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A NOTE ON QUASI-SPLITTING OF ABELIAN GROUPS

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In the present note, some relations between the papers [1],[3] and [4] will be investigated.

A group shall always mean an additively written abelian group. If G is such a group, then G_t denotes the maximal torsion subgroup of G. A group G is said to be split if G_t is a direct summand of G. In general, we adopt the notation used in [2].

Lemma 1. Let L be a torsion free group, S a subgroup such that $nL \subseteq S \subseteq L$ for some positive integer n. Then

(1) $$Ext(L,K) \cong Ext(S,K)$$

for every group K.

Proof. Let φ be the isomorphism of L onto nL defined by $\varphi(x) = nx$ $(x \in L)$ and let ψ be the natural homomorphism of S onto S/nL. The sequence

$$0 \rightarrow L \xrightarrow{\varphi} S \xrightarrow{\psi} S/nL \rightarrow 0$$

is exact, and therefore the sequence

$$Ext(S/nL, K) \xrightarrow{\psi^*} Ext(S,K) \xrightarrow{\varphi^*} Ext(L,K) \rightarrow 0$$

is exact for any group K. Thus there is an isomorphism

(2) $Ext(L,K) \cong Ext(S,K)/\psi^*(Ext(S/nL, K))$.

Since $n(S/nL) = 0$, there is also (see [2], § 63, D)) $n(Ext(S/nL, K)) = 0$, and hence $n\psi^*(Ext($

$(S/nL, K)) = 0$. This means that $\psi^*(Ext(S/nL, K))$ is a bounded subgroup of the divisible group $Ext(S, K)$, S being torsion free (see [1], theorem 3.1, or [2], § 63, I)). From this it follows easily that

(3) $Ext(S, K)/\psi^*(Ext(S/nL, K)) \cong Ext(S, K)$.

(1) is now a consequence of (2) and (3). Recall that two groups G, H are said to be quasi-isomorphic (denoted as $G \dot{\cong} H$) if there exist positive integers m, n and also subgroups S and T of G and H respectively with $mG \leq S \leq G$, $nH \leq T \leq H$, and $S \cong T$.

__Lemma 2__. If L_1, L_2 are quasi-isomorphic torsion free groups, then $Ext(L_1, K) \cong Ext(L_2, K)$ for every group K.

__Proof__. Since $L_1 \dot{\cong} L_1$, there are subgroups $S_i \leq$ $\leq L_i$ $(i = 1, 2)$ such that $S_1 \cong S_2$ and $n_i L_i \leq S_i$ $(i = 1, 2)$ for some integers n_1, n_2. By lemma 1,

(4) $Ext(L_i, K) \cong Ext(S_i, K)$ $(i = 1, 2)$

for any group K. Since $S_1 \cong S_2$, there is $Ext(S_1, K) \cong$ $\cong Ext(S_2, K)$ and the required assertion follows hence by (4).

The following definitions were introduced in [4].

__Definition 1__. A torsion free group A is called a K-group if, for every torsion group P, any group G splits whenever G is an extension of the group $H = A \dotplus P$ by a bounded group.

__Definition 2__. A torsion free group A is said to be of locally finite π-rank if A/pA is a finite group for every prime p.

For example, any torsion free group of finite rank, and also the additive group of p-adic integers, are groups of locally finite π-rank.

The following propositions can be easily deduced from definition 1 (see [4]).

Lemma 3. a) Let A be a K-group and P a torsion group. If H is a subgroup of $G = A \dot{+} P$ such that $n G \leq H \leq G$ for some positive integer n, then H is a splitting group.

b) If A_1, A_2 are quasi-isomorphic torsion free groups, then A_1 is a K-group if and only if A_2 is a K-group.

Lemma 4. If groups G and H have $G \doteq H$, then $G/G_t \doteq H/H_t$.

Proof. Consider two subgroups S and T of G and H respectively with $S \cong T$ and $m G \leq S \leq G$, $n H \leq T \leq H$ for some positive integers m, n. From $S \cong T$ it follows that $S/S_t \cong T/T_t$. Since

$$S/S_t = S/(G_t \cap S) \cong \{S, G_t\}/G_t ,$$

$$T/T_t = T/(H_t \cap T) \cong \{T, H_t\}/H_t ,$$

there is an isomorphism

$$\{S, G_t\}/G_t \cong \{T, H_t\}/H_t .$$

Hence and from

$$m(G/G_t) = \{mG, G_t\}/G_t \leq \{S, G_t\}/G_t \leq G/G_t ,$$

$$n(H/H_t) = \{nH, H_t\}/H_t \leq \{T, H_t\}/H_t \leq H/H_t ,$$

one obtains the assertion of the lemma.

Lemma 5. A torsion free group A is a K-group if and only if, for every torsion group P, every group G with $G \doteq A \dot{+} P$ splits.

Proof. Let A be a K-group, let P be any torsion group, and consider a group G with $G \doteq A \dotplus P$. Then there are positive integers m, n and subgroups S, T such that $mG \leq S \leq G$, $nH \leq T \leq T$, and $S \cong T$. By lemma 3 a), T splits; therefore $T = A_1 \dotplus T_t$ and $S = A_2 \dotplus S_t$ with $A_1 \cong A_2$. Clearly, $T \doteq H$, and hence $A_1 \doteq A$ by lemma 4. According to lemma 3 b), both A_1, A_2 are K-groups. From $mG \leq A_2 \dotplus S_t \leq G$ it follows that G splits.

On the other hand, suppose that for every torsion group P, any group G with $G \doteq A \dotplus P$ splits. Let there be given a torsion group P, and consider an extension G of $A \dotplus P$ by a bounded group. Then $mG \leq A \dotplus P \leq G$ for some positive integer m, and therefore $G \doteq A \dotplus P$. By assumption G splits, so that in accordance with definition 1, A is a K-group.

Theorem 1. A torsion free group A is a K-group if and only if $Ext(A, P)$ is a torsion free group (possibly trivial) for every torsion group P.

Proof. Suppose first that A is a K-group, and take a torsion group P. Let the exact sequence

(5) $$0 \to P \to G \to A \to 0$$

represent an element of finite order in $Ext(A, P)$. By [3, theorem 3] for some positive integer n the sequence

$$0 \to P \to \{nG, P\} \to nA \to 0$$

is splitting exact. This means that $\{nG, P\} = A^* \dotplus P$, where $A^* \cong nA \cong A$. Thus A^* is also a K-group.

Since $n G \subseteq A^* \dot{+} P \subseteq G$, there is $G \doteq A^* \dot{+}$ $\dot{+} P$. By lemma 5, this implies that G is a splitting group, i.e. that the sequence (5) represents the zero element of $Ext(A, P)$. Thus we have proved that $Ext(A, P)$ is torsion free.

Now suppose that $Ext(A, P)$ is torsion free for every torsion group P. Take a torsion group P, and consider a group G with $G \doteq A \dot{+} P$. It will be shown that G splits. By [3, theorem 5] the exact sequence

(6) $$0 \to G_t \to G \to G/G_t \to 0$$

represents an element of finite order in the group $Ext(G/G_t , G_t)$. From lemma 4 it follows that $G/G_t \doteq$ $\doteq A$; thus, in view of lemma 2,

(7) $$Ext(G/G_t , G_t) \cong Ext(A, G_t) .$$

By assumption, $Ext(A, G_t)$ is a torsion free group, hence, by (7), the group $Ext(G/G_t , G_t)$ is also torsion free. Thus (6) necessarily represents the zero element of $Ext(G/G_t , G_t)$. This means that (6) is a splitting sequence, i.e. that the group G splits. By lemma 5 this proves that A is a K-group.

Corollary 1. Let A^* be a subgroup of a torsion free group A such that A/A^* is a reduced Π-primary group with finite Π . Let A_n $(n = 1, 2, \ldots)$ be torsion free groups of locally finite κ-rank.

a) If $A = \sum_{n=1}^{\infty} A_n$, then $Ext(A^*, P)$ is torsion-free for any torsion group P .

b) If $A^* = \sum_{n=1}^{\infty} A_n$, then $Ext(A, P)$ is torsion free for any torsion group P.

Proof. If $A = \sum_{n=1}^{\infty} A_n$, then A^* is a K-group; this follows from [4, theorem 7]. Now one may apply theorem 1. Analogously, for $A^* = \sum_{n=1}^{\infty} A_n$.

Theorem 2. Let A be a torsion-free group represented as the union of an increasing chain of subgroups A_n $(n = 1, 2, \dots)$ with $A_1 = 0$. If every A_{n+1}/A_n $(n = 1, 2, \dots)$ is a torsion free group of locally finite r-rank, then A is a K-group.

Proof. By [4, lemma 5], all A_n $(n = 1, 2, \dots)$ are torsion-free groups of locally finite r-rank. Applying [1, theorem 3.3], one obtains that $\mathrm{Ext}(A, P)$ is torsion free for every torsion group P.

Theorem 3. If a torsion free group A is a direct sum of K-groups, then A is again a K-group.

Proof. Consider $A = \sum_{\iota \in I} A_\iota$, where all A_ι $(\iota \in I)$ are K-groups, and let P be a torsion group. Then

(8) $$\mathrm{Ext}(A, P) \cong \sum_{\iota \in I}^{*} (\mathrm{Ext}(A_\iota, P));$$

here the symbol $\sum^*$ denotes the complete direct sum. By theorem 1, all groups $\mathrm{Ext}(A_\iota, P)$ $(\iota \in I)$ are torsion free; thus, in view of (8), $\mathrm{Ext}(A, P)$ must also be torsion free. Now the required assertion follows from theorem 1.

Remark. Theorem 3 was also proved in [4, theorem 2], directly from the definition 1 of a K-group. In [4], it is also shown that every torsion free group of locally finite r-rank is a K-group. This proposition is obtained as a consequence of the following general theorem, [5, theorem 3]: If H is a subrgoup of finite index in a group G, then G splits if and only if H splits. From this it follows that every torsion free group of finite rank is a K-group, [6, theorem 5], and also that if a group G of finite torsion free rank is quasi-isomorphic to a splitting group then G splits.

A group G is called quasi-splitting if it is quasi-isomorphic to a splitting group.

<u>Theorem 4</u>. If A is a torsion free group, then every quasi-splitting group G with $G/G_t \cong A$ is splitting if and only if A is a K-group.

<u>Proof</u>. Suppose first that A is a K-group. Consider a quasi-splitting group G such that $G/G_t \cong A$, and let $G \doteq A^* \dotplus P$, where P is a torsion group and A^* is torsion free. By lemma 4, $A \cong G/G_t \doteq$ $\doteq A^*$; thus, in view of lemma 3 b), A^* is a K-group. On applying lemma 5 we obtain that G splits.

Now suppose that A is not a K-group. By theorem 1, there is a torsion group P for which $\mathrm{Ext}(A, P)$ is not torsion free. Take an exact sequence

(9) $$0 \to P \to G \to A \to 0$$

representing an element of finite order $n > 1$ in $\mathrm{Ext}(A, P)$. Hence G does not split. By [3, theorem 3], the sequence

$$0 \to P \to \{nG, P\} \to nA \to 0$$

is splitting exact, therefore the group $G_1 = \{n G, P\}$ splits. Since $nG \leq G_1 \leq G$, there is $G_1 \doteq G$, and thus G is quasi-splitting. Hence one has a quasi-splitting group G with $G/G_t \cong A$ (see (9)), but which does not split.

This completes the proof of the theorem.

Corollary 2. Let A be a torsion free group and let G be a quasi-splitting group with $G/G_t \cong A$.

a) If $A = \sum_{\iota \in I} A_\iota$, where every A_ι $(\iota \in I)$ is either countable or of locally finite π-rank, then G splits.

b) If A is the union of an increasing chain of subgroups A_n $(n = 1, 2, \dots ; A_1 = 0)$ such that every A_{n+1} / A_n is a torsion free group of locally finite π-rank, then G splits.

Proof. In both cases A is a K-group. Indeed, in case a) this is a consequence of [1, corollary 3.4] and of theorems 1 and 3; in case b) this follows from theorem 2. Now one may apply theorem 4.

Corollary 3. Let A, A^* and A_n $(n = 1, 2, \dots)$ be groups as in corollary 1, and consider a quasi-splitting group G.

a) If $A = \sum_{n=1}^{\infty} A_n$ and $G/G_t \cong A^*$, then G splits.

b) If $A^* = \sum_{n=1}^{\infty} A_n$ and $G/G_t \cong A$, then G splits.

Proof. If $A = \sum_{n=1}^{\infty} A_n$, then by [4, theorem 7], A^* is a K-group. Now our assertion follows from theorem 4; analogously for $A^* = \sum_{n=1}^{\infty} A_n$.

R e f e r e n c e s

[1] R. BAER: Die Torsionsuntergruppe einer Abelschen Gruppe, Math.Ann.,135(1958),219-234.

[2] L. FUCHS: Abelian Groups, Budapest, 1958.

[3] C.P. WALKER: Properties of Ext and quasi-splitting of abelian groups, Acta Math.Acad.Sci. Hung.XV(1964),157-160.

[4] Л. ПРОХАЗКА: Расширения абелевых групп при помощи групп периодических, to appear in Чех.мат.журн.

[5] Л. ПРОХАЗКА: Заметка о расщепляемости смешанных абелевых групп, Чехосл.мат.ж.,10(85), 1960,479-492.

[6] Л. ПРОХАЗКА: К проблеме расщепления некоторых абелевских расширений расщепляемых абелевых групп, Чехосл.мат.ж.,11(86),1961, 365-380.

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