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MODEL $\nabla[\omega_\alpha \to \omega_\beta]$ IN WHICH β IS LIMIT NUMBER

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In [3], some models $\nabla[\omega_\alpha \to \omega_{\beta+1}]$ were constructed. In this note the discussion of existence of models $\nabla[\omega_\alpha \to \to \omega_\beta]$ is performed for the case that β is a limit number.

We shall use the notation from [1],[2],[3].

(1) Metadefinition: A model ∇ of the Gödel-Bernays set theory Σ is said to be of type $[\omega_\alpha \to \omega_\beta]$ if the following holds in the set theory: $\omega_\alpha \in \omega_\beta$, and

(a) if f is a relatively cardinal number of the model ∇, $f \subseteq^* k_{\omega_\alpha}$ or $k_{\omega_\beta} \subseteq^* f$ then f is a cardinal number of the model ∇,

(b) k_{ω_β} is the first cardinal number of the model ∇ greater than k_{ω_α}. (See [3].)

(2) Metatheorem: If the singularity of ω_β is provable, then there exists no model of type $[\omega_\alpha \to \omega_\beta]$.

Proof. In such a model $k_{\omega_\alpha} \subseteq^* f \in^* k_{\omega_\beta}$ implies $\bar{\bar{f}}^* \stackrel{*}{=} k_{\omega_\alpha}$. But ω_β is a singular number, hence there is an a such that $a \subseteq \omega_\beta$, $\bar{\bar{a}} \in \omega_\beta$, $\omega_\beta = \bigcup a$. From this we obtain easily $k_{\omega_\beta} \stackrel{*}{=} \bigcup^* k_a$, $\bar{\bar{k}}_a^* \subseteq^* k_{\omega_\alpha}$, $(f)[f \in^* k_a \to \to \bar{\bar{f}}^* \subseteq^* k_{\omega_\alpha}]$. Hence $\bar{\bar{k}}_{\omega_\beta}^* \subseteq^* k_{\omega_\alpha}$, a contradiction.

In the following, we assume the generalized continuum hypothesis to be valid in the set theory. Let $\mathcal{A}$ be an inaccesible cardinal number of the set theory, θ a set of power ϑ. Let ℓ be an ideal on θ such that there is a $\omega_\gamma \in \vartheta$ with $(x)[x \in \ell \rightarrow \overline{\overline{x}} \in \omega_\gamma]$. For every $x \in \theta$ let $\langle c(x), t(x) \rangle$ be a topological space, and $\langle \mathcal{K}_{x \in \theta} c(x), \mathrm{T}^{\ell}_{x \in \theta} t(x) \rangle$ the topological product of the spaces $\langle c(x), t(x) \rangle$ by the ideal ℓ as defined in [3]. If $\langle c, t \rangle$ is a topological space, we define

$\mu(c,t) = \min\{\omega_\gamma; \neg(\exists a)[\mathcal{E}x(a) \& a \subseteq t \& \overline{\overline{a}} = \omega_\gamma\}$ (see [2]),

$\chi_t(c,t) = \min\{\omega_\delta; \omega_\delta = \overline{\overline{\mathcal{b}}}, \mathcal{b}$ some basis of $\langle c, t \rangle\}$.

(3) Theorem. Let $\chi_t(\langle c(x), t(x) \rangle) \in \vartheta$ for every $x \in \theta$. Then $\mu(\langle \mathcal{K}_{x \in \theta} c(x), \mathrm{T}^{\ell}_{x \in \theta} t(x) \rangle) \subseteq \vartheta$.

Proof. Let a be a system of open sets in the space $\langle \mathcal{K}_{x \in \theta} c(t), \mathrm{T}^{\ell}_{x \in \theta} t(x) \rangle$, $\mathcal{E}x(a)$, $\overline{\overline{a}} = \vartheta$. Denote by $\beta(x)$ some basis composed of open sets in the space $\langle c(x), t(x) \rangle$, such that $\overline{\overline{\beta(x)}} = \chi_t(\langle c(x), t(x) \rangle)$. In $\langle \mathcal{K}_{x \in \theta} c(x), \mathrm{T}^{\ell}_{x \in \theta} t(x) \rangle$ choose the basis generated by the sets $\overline{g}$ with $g \in \mathcal{L}^{\ell} \beta(x)$ (see [2], def. 3,4). We may assume that a contains elements of this basis only. Define

$$(*)\ f \in \mathcal{b} \equiv \mathcal{F}nc\, f \,\&\, \mathcal{D}(f) \subset \theta \,\&\, \overline{\overline{\mathcal{D}(f)}} \in \omega_\gamma \,\&\, (x)[x \in \mathcal{D}(f) \rightarrow$$
$$\rightarrow f(x) \in \beta(x)] \,\&\, (\exists g)[\overline{g} \in a \,\&\, (x)(x \in \mathcal{D}(f) \cap \mathcal{D}(g) \rightarrow$$
$$\rightarrow f(x) = g(x))].$$

The axiom of choice implies the existence of a function B such that

$$\mathcal{F}nc\, B \,\&\, \mathcal{D}(B) = \mathcal{b} \,\&\, (f)\{f \in \mathcal{D}(B) \rightarrow [\overline{B(f)} \in a \,\&$$
$$\&\, (x)(x \in \mathcal{D}(f) \cap \mathcal{D}(B(f)) \rightarrow f(x) = (B(f))(x))]\}.$$

Now choose $\overline{g}_0 \in a$ and put $G_0 = \{g_0\}$. Having defined G_ρ for all $\rho \in \sigma \in \omega_{\gamma+1}$, we define G_σ by $(**)\ g \in G_\sigma \equiv (\exists f)[f \in b \ \&\ \mathcal{D}(f) \subseteq \mathcal{D}\bigcup_{\rho\in\sigma} \bigcup G_\rho \ \&\ g = B(f)]$, and $G = \bigcup_{\sigma\in\omega_{\gamma+1}} G_\sigma$. By induction one proves easily that $\overline{\overline{G_\sigma}} \in \vartheta$, $\overline{\overline{G}} \in \vartheta$. Then there must exist a $\bar{h} \in a$ with $h \notin G$. Set $y = \mathcal{D}(h) \cap \mathcal{D} \bigcup G$, then $y \neq \emptyset$ because $\bar{h} \cap \bar{g}_0 = \emptyset$, also $\overline{\overline{y}} \in \omega_\gamma$. Set $h_0 = h \upharpoonright y$. From $(*)$ we obtain $h_0 \in b$, and there is a σ such that $\mathcal{D}(h_0) = y \subseteq \mathcal{D}\bigcup_{\rho\in\sigma} \bigcup G_\rho$, then $B(h_0) \in G_\sigma \subset G$ is an obvious consequence of $(**)$. Next, $h \neq B(h_0)$ implies $\bar{h} \cap \overline{B(h_0)} \neq \emptyset$, because $(x)[x \in \mathcal{D}(h) \cap \mathcal{D}(B(h_0)) \rightarrow h(x) = (B(h_0))(x)]$. Hence $h = B(h_0) \in G$. This is a contradiction.

We shall construct models $\nabla[\omega_\alpha \rightarrow \vartheta]$ under the assumption that ω_α is a regular cardinal. The parameters of the model ∇ will be chosen as follows:

Set $\theta = \{\iota ;\ \omega_\alpha \in \iota \in \vartheta\}$, and define:

I. $ind = \{a_\iota\}_{\iota\in\theta}$,

II. $G(a_\iota) = \omega_\iota \times \omega_\alpha$ for every $\iota \in \theta$.

III. Definition of the space $\langle c, t \rangle$: for every $\iota \in \theta$ define

$$f \in c(\iota) \equiv \mathcal{F}nc\ f \ \&\ \mathcal{D}(f) = \omega_\alpha \ \&\ \mathcal{W}(f) \subseteq \omega_\iota$$

$$f \in b(\iota) \equiv \mathcal{F}nc\ f \ \&\ \mathcal{D}(f) \subseteq \omega_\alpha \ \&\ \mathcal{W}(f) \subseteq \omega_\iota \ \&\ \overline{\overline{\mathcal{D}(f)}} \in \omega_\alpha$$

$$\bar{f} = \{g ;\ g \in c(\iota) \ \&\ f \subseteq g\} \quad \text{for } f \in b(\iota) ,$$

$t(\iota)$ is the topology on $c(\iota)$ generated by basis $b(\iota)$.

Let ℓ be the ideal: $x \in \ell \equiv x \subseteq \theta \ \&\ \overline{\overline{x}} \in \omega_\alpha$ and $\langle c, t \rangle$ the space $\langle \mathcal{K}_{\iota\in\theta}\, c(\iota),\ \mathrm{T}^{\ell}_{\iota\in\theta}\, t(\iota) \rangle$.

IV. For $\iota \in \theta$ and $\langle \gamma \sigma \rangle \in \omega_\iota \times \omega_\alpha$ we define

$\varkappa\ (\langle \gamma \sigma \rangle a_\iota) = \{f;\ f \in c\ \&\ (f(\iota))(\sigma) = \gamma\}$.

It is then easy to prove the following statements:

For every $\iota \in \theta$ there is $\chi_t\ (\langle c(\iota), t(\iota)\rangle) \in \vartheta$, and then by theorem (3),

(1) $\mu(c, t) \in \vartheta$,

(2) $\omega_\alpha \in \nu(c, t)$ (since the intersection of a monotone system of ω_β sets from the basis of the space $\langle c, t\rangle$ is open set, if $\omega_\beta \in \omega_\alpha$, and it is non-void set, if $\omega_\beta = \omega_\alpha$).

By [2] theorem 4, condition (a) from metadefinition (1) holds in the model $\nabla(ind, G, \langle c, t\rangle, \varkappa, j)$.

Obviously k_{a_ι} is a 1-1 mapping of k_{ω_α} onto k_{ω_ι} in the model ∇.

Hence condition (b) from metadefinition (1) also holds in the model ∇.

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