\| M_\omega^{k \cdot q + p} x \| = \| M_\omega^q \| . \| M_\omega^{(k-1) \cdot q + p} x_\omega \| < C \| M_\omega^{(k-1) \cdot q + p} x_\omega \|$, for all positive integers k and for $p = 0, 1, \dots, q-1$. Therefore every sequence defined by (15) converges uniformly to 0 on $\langle \omega_0 - \vartheta, \ \omega_0 + \vartheta \rangle$; indeed for every $\varepsilon > 0$ there exist $k_0, k_1, \dots, k_{q-1}$ such that $\| M_\omega^{kq+p} x_\omega \| < \varepsilon$ for $k > k_p$ $(p = 0, 1, \dots, q-1)$. Let $k_\varepsilon = \max\limits_{p=1,\dots,q-1} k_p$. Therefore $\| M_\omega^k x_\omega \| < \varepsilon$ for $k \geq (k_\varepsilon + 1) \cdot q$. The intervals $(\omega - \vartheta, \ \omega + \vartheta)$ constitute an open covering of the segment $\langle \alpha, \beta \rangle$ (in general ϑ depends on ω). Using the Borel theorem we obtain $\| M_\omega^k x_\omega \|$ converges uniformly to 0 in $\langle \alpha, \beta \rangle$. Since $| |\eta_\omega| - Q_k(\omega) | \leq | M_\omega^k x_\omega |$, also $Q_k(\omega)$ converge uniformly to $|\eta_\omega|$ in $\langle \alpha, \beta \rangle$.

3. The influence of the relaxation factor on the number of iterations necessary for concluding the iteration process.

Let us introduce the following notation: for any $\varepsilon > 0$ and $\omega \in (0, 2)$ denote by $k_\varepsilon(\omega)$ a positive integer for which $\| \mathcal{L}_\omega^{k_\varepsilon(\omega)} y \| \leq \varepsilon$ and such that $\| \mathcal{L}_\omega^k y \| > \varepsilon$ for all $k < k_\varepsilon(\omega)$.

Theorem 1: Let A fulfill conditions (4) and B fulfill conditions (6); choose a vector $y \in V$. Let there exist a segment $\langle \alpha, \beta \rangle \in (0, \omega_b)$ such that $\eta_\omega \neq 0$ for every $\omega \in \langle \alpha, \beta \rangle$. Then there exists an $\varepsilon_0 > 0$ such that for every $\varepsilon \in (0, \varepsilon_0)$ the following assertion is true: there is a $\varkappa > 0$ such that $|k_\varepsilon(\omega_1) - k_\varepsilon(\omega_2)| \leq 1$ for all $\omega_1, \omega_2 \in \langle \alpha, \beta \rangle$ with $|\omega_1 - \omega_2| < \varkappa$.

Proof: Let σ be arbitrary with $0 < \sigma < \min_{\omega \in \langle \alpha, \beta \rangle} |\eta_\omega|$. According to (13) there exists a positive integer k_σ such that

(16) $$|\eta_\omega| - \sigma < Q_k(\omega) < |\eta_\omega| + \sigma$$

whence, using (12),

(17) $$(|\eta_\omega| - \sigma)\, \lambda_1^k(\omega) < \| \mathcal{L}_\omega^k y \| < (|\eta_\omega| + \sigma)\, \lambda_1^k(\omega)$$

for $k > k_\sigma$ and $\omega \in \langle \alpha, \beta \rangle$. The functions $f_1(\xi) = (|\eta_\omega| - \sigma) . \lambda_1^\xi(\omega)$ and $f_2(\xi) = (|\eta_\omega| + \sigma)\lambda_1^\xi(\omega)$ decrease. Now choose an $\varepsilon > 0$ and determine the points $\xi_1(\omega)$ and $\xi_2(\omega)$ in which $f_1(\xi_1(\omega)) = f_2(\xi_2(\omega)) = \varepsilon$:

(18)
$$\xi_1(\omega) = \frac{\log \left(\frac{\varepsilon}{|\eta_\omega| - \sigma}\right)}{\log \lambda_1(\omega)} \; ;$$
$$\xi_2(\omega) = \frac{\log \frac{\varepsilon}{|\eta_\omega| + \sigma}}{\log \lambda_1(\omega)} \; .$$

Choose a real Δ, $0 < \Delta < 1$. We shall prove that there exists a $\sigma > 0$ with $\xi_2(\omega) - \xi_1(\omega) < \Delta$ for all admissible ω and ε. The difference may be estimated as follows:

$$\xi_2(\omega) - \xi_1(\omega) = \frac{\log \frac{|\eta_\omega| - \sigma}{|\eta_\omega| + \sigma}}{\log \lambda_1(\omega)} \leqq \frac{\log \frac{|\eta_\omega| - \sigma}{|\eta_\omega| + \sigma}}{\max\limits_{\omega \in \langle \alpha, \beta \rangle} \log \lambda_1(\omega)} \leqq$$

(19)

$$\leqq \frac{\log \frac{\min\limits_{\omega \in \langle \alpha, \beta \rangle} |\eta_\omega| - \sigma}{\min\limits_{\omega \in \langle \alpha, \beta \rangle} |\eta_\omega| + \sigma}}{\max\limits_{\omega \in \langle \alpha, \beta \rangle} \log \lambda_1(\omega)} .$$

We used the fact that $\log \frac{\xi - \sigma}{\xi + \sigma}$ is an increasing function of ξ in the interval $(\sigma, +\infty)$. The right hand term of the last inequality is a continuous function of σ with value 0 at $\sigma = 0$. It is therefore possible to find a $\sigma > 0$ such that $|\xi_2(\omega) - \xi_1(\omega)| < \Delta$ independently of ω and ε.

Let σ have this property, and take the corresponding k_σ. Let $\varepsilon_0 = \min\limits_{k=0,1,\dots,k_\sigma} \min\limits_{\omega \in \langle \alpha, \beta \rangle} \|\mathcal{L}_\omega^k y\|$, and choose $\varepsilon < \varepsilon_0$. Since the functions $\xi_1(\omega)$ and $\xi_2(\omega)$ are continuous on $\langle \alpha, \beta \rangle$, they are uniformly continuous. Hence there exists a $\varkappa > 0$ such that for $|\omega_1 - \omega_2| < \varkappa$, $\omega_1, \omega_2 \in \langle \alpha, \beta \rangle$ there is $|\xi_1(\omega_1) - \xi_1(\omega_2)| < \frac{1-\Delta}{2}$ and $|\xi_2(\omega_1) - \xi_2(\omega_2)| < \frac{1-\Delta}{2}$.

Choose $\omega_1, \omega_2 \in \langle \alpha, \beta \rangle$ with $|\omega_1 - \omega_2| < \varkappa$. For all $\omega \in \langle \alpha, \beta \rangle$ and $k = 0, 1, \dots, k_\sigma$ there is $\|\mathcal{L}_\omega^k y\| > \varepsilon$. This implies, that both $k_\varepsilon(\omega_1) > k_\sigma$ and $k_\varepsilon(\omega_2) > k_\sigma$, and these constitute a sufficient condition for the validity of (17). Therefore both $k_\varepsilon(\omega_1)$ and $k_\varepsilon(\omega_2)$ satisfy the following inequalities:

(20) $$\xi_1(\omega_i) \leqq k_\varepsilon(\omega_i) \leqq \xi_2(\omega_i) + 1 ,$$

$$(21) \quad \min(\xi_1(\omega_1),\ \xi_1(\omega_2)) \leqq k_\varepsilon(\omega_i) \leqq \max(\xi_2(\omega_1),\ \xi_2(\omega_2)) + 1 .$$

We shall now estimate the difference of the upper and lower bounds for $k_\varepsilon(\omega_i)$:

$$|\max(\xi_2(\omega_1), \xi_2(\omega_2)) + 1 - \min(\xi_1(\omega_1), \xi_1(\omega_2))| \leqq$$

$$(22) \leqq |\xi_1(\omega_1) - \xi_1(\omega_2)| + \min(|\xi_2(\omega_1) - \xi_1(\omega_1)|, |\xi_2(\omega_2) - \xi_1(\omega_2)|) + |\xi_2(\omega_1) - \xi_2(\omega_2)| < \frac{1-\Delta}{2} + \Delta + \frac{1-\Delta}{2} + 1 = 2 .$$

Thus most two integers lie between these bounds and therefore $|k_\varepsilon(\omega_1) - k_\varepsilon(\omega_2)| \leqq 1$.

Theorem 2: Let A fulfill conditions (4) and B fulfill conditions (6). Then

a) Take any finite sequence of real numbers $\omega_1 < \omega_2 <, \dots, < \omega_p$ from the interval $(0, \omega_b)$. Let y be a vector such that $\eta_{\omega_i} \neq 0$ for $i = 1, \dots, p$. Then there exists an $\varepsilon_0 > 0$ such that

$$(23) \qquad k_\varepsilon(\omega_i) \geqq k_\varepsilon(\omega_{i+1})$$

for $\varepsilon < \varepsilon_0$, $\quad i = 1, 2, \dots, p - 1$.

b) Take any finite sequence of real numbers $\omega_1 < \omega_2 \dots < \omega_p$ from the interval $\langle \omega_b, 2)$. Let y be an arbitrary vector, $y \neq 0$.

Then there exists an ε_0 such that, for $\varepsilon < \varepsilon_0$,

$$(24) \qquad k_\varepsilon(\omega_i) \leqq k_\varepsilon(\omega_{i+1}), \quad i = 1, 2, \dots, p - 1 .$$

Proof: It is obviously sufficient to prove the theorem for $p = 2$.

a) If two numbers $0 < \omega_1 < \omega_2 < \omega_b$ are chosen, then, according to (8), $\lambda_1(\omega_1) > \lambda_1(\omega_2)$. Choose $\delta < \min(|\eta_{\omega_1}|, |\eta_{\omega_2}|)$.

Then a positive integer k_1 exists such that for $k > k_1$

(25) $\|\mathcal{L}^k_{\omega_1} y\| > (|\eta_{\omega_1}| - \sigma)\lambda_1^k(\omega_1)$; $\|\mathcal{L}^k_{\omega_2} y\| < (|\eta_{\omega_2}| + \sigma)\lambda_1^k(\omega_2)$.

There exists a positive integer k_2 such that for $k > k_2$

(26) $(|\eta_{\omega_2}| + \sigma)\lambda_1^k(\omega_2) < (|\eta_{\omega_1}| - \sigma)\lambda_1^k(\omega_1)$.

Let $k_0 = max(k_1, k_2)$ and $\varepsilon_0 = \min_{k=0,1,\dots,k_0} \|\mathcal{L}^k_{\omega_1} y\|$.

Now choose $\varepsilon < \varepsilon_0$. For every $k \leq k_0$ there is $\|\mathcal{L}^k_{\omega_1} y\| > \varepsilon$, therefore $k_\varepsilon(\omega_1) > k_0$. If for some k the inequality $\|\mathcal{L}^k_{\omega_1} y\| < \varepsilon$ is satisfied, then $k > k_0$, whence

(27) $\|\mathcal{L}^k_{\omega_2} y\| < (|\eta_{\omega_2}| + \sigma)\lambda_1^k(\omega_2) < (|\eta_{\omega_1}| - \sigma)\lambda_1^k(\omega_1) < \|\mathcal{L}^k_{\omega_1} y\| < \varepsilon$.

Therefore $k_\varepsilon(\omega_2) \leqq k_\varepsilon(\omega_1)$.

b) Let two real numbers ω_1 and ω_2 be chosen, $\omega_b \leq \omega_1 < \omega_2 < 2$. There is $\|\mathcal{L}^k_{\omega_i} y\| = (\omega_i - 1)^k \|(\frac{1}{\omega_i - 1}\mathcal{L}_{\omega_i})^k y\|$ $(i = 1, 2)$.

It is easy to prove that both sequences $\|(\frac{1}{\omega_i-1}\mathcal{L}_{\omega_i})^k y\|$ $(i = 1, 2)$ have finite nonzero upper and lower limits as k tends to infinity. Assertion b) can be then proved analogously as case a) if we note that $0 < \sigma < \min_{i=1,2} \liminf_{k\to+\infty} \|(\frac{1}{\omega_i - 1}\mathcal{L}_\omega)^k y\|$ implies the existence of a positive integer k_0 such that, for $k > k_0$,

$$(28)\quad \|\mathcal{L}^k_{\omega_1} y\| < (\limsup_{k\to+\infty} (\tfrac{1}{\omega_1-1}\mathcal{L}_{\omega_1})^k y + \sigma)(\omega_1 - 1)^k <$$
$$< (\liminf_{k\to+\infty} (\tfrac{1}{\omega_2-1}\mathcal{L}_{\omega_2})^k y - \sigma)(\omega_2 - 1)^k < \|\mathcal{L}^k_{\omega_2} y\|.$$

Remark. If the matrix A is symmetric, then the assertions of theorems 1 and 2 hold even if (6) b) is replaced by $1 > \mu_1 \geq \mu_2 \geq \dots \geq \mu_s > 0$.

4. Conclusion. In theorems 1 and 2, two basic properties of

the characteristic $k_\varepsilon(\omega)$ were proved for small ε.

Theorem 2 expresses the"monotonical" dependence of $k_\varepsilon(\omega)$ on the relaxation factor in the intervals $(0, \omega_b)$ and $\langle \omega_b, 2)$. In theorem 1 the "quasi-contimuity" of $k_\varepsilon(\omega)$ (defined in the statement of theorem 1) as a function of ω in the interval $(0, \omega_b)$ is treated. It seems very unlikely that a similar result holds in the interval $\langle \omega_b, 2)$. In the proof of Theorem 1, the fact that the sequence $Q_k(\omega) = \| ((\lambda_1(\omega))^{-1} \mathcal{L}_\omega)^k y \|$ converges as k tends to infinity for every $\omega \in (0, \omega_b)$, is substantially exploited. However, in the interval $\langle \omega_b, 2)$ all eigenvalues of the matrix $((\omega - 1)^{-1} \mathcal{L}_\omega)^k$ have unit moduli hence in the general case the sequence $\| ((\omega - 1)^{-1} \mathcal{L}_\omega)^k y \|$ is not convergent.

The dependence of $k_\varepsilon(\omega)$ on $\omega \in (0, 2)$ for a 16×16 matrix with properties (4) and (6) was investigated by J. Zitko. A preliminary result is the following: while in the interval $(0, \omega_b)$ the dependence of $k_\varepsilon(\omega)$ on ω was "quasi-continuous", the changes of $k_\varepsilon(\omega)$ for $\omega \in \langle \omega_b, 2)$ are step-like. (The report on further investigations will be published in the Czechoslovak Mathematical Journal.)

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