Commentationes Mathematicae Universitatis Carolinae

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Commentationes Mathematicae Universitatis Carolinae, Vol. 7 (1966), No. 1, 105--108

Persistent URL: http://dml.cz/dmlcz/105044

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ON A THEOREM OF M. KATĚTOV

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In the present note we shall prove a theorem of M. Katětov (c.f.[4],[5]).

Let X be an infinite completely regular topological space. We denote by $E(X)$ the linear space of all finite formal linear combinations $\sum \lambda_i x_i$, where $x_i \in X$ and λ_i are real numbers. If λ is a locally convex topology on $E(X)$, we write $(E(X), \lambda)$. $C(X)$ means the Banach space of all bounded continuous functions on X with the usual norm. Putting

$$\langle x, f \rangle = \sum \lambda_i \langle x_i , f \rangle$$

for any $f \in C(X)$ and any $x = \sum \lambda_i x_i \in E(X)$, we define a locally convex topology $\sigma = \sigma(E(X), C(X))$ on $E(X)$. Every $f \in C(X)$ defines a linear function on $E(X)$ continuous in the weak topology σ (we denote this linear extension of f by the same letter). Consequently we may consider on the adjoint space $C(X)$ to $(E(X), \sigma)$ all concepts defined on $C(X)$ by means of the duality between $C(X)$ and $E(X)$ (e.g., the polar set $\mathcal{U}^\circ$, the Mackey topology etc.).

Notation. In further discussion $\mathcal{H}$ means the system of all uniformly bounded and equicontinuous subsets H of $C(X)$, $\mathcal{H}^*$ stands for the collection of all equicontinuous subsets H of $C(X)$ bounded in the weak topology $\sigma(C(X), E(X))$.

The locally convex topology on $E(X)$ defined by the collection $\{H^\circ, H \in \mathcal{H}\}$ ($\{H^\circ, H \in \mathcal{H}^*\}$) we denote by t (by t^*). τ represents the Mackey topology on $E(X)$. It is well known that all spaces $(E(X), t)$, $(E(X), t^*)$ and $(E(X), \sigma)$ have the same dual space $C(X)$ (c.f.[7]). This implies

$$\sigma(E(X), C(X)) \leq t \leq t^* \leq \tau(E(X), C(X)).$$

Obviously $E(X)$ may be considered as a subset of the space $C'(X)$ of all linear functions continuous on $C(X)$. Any topology $\lambda = \sigma, t, t^*$ on $E(X)$ may be extended in a natural manner on $C'(X)$. If X is a compact space, then $\mathcal{H} = \mathcal{H}^*$ and $t = t^*$. This follows directly from the classical theorem of Ascoli.

Now we recall the following result of M. Katětov (c.f.[4], [5]):

for any compact space X the completion $(\hat{E}(X), t)$ of $(E(X), t)$ is algebraically isomorphic with $C'(X)$.

Remark. It should be noticed that the preceding theorem is true in a more general case. For a pseudocompact space it can be proved by the Grothendieck's method of completion (for the topology $t = t^*$). The proof of this statement will appear in a study on Λ-structures.

The following theorem is due to M. Katětov (c.f.[4]).

Theorem. Let X be a compact space; then the completion of $E(X)$ with the topology $\tau(E(X), C(X))$ is isomorphic to $C'(X)$ with the topology $\tau(C'(X), C(X))$.

Proof. 1° Let X be a compact space. In this case X is a bounded subset of $E(X)$ for any locally convex topology λ compatible with the duality. From a theorem of Mackey it follows that any absolute convex and $\sigma(C(X), E(X))$-compact subset K

of $C(X)$ is bounded on X. Making use of a theorem of Grothendieck (c.f.[2]), we may conclude that K is compact in the weak topology $\sigma(C(X), C'(X))$. This implies that $\tau(\hat{E}(X), C(X))$ is induced by the topology $\tau(C'(X), C(X))$ on the subspace $E(X) \subseteq C'(X)$.

2^o We shall prove that $C'(X)$ is a complete space with the topology t. It was recalled that $(\hat{E}(X), t)$ is algebraically isomorphic with $C'(X)$. The topology on $C'(X)$ defined by this algebraical isomorphism we denote by λ_o. It is evident that both locally convex spaces $(C'(X), \lambda_o)$ and $(E(X), t)$ have the same adjoint space $C(X)$. From this it follows that the topology λ_o is compatible with the duality between $C(X)$ and $C'(X)$.

Any neighborhood of the origin in $C'(X)$ for the topology λ_o is of the form $\overline{\mathcal{U}}$, where $\mathcal{U}$ is an absolute convex neighborhood of the origin in $(E(X), t)$; the closure $\overline{\mathcal{U}}$ is taken in an arbitrary locally convex topology compatible with the duality. The statement follows from the fact that the polar set $\mathcal{U}^o$ is an element of $\mathcal{K}$ and $\mathcal{U}^{oo} = \overline{\mathcal{U}}$.

3^o We prove that $(C'(X), \tau)$ is complete. The last statement is evident. The neighborhood basis of the origin for the topology τ is formed by absolute convex subsets closed in the topology t. From $t \leq \tau$ and 2^o it follows that $(C'(X), \tau)$ is complete (c.f.[6]).

The proof of the theorem will be complete if we note that $E(X)$ is a dense subset of $(C'(X), \tau)$.

<u>Remark</u>. In general the theorem is not true for any completely

regular space X. Let X be, for example, an infinite discrete space. It is well known that in this case $(\hat{E}(X), t)$ is isomorphic to $\ell^1(X)$ (c.f.[3]). Obviously the space $(\hat{E}(X), \tau)$ is isomorphic to $\ell^1(X)$, too. The dual space to $(E(X), \tau)$ is in this case identical with $\ell^\infty(X)$.

R e f e r e n c e s :

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(Received October 9, 1965)