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1. The purpose of this note is to establish the existence of a class of maximal subgroups in the simple groups $G(F)$ and $G(\mathcal{J}')$.

Terminology, notation, and fundamental facts about these groups to be used later are given in [1].

2. Following Chehata [1], we denote by $\Phi(\mathcal{J})$, $\mathcal{J} = [\alpha, \beta]$, the subset of $G(\mathcal{J})$ of all elements which have three breaks only, by $S(\mathcal{J})$ the subset of $\Phi(\mathcal{J})$ of elements with three breaks, the first at α, the second between α and β, and the third at β.

Let $\mathcal{J}' = (\alpha, \beta)$, $\alpha < \beta$, be an open interval, $\alpha, \beta \in F$, $\xi_o \in (\alpha, \beta)$. The set of all elements $f \in G(\mathcal{J}')$ such that $f(\xi_o) = \xi_o$ we denote by $\mathfrak{M}(\xi_o, \mathcal{J}')$. Similarly is defined $\mathfrak{M}(\xi_o, F)$.

3. The following assertions will be used in the proof of the main theorems:

Lemma (3,1). Let $\alpha < \gamma < \xi < \delta < \beta$ and let f be an element of $G((\alpha, \beta))$ which has the break at ξ, but in (γ, ξ) and (ξ, δ) it has no breaks.

Then there is an element $k \in S([\gamma, \delta])$ such that

(i) $\hat{f} = fk$

(ii) $\hat{f}(\xi) = f(\xi)$ if $\xi \notin (\gamma, \delta)$

(iii) $\hat{f}$ has no breaks in (γ, δ).

This Lemma is a generalization of Chehata's Lemma 5 and can be proved by similar methods.

Corollary (3,2). Let $\alpha < \xi_o < \beta$, $\mathcal{J}' = (\alpha, \beta)$ and let f be an element of $G(\mathcal{J}') \doteq \mathfrak{M}(\xi_o, \mathcal{J}')$,

$$f = \{ {}_1 \; {}^{\lambda}{}_{\eta_1} \; {}^{\xi_1 \cdots}_{\cdots} \; {}^{\xi_{n-2}}{}_{\eta_{n-1}} \; {}^{\mu}{}_1 \} .$$

Then there are $g_i \in \Phi([\xi_o, \mu))$, $i = 1, \ldots, s$ and $h_j \in \Phi((\lambda, \xi_o])$, $j = 1, \ldots, t$, $s+t \leq n-2$ such that

(i*) $f^* = f g_1 \cdots g_s h_1 \cdots h_t$

(ii*) $f^*(\xi_o) = f(\xi_o)$

(iii*) $f^*(\xi_o) = \xi$ for $\xi \notin (\lambda', \mu')$, $\lambda \leq \lambda' < \xi_o < \mu' \leq \mu$

(iv*) $f^* \in S([\lambda', \mu'])$.

Corollary (3,3). If we assume in Lemma (3,1) that moreover $f(\xi) > \xi$ for all $\xi \in (\gamma, \sigma)$, then $\hat{f}(\xi) > \xi$ for $\xi \in$ $\in (\gamma, \sigma)$.

Lemma (3,4). Let $f^+ \in S([\gamma, \sigma]) \doteq \mathfrak{M}(\xi_o, (\alpha, \beta))$, $\alpha < \gamma < \xi_o < \sigma < \beta$, be an element of the form

$$f^+ = \{ {}_1 \; {}^{\gamma}{}_{\omega_1} \; {}^{\xi_o} \; {}_{\omega_2} \; {}^{\sigma} \; {}_1 \}$$

and let $f^+(\xi_o) > \xi_o$.

If $\rho \in F$, $\xi_o < \rho < \sigma$, then there are elements ℓ_1, $\ell_2 \in \mathfrak{M}(\xi_o, (\alpha, \beta))$ which have the following properties:

1) $f' = \ell_1 f^+ \ell_2 \in S([\gamma, \sigma])$

2) $f'(\xi_o) = \rho$.

Proof: If we take

$$\ell_1 = \{ {}_1 \; {}^{\xi_o} \; {}_{\varphi} \; {}^{f^+(\xi_o)} \; {}_{\psi} \; {}^{\sigma} \; {}_1 \}$$

and choose φ, ψ such that

$$\ell_1(f^+(\xi_o)) = \rho ,$$

then

$$\ell_1 f^+ = \{ 1 \ \gamma \ \omega_1 \ f^{+-1}(\xi_0) \ \varphi\omega_1 \ \xi_0 \ \psi\omega_2 \ \sigma \ 1 \} .$$

This element we can transform by a suitable ℓ_2 to an element which has its break in (γ, σ) at $f^+(\xi_0)$ only; this is possible by Lemma (3,1). Hence we obtain

$$f' = \ell_1 f^+ \ell_2 \in \Phi([\gamma, \sigma])$$

and

$$(\ell_1 f^+ \ell_2)(\xi_0) = (\ell_1 f^+)(\ell_2(\xi_0)) = (\ell_1 f^+)(\xi_0) = \rho .$$

Since

$$(\ell_1 f^+)(f^{+-1}(\xi_0)) = \xi_0 > f^{+-1}(\xi_0)$$

and

$$(\ell_1 f^+)(\xi_0) = \rho > \xi_0 ,$$

it follows that

$$(\ell_1 f^+)(\xi) > \xi \quad \text{if} \quad \xi \in (\gamma, \sigma).$$

Consequently, by Corollary (3,3)

$$f'(\xi) > \xi \quad \text{if} \ \xi \in (\gamma, \sigma)$$

and we conclude

$$f' \in S([\gamma, \sigma]).$$

Lemma (3,5). If $\alpha < \gamma' < \xi_0 < \sigma' < \beta, \alpha < \gamma < \xi_0 < \sigma < \beta$, and $g \in G([\gamma', \sigma'])$, then there is an element $m \in \mathfrak{M}(\xi_0, (\alpha, \beta))$ such that $m g m^{-1} \in G([\gamma, \sigma])$.

The proof is analogous to that of Lemma 1 in [1] and is here omitted.

4. The main results are expressed in the following theorems:

Theorem (4,1). The group $\mathfrak{M}(\xi_0, (\alpha, \beta)), \xi_0 \in (\alpha, \beta)$, is a maximal subgroup of $G((\alpha, \beta))$.

Proof: Obviously $\mathfrak{M}(\xi_0,(\alpha,\beta)) \subset G((\alpha,\beta))$. Suppose $f \notin \mathfrak{M}(\xi_0,(\alpha,\beta))$, i.e. $f(\xi_0) \neq \xi_0$. We can assume without loss of generality that $f(\xi_0) > \xi_0$. Using Corollary (3,2), we transform f to the element f^* and since $f^*(\xi_0) = f(\xi_0) > \xi_0$ it implies $f^* \in \Phi((\alpha,\beta)) \dot{-} \mathfrak{M}(\xi_0,(\alpha,\beta))$. Let us denote by $\mathfrak{N} = \{\mathfrak{M}(\xi_0,(\alpha,\beta)), f\}$ the subgroup generated by elements of $\mathfrak{M}(\xi_0,(\alpha,\beta))$ and f. Then $f^* \in \mathfrak{N}$ and by Lemmas (3,4) and (3,1) $f' \in \mathfrak{N}$ if $f' \in S([\gamma,\sigma])$ and $\xi_0 < f'(\xi_0) = \rho < \sigma$, where $\gamma = \lambda(f^*)$, $\sigma = \mu(f^*)$.

Now it is easy to show that $\mathfrak{N} \supset S([\gamma,\sigma])$. Indeed, if $f'' \in S([\gamma,\sigma])$, then either $f''(\xi_0) > \xi_0$ or $f''^{-1}(\xi_0) > \xi_0$. But if $f''(\xi_0) > \xi_0$, we know already that $f'' \in \mathfrak{N}$. If $f''^{-1}(\xi_0) > \xi_0$, let us multiply this element on the right by a suitable element $\ell \in \mathfrak{M}(\xi_0,(\alpha,\beta)) = \mathfrak{M}$, using Corollary (3,2) and Lemma (3,4). We get $f''^{-1}\ell \in S([\gamma,\sigma])$ and since $(f''^{-1}\ell)(\xi_0) = f''^{-1}(\xi_0) > \xi_0$, it follows that $f''^{-1}\ell \in \mathfrak{N}$ and thus $f'' \in \mathfrak{N}$. Hence

$$\mathfrak{N} \supset \{S([\gamma,\sigma])\} = G([\gamma,\sigma]).$$

If g is an element of $G((\alpha,\beta))$ and $g \notin \mathfrak{M}$, then $\lambda(g) < \xi_0$, $\mu(g) > \xi_0$. Let us write $\lambda(g) = \gamma'$, $\mu(g) = \sigma'$. By Lemma (3,5) there is an element $m \in \mathfrak{M}$ such that

$$m g m^{-1} \in G([\gamma,\sigma]) \subset \mathfrak{N}.$$

This implies

$$g \in m^{-1}\mathfrak{N}m \subset \mathfrak{N},$$

since

$$m \in \mathfrak{M} \subset \mathfrak{N}.$$

But then

$$G((\alpha,\beta)) \subset \mathfrak{N} = \{\mathfrak{M}, f\} \subset G((\alpha,\beta)),$$

hence

$$G((\alpha,\beta)) = \{\mathfrak{M}, f\}.$$

The proof is complete.

Using Theorem (4,1), we infer the

Theorem (4,2). The group $\mathfrak{M}(\xi_0, F)$, $\xi_0 \in F$ is a maximal subgroup of $G(F)$.

B i b l i o g r a p h y :

[1] C.G. CHEHATA: An algebraically simple ordered group, Proc.Lond. Math.Soc.,2(1952),183-197.

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