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# SOME EXISTENCE THEOREMS FOR NONLINEAR PROBLEMS

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1. Let $X, X_1, X_2$ be Banach spaces. A mapping $F: X_1 \to X_2$ is said to be bounded if $F$ transforms bounded sets in $X_1$ into bounded sets in $X_2$. We shall say that a mapping $F: X_1 \to X_2$ is linearly bounded if there exists a positive number $\gamma$ such that $\|F(x)\| \leqq \gamma \|x\|$ for every $x \in X_1$. A mapping $F: X_1 \to X_2$ is said to be linearly upper bounded [1], if there exist numbers $\alpha, \gamma > 0$ such that $\|F(x)\| \leqq \gamma \|x\|$ whenever $\|x\| \geqq \alpha$. Similarly we shall say that $F: X_1 \to X_2$ is linearly lower bounded if there exist numbers $\beta, \gamma > 0$ such that $\|F(x)\| \leqq \gamma \|x\|$ whenever $\|x\| \leqq \beta$. Let $F: X_1 \to X_2$ be a mapping of $X_1$ into $X_2$. If the number $|F| = \inf_{0<\rho<+\infty} \{\sup_{\|x\| \geqq \rho} \|F(x)\| \, \|x\|^{-1}\}$ is finite, then the mapping $F$ is linearly upper bounded. The number $|F|$ is called the quasinorm of $F$, see [2]. The mapping $F: X_1 \to X_2$ is said to be asymptotic close to a linear continuous mapping $L: X_1 \to X_2$ if $\lim_{\|x\| \to \infty} (\|F(x) - L(x)\| \, \|x\|^{-1}) = 0$. In particular, a mapping $F: X_1 \to X_2$ is asymptotic close to zero if $\lim_{\|x\| \to \infty} (\|F(x)\| \, \|x\|^{-1}) = 0$. This definition is due to M.A. Krasnoselskij [3] and the following theorem to V.M. Dubrovskij [3]: If $F: X \to X$ is completely continuous (i.e. compact and continuous) and asymptotic close to

zero, then $(I + F)X = X$ . The results of A. Granas [2] and M.D. George [4] are closely related to this theorem. The purpose of this note is to give some further existence theorems for nonlinear functional equations without assumptions of complete continuity of $F$ .

2. In the sequel $E$ denotes the identity mapping of a real or complex separable and complete Hilbert space $X$, $X \neq (0)$ .

We shall say that a linear continuous mapping $A : X \to X$ of Hilbert space $X$ is normal if $AA^* = A^*A$ , where $A^*$ denotes the mapping adjoint to $A$ .

Theorem 1. Let $F: X \to X$ be a mapping of a Hilbert space $X$ into $X$ such that, for every $x \in X$ it has the Gateaux derivative $F'(x)$ . Let $PF'(x)$ be a normal mapping for every $x \in X$ such that $(PF'(x)h, h) \geq 0$ for every $x \in X$, $h \in X$ , where $P$ is a linear mapping of $X$ onto $X$ having an inverse $P^{-1}$, $\|P\| \leq (\sup_{x \in X} \|F'(x)\|)^{-1}$ .

If $|E - PF| < 1$ , then the equation $F(x) = y$ has at least one solution for every $y \in X$ .

Proof. For every $x \in X$ the mapping $G(x) = x - PF(x)$ has the Gateaux derivative $G'(x)$ and $G'(x) = E - PF'(x)$ . Because $G'(x)$ is a normal mapping for every $x \in X$ , then (see [5])

$$\|G'(x)\| = \sup_{\|h\|=1} |(G'(x)h, h)| = \sup_{\|h\|=1} |(h - PF'(x)h, h)| =$$

$$= \sup_{\|h\|=1} [1 - (PF'(x)h, h)] \leq 1, \quad \text{since}$$

$0 \leqq (PF'(x)h,h) \leqq \|P\| (\sup_{x \in X} \|F'(x)\|) < 1$ for every $x \in X$ and $h \in X$ with $\|h\| = 1$. Because $\|G(x) - G(y)\| \leqq \|G'(\bar{x})\| \, \|x - y\|$, where $\bar{x}$ is an element which lies on the line-segment connecting the points $x$, $y \in X$ and $\sup_{x \in X} \|G'(x)\| \leqq 1$, we conclude that $G : X \to X$ is Lipschitzian mapping with constant one. Now let $y^*$ be an arbitrary point in $X$ and set $z^* = P(y^*)$. The equation $F(x) = y^*$ is equivalent to $x - G(x) = z^*$. We shall show that there exists an element $x^* \in X$ such that $F(x^*) = y^*$. Define a mapping $\bar{G} : X \to X$ by $\bar{G}(x) = G(x) + z^*$ for $x \in X$. Since $|G| < 1$, it follows that the inequality $\|G(x)\| \, \|x\|^{-1} < \varepsilon < 1$ holds for all $x$ with norm $\|x\| \geqq \rho_1$, where $\varepsilon$, $\rho_1$ are some constants. Now choose a positive number $\gamma$ such that $\varepsilon + \gamma < 1$ and let $\rho_2 = \|z^*\| \, \gamma^{-1}$. Put $r = \rho_1 + \rho_2$, $D = \{x \in X ; \|x\| \leqq r\}$, $S = \{x \in X ; \|x\| = r\}$. Let $x \in S$, then $\|\bar{G}(x)\| \leqq \|z^*\| + \|G(x)\| \leqq (\varepsilon + \gamma)\|x\| < \|x\|$. Thus $\|\bar{G}(x)\| < \|x\|$ for every $x \in S$. Also $\|\bar{G}(x_1) - \bar{G}(x_2)\| \leqq \|x_1 - x_2\|$ for every $x_1, x_2 \in D$. Hence $\bar{G}$ is Lipschitzian with constant one on $D$, $\bar{G} : D \to X$ and $\bar{G}(S) \subset D$. Since all the assumptions of Browder's theorem [6] are fulfilled, there exists at least one $x^* \in D$ such that $\bar{G}(x^*) = x^*$. Hence $x^* = G(x^*) + z^*$ and therefore $F(x^*) = P^{-1}(z^*)$. Because $P^{-1}(z^*) = y^*$, there is $F(x^*) = y^*$, which completes the proof.

__Remark 1__. Every bounded linear symmetric mapping is normal.

If $F'(x)$ is continuous on $X$, $P = E$ or $P = \vartheta E$, $\vartheta > 0$, then the theorem 1 holds in particular for the equations with poteneial operators (cf.[7], § 5, theorem 5.1). For some classes of potencial operators see [7], chapt. VI. and [8], chapt. VI.

Remark 2. The condition $| E - PF | < 1$ is equivalent to the following assumption: there exist numbers $\alpha, \gamma > 0$, $\gamma < 1$ such that

$$\| x - PF(x) \| \leqq \gamma \| x \| \text{ whenever } \| x \| \geqq \alpha .$$

Corollary 1. Let $\phi : X \to X$ be a mapping of a Hilbert space $X$ into $X$ such that, for every $x \in X$ it has the Gateaux derivative $\phi'(x)$. Let $\phi'(x)$ be a normal mapping for every $x \in X$ such that $|(\lambda\phi'(x)h, h)| \leqq \leqq \|h\|^2$ for every $x \in X$, $h \in X$. If the mapping $\lambda\phi$ is linearly upper bounded with a constant $\gamma < 1$ ($\lambda$ is a real parameter), then the equation $x - \lambda\phi(x) = y$ has at least one solution for every $y \in X$.

Theorem 2. Let $X$ be a Hilbert space, $A$ a linear (not necessarily continous) mapping with domain $\mathcal{D}(A) \subset X$ and $A(X) = X$. Suppose that $A$ has a continuous inverse $A^{-1}$. Let $\phi : X \to X$ be a mapping of $X$ into $X$ asymptotic close to zero having the Gateaux derivative $\phi'(x)$ for every $x \in X$. If $\sup_{x \in X} \| A^{-1}\phi'(x) \| \leqq 1$, then $(A + \phi)\mathcal{D}(A) = X$, i.e. the equation $A(x) + \phi(x) = y$, $x \in \mathcal{D}(A)$, has at least one solution for every $y \in X$.

**Proof.** The equation $A(x)+\phi(x)=y$ $(x\in\mathcal{D}(A),\ y\in X)$ is equivalent to $x+A^{-1}\phi(x)=A^{-1}(y)$. We have $\mathcal{D}(A+\phi)=$ $=\mathcal{D}(A)$, $\mathcal{D}(E+A^{-1}\phi)=X$. Because $\phi$ is asymptotic close to zero, then for an arbitrary $\varepsilon>0$ there exists a number $N>0$ such that for every $x\in X$ with $\|x\|>N$ we have $\|\phi(x)\|\ \|x\|^{-1}<\varepsilon$. Then for every $x\in X$ with $\|x\|>N$

$0\leqq\|A^{-1}\phi(x)\|\ \|x\|^{-1}\leqq\|A^{-1}\|\ \|\phi(x)\|\ \|x\|^{-1}<\varepsilon\ \|A^{-1}\|$.

Thus $A^{-1}\phi : X\to X$ is asymptotic close to zero. Further, similarly as the proof of theorem 1.

**Theorem 3.** Let $F: X\to X$ be a mapping of a Hilbert space $X$ into $X$ such that, for every $x\in X$, it has the Gateaux derivative $F'(x)$, $F$ is Fréchet-differentiable at $0$, $F(0)=0$. Let $PF'(x)$ be a normal mapping for every $x\in X$ and such that $(PF'(0)h,h)\geqq$ $\geqq m\|h\|^2, m>0, (PF'(x)h,h)\geqq 0$ for every $x\in X$, $x\neq 0$, $h\in X$, where $P$ is a linear mapping of $X$ onto $X$ having an inverse $P^{-1}$, $\|P\|<(\sup_{x\in X}\|F'(x)\|)^{-1}$.

Let $\varepsilon$ be an arbitrary positive number such that $\varepsilon<1-$ $-\|G'(0)\|$, where $G=E-PF$. Then there exists a positive number $\delta$ such that for any $y\in X$ with $\|y\|\leqq$ $\leqq\delta(1-d)\|P\|^{-1}$, where $d=\|G'(0)\|+\varepsilon$, the equation $F(x)=y$ has at least one solution in the ball $D=\{x\in X;\ \|x\|\leqq\delta\}$.

**Proof.** Again set $G(x)=x-PF(x)$ for $x\in X$. The mapping $G: X\to X$ is Gateaux-differentiable on $X$ and Fréchet-differentiable at $0$. Moreover $G'(x)=E-PF'(x)$

for every $x \in X$ in the sense of Gateaux, $G'(0) = E - PF'(0)$ in the sense of Fréchet and $G(0) = 0$. We have $\|G'(0)\| \leqq 1-m < 1$, since
$0 < m \leqq (PF'(0)h, h) \leqq \|P\| \, \|F'(0)\| \leqq \sup_{x \in X} \|F'(x)\| \, \|P\| < 1$
for every $h \in X$ with $\|h\| = 1$. Because $\sup_{x \in X} \|G'(x)\| \leqq 1$, the mapping $G: X \to X$ is Lipschitzian on $X$ with constant one. Now choose a positive number $\varepsilon$ such that $\varepsilon < 1 - \|G'(0)\|$. Then there exists $\delta' > 0$ such that $G'(x) - G(0) = G'(0)x + \omega(0, x)$, where $\|\omega(0, x)\| < \varepsilon \|x\|$ if $\|x\| < \delta'$. Taking $0 < \delta < \delta'$, then for every $x \in S$, where $S = \{x \in X ; \|x\| = \delta\}$ we have that
$\|G(x)\| \leqq \|G'(0)x\| + \|\omega(0,x)\| < \|G'(0)\| \, \|x\| + \varepsilon \|x\| = d\|x\| = d\delta$.
Define a mapping $\bar{G}: X \to X$ by $\bar{G}(x) = G(x) + P(y)$, where $y \in X$ is an arbitrary (but fixed) element with $\|y\| \leqq \delta(1-d) \|P\|^{-1}$. Let $x \in S$, then $\|\bar{G}(x)\| \leqq$
$\leqq \|G(x)\| + \|Py\| < d\delta + \delta(1-d) = \delta$. Thus $\bar{G}(S) \subset D$,
$\|\bar{G}(x_1) - \bar{G}(x_2)\| \leqq \|x_1 - x_2\|$ for every $x_1, x_2 \in D$ and $\bar{G}: D \to X$. According to Browder's theorem there exists at least one $x^* \in D$ such that $\bar{G}(x^*) = x^*$. Hence $F(x^*) = y$, which completes the proof.

Theorem 4. Let $F$ be a weakly continuous mapping of Hilbert space $X$ into $X$ such that $F$ has the Fréchet derivative at $0$, $F(0) = 0$ and $(P_1 F'(0)h, h) \geqq m \|h\|^2$, $(m > 0)$, $h \in X$, where $P_1$ is a linear continuous mapping of $X$ onto $X$ having an inverse. Let $\vartheta$ be a number satisfying $0 < \vartheta < 2m \|P_1 F'(0)\|^{-2}$. If $\varepsilon$ is an arbitrary positive number such that $\varepsilon < 1 - \|G'(0)\|$ , where $G = E - PF$, $P = \vartheta P_1$ , then there exists a positive number $\delta$ such that for any $y \in X$ with $\|y\| \leqq \delta(1-(\|G'(0)\|+\varepsilon))\|P\|^{-1}$ the equation $F(x) = y$ has at least one solution in the ball $D = \{x \in X ; \|x\| \leqq \delta\}$ .

<u>Proof</u>. Set $G(x) = x - PF(x)$, $P = \vartheta P_1$ . Then $G(0) = 0$, $G$ is Fréchet-differentiable at $0$ and $\|G'(0)\| < 1$ . Choose $\varepsilon > 0$ such that $\varepsilon < 1 - \|G'(0)\|$ . Then there exists $\delta' > 0$ such that if $\|x\| < \delta'$ then $G(x) = G'(0)x + \omega(0,x)$ , where $\|\omega(0,x)\| < \varepsilon \|x\|$ . Taking $0 < \delta < \delta'$, then for every $x \in D$ we have $\|G(x)\| < (\|G'(0)\|+\varepsilon)\delta$. Define $\bar{G} : X \to X$ by $\bar{G}(x) = G(x) + P(y)$, $x \in X$, $y \in X$ is an arbitrary (but fixed) element with $\|y\| \leqq \delta(1-(\|G'(0)\|+\varepsilon))\|P\|^{-1}$ . Then for every $x \in D$ we have that $\|\bar{G}(x)\| \leqq \|G(x)\| + \|P\| \|y\| < \delta$ . Thus $\|\bar{G}(x)\| < \delta$ for every $x \in D$ and $\bar{G} : D \to D$ is weakly continuous mapping of $D$ into $D$ . But every bounded set in $X$ is weakly compact and every convex closed set in $X$ is weakly closed. Hence according to Schauder's principle there exists at least one $x^* \in D$ such that $x^* = \bar{G}(x^*)$ . Hence $F(x^*) = y$ , which completes the proof.

3. Let $X$ be a Hilbert space, $Y, Z$ non-trivial subspaces of $X$ such that $X$ is their direct sum, $X = Y \oplus Z$ . Denote by $P_Y$, $P_Z$ the linear projection of $X$ onto $Y$, $Z$ respectively. Set $f(x) = x + AF(x)$, $g(x) = x + AG(x)$ , where $A : X \to X$ is a linear continuous mapping of $X$

into $X$ and $F$, $G$ are non-linear mappings of $Y, Z$ into $X$ respectively.

<u>Theorem 5</u> (on intersection). Let $X = Y \oplus Z$ and let $f: Y \to X, g: Z \to X$ be defined as above, where $F: Y \to X$, $G: Z \to X$ are Lipschitzians mappings with constants $\alpha_1$, $\alpha_2$ respectively. Furthermore let $F$, $G$ be linearly upper bounded with bounds $\beta_1$, $\beta_2$ respectively such that $\varepsilon = \|A\|(\beta_1 \|P_Y\| + \beta_2 \|P_Z\| \leqq 1$. If $(\alpha_1 \|P_Y\| + \alpha_2 \|P_Z\|) \|A\| \leqq$ $\leqq 1$, then the intersection $f(Y) \cap g(Z)$ is non-void.

<u>Proof</u>. Put $\phi(x) = A(G(-P_Z x) - F(P_Y x))$ for every $x \in X$. Then for all $x_1, x_2 \in X$ there is

$$\|\phi(x_1) - \phi(x_2)\| \leqq \|A\| (\|G(-P_Z x_1) - G(-P_Z x_2)\| +$$
$$+ \|F(P_Y x_2) - F(P_Y x_1)\|) \leqq \|A\| (\alpha_2 \|P_Z x_1 - P_Z x_2\| + \alpha_1 \|P_Y x_2 -$$
$$- P_Y x_1\|) \leqq \|A\| (\alpha_1 \|P_Y\| + \alpha_2 \|P_Z\|) \|x_1 - x_2\| \leqq \|x_1 - x_2\| .$$

Thus the mapping $\phi: X \to X$ is Lipschitzian with constant one. Under our assumptions, $F$, $G$ are linearly upper bounded with constants $\beta_1$, $\beta_2$ respectively. Therefore

(1) $\|F(y)\| \leqq \beta_1 \|y\|$ for every $y \in Y$ with $\|y\| \geqq \varrho_1$,

(2) $\|G(z)\| \leqq \beta_2 \|z\|$ for every $z \in Z$ with $\|z\| \geqq \varrho_2$

for some $\varrho_1, \varrho_2 > 0$. Put $\varrho = \max(\varrho_1, \varrho_2)$; then (1),(2) are fulfilled for every $y \in Y$ with $\|y\| \geqq \varrho$ and every $z \in Z$ with $\|z\| \geqq \varrho$. Put $K_\varrho = \{y \in Y, \|y\| \leqq \varrho\}$, $\Omega_\varrho = \{z \in Z, \|z\| \leqq \varrho\}$. Since $F$, $G$ are Lipschitzians on $Y$, $Z$ respectively, then there exist positive numbers $K_1$, $K_2$ such that $\|F(y)\| \leqq K_1$, $\|G(z)\| \leqq K_2$ for all $y \in K_\varrho$, $z \in \Omega_\varrho$ respectively (cf.[7], § 1). Set $K = \max(K_1, K_2)$, $N = \max(\beta_1 \|A\| \|P_Y\|, \beta_2 \|A\| \|P_Z\|)$, $\varrho_0 = \max(2\varrho, \|A\| K/(1-N))$

$K_{\rho_0}=\{x\in X, \|x\|\leqq \rho_0\}$, $S_{\rho_0}=\{x\in X, \|x\|=\rho_0\}$. If for $x = P_Y x + P_Z x \in X$ with $\|x\| \geqq \rho_0$ there is also $\|P_Y x\| \geqq \rho$, $\|P_Z x\| \geqq \rho$, then

$$\|\Phi(x)\| \leqq \|A\|(\|F(P_Y x)\| + \|G(-P_Z x)\|) \leqq \varepsilon \|x\| \leqq \|x\| .$$

If for $\|x\| \geqq \rho_0$ one of the inequalities $\|P_Y x\| \geqq \rho$ $\|P_Z x\| \geqq \rho$ is not fulfilled (for instance the first), then $\|\Phi(x)\| \leqq \|A\| K + N\|x\| \leqq \|x\|$ . Hence for every $x \in S_{\rho_0}$ there is $\|\Phi(x)\| \leqq \rho_0$ . Therefore $\Phi(S_{\rho_0}) \subset K_{\rho_0}$ . According to Browder´s theorem, the mapping $\Phi: K_{\rho_0} \to X$ has at least one point $x^* \in K_{\rho_0}$ such that $\Phi(x^*) = x^*$. Hence $P_Y x^* + P_Z x^* = A(G(-P_Z x^*) - F(P_Y x^*))$ and $f(P_Y x^*) = g(-P_Z x^*)$ . This concludes the proof.

The "intersection problem" was studied for completely continuous and weakly continuous mappings in [2],[10] respectively. For $A$ one may set either

1) $A = E$, $A = \lambda E$ and for $F$ some Uryhson operator, or

2) $Ax = \int_G K(s,t)\, x(t)\, dt$ and for $F$ some operator of Nemyckij.

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