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UNIFORM DIMENSION OF MAPPINGS
(Preliminary communication)

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By the dimension of a mapping $f : P \rightarrow Q$, where P, Q are topological spaces, the number $\sup\{\dim f^{-1}[y] ; y \in Q\}$ is usually understood. Some authors consider in a certain sense stronger definitions of the dimension of mappings for metric spaces, e.g. uniformly zero-dimensional mappings [2] or, as a generalization, the strong dimension of mappings [5]. We define the uniform dimension of uniformly continuous mappings for uniform spaces. It is closely connected with the uniform dimension Δd (see[1]).

For uniform spaces, we use the terminology of [3]. If $(X, \mathcal{U})$ is a uniform space, $U \in \mathcal{U}$, $\mathcal{K}$ is a collection of subsets of X, we say that $\mathcal{K}$ is U-discrete if $U[K] \cap L = \emptyset$ for any K, L in $\mathcal{K}$, $K \neq L$; we say that $\mathcal{K}$ is a U-cover of a subset M of X, if for each point x of M there exists a K in $\mathcal{K}$ such that $U[x] \cap M \subset K$. Further, all mappings are supposed to be uniformly continuous.

Definition. Let $(X, \mathcal{U})$, $(Y, \mathcal{V})$ be uniform spaces, $f : X \rightarrow Y$ a mapping. The uniform dimension of f is defined as the smallest non-negative integer n with the following property: for each U in $\mathcal{U}$ there exist V in $\mathcal{V}$ and W in $\mathcal{U}$ such that, if M is a subset of Y and

$M \times M \subset V$, then there exists a collection $\mathcal{K}$ of subsets of X such that $\mathcal{K}$ is a W-cover of $f^{-1}[M]$, $K \times K \subset U$ for each K in $\mathcal{K}$, and each point x of $f^{-1}[M]$ is contained in at most $n + 1$ sets of $\mathcal{K}$. The uniform dimension of f will be denoted by $\Delta d f$. If such a number n does not exist we set $\Delta d f = \infty$.

It is easy to prove that the definition may be expressed in a formally stronger manner, in that the collection $\mathcal{K}$ may be supposed to be the union of $n + 1$ W-discrete subcollections.

First we introduce some elementary properties of $\Delta d f$. If X is a non-void uniform space, S is a one-point space, $f : X \rightarrow S$ is a mapping, then $\Delta d f$ is equal to the mentioned Δ d-dimension of the space X ; shortly $\Delta d f = \Delta d X$. Thus Δ d-dimension of a uniform space may be considered as the Δ d-dimension of a certain mapping. If X, Y are uniform spaces, $f : X \rightarrow Y$ is a mapping, Y' is a subspace of Y such that $Y' \supset f[X]$ and $f' = f : X \rightarrow Y'$, then $\Delta d f = \Delta d f'$. If g is the restriction of a mapping f then $\Delta d g \leq \Delta d f$. If j is a uniform embedding then $\Delta d j = 0$.

<u>Theorem 1</u>. Let X, Y be non-void uniform spaces, p the canonical projection of $X \times Y$ onto X . Then $\Delta d p = \Delta d Y$.

<u>Theorem 2</u>. Let X, Y be uniform spaces, $f : X \rightarrow Y$, g the restriction of f to a dense subspace of X . Then $\Delta d f = \Delta d g$.

Every compact space has a uniquely determined uniformity and every continuous mapping is uniformly continuous.

Theorem 3. Let X, Y be compact Hausdorff spaces, $f : X \longrightarrow Y$. Then $\Delta d f \leqq n$ if and only if $\dim f^{-1}[y] \leqq n$ for all y in Y.

The following theorems concern some non-trivial properties of the uniform dimension of mappings.

Theorem 4. Let X, Y, Z be uniform spaces, $f : X \to$ $\to Y$, $g : Y \to Z$. Then $\Delta d(g \circ f) \leqq \Delta d f + \Delta d g$.

From Theorem 4 we obtain immediately

Theorem 5. Let X, Y be uniform spaces, $f : X \longrightarrow Y$. Then $\Delta d X \leqq \Delta d Y + \Delta d f$.

Theorem 6. Let $\{X_\alpha ; \alpha \in A\}$, $\{Y_\alpha ; \alpha \in A\}$ be families of uniform spaces, $\{f_\alpha ; \alpha \in A\}$ a family of mappings, $f_\alpha : X_\alpha \to Y_\alpha$. Let $f : \prod\{X_\alpha ; \alpha \in A\} \to$ $\to \prod\{Y_\alpha ; \alpha \in A\}$ be defined by the formula $f\{x_\alpha\} =$ $= \{f_\alpha x_\alpha\}$. Then $\Delta d f \leqq \sum \Delta d f_\alpha$.

If X is a uniform space and (R, ρ) is a metric space, we shall denote by $C_u(X, R)$ the set of all uniformly continuous mappings of X into R, endowed with the distance σ defined by
$\sigma(f, g) = \min(1, \sup\{\rho(fx, gx) ; x \in X\})$. If R is complete, then $C_u(X, R)$ is also a complete metric space. The following theorem (which is first proved for $k = 0$) characterizes the dimension Δd of pseudometric spaces by means of mappings into Euclidean spaces.

Theorem 7. Let P be a pseudometric space, k, n integers, $0 \leqq k \leqq n$. Then the following properties are equivalent:

(1) $\Delta d P \leqq n$,

(2) there exists a mapping $f : P \rightarrow E_{n-k}$ with $\Delta d f \leq k$,

(3) the set of all mappings $f : P \rightarrow E_{n-k}$ with $\Delta d f \leq k$ is a dense G_{δ}-set in the space $C_u(P, E_{n-k})$.

It can be proved that the assumption of pseudometrizability of P is essential even for the implication (1) ⟹ (2). Thus every metric space with finite dimension Δd can be mapped by a uniformly zero-dimensional mapping into a compact space (e.g. into the Hilbert cube). One may ask whether this holds for any metric space. We shall show that the answer is negative. First, let us introduce a theorem of another character, which is also concerned with the equality of the dimensions Δd and δd (see [4] or [1]).

Theorem 8. Let a uniform space $(Y, \mathcal{V})$ have the following property:

(f) for each V in $\mathcal{V}$ there exist a uniform cover $\mathcal{K}$ of Y and a number n such that $K \times K \subset V$ for each K in $\mathcal{K}$, and each point of Y is contained in at most n sets of $\mathcal{K}$.

Let X be a uniform space and $f : X \quad Y$ a mapping with finite $\Delta d f$. Then the space X also has the property (f).

If a uniform space X fulfils condition (f), then $\Delta d X = \delta d X$. Condition (f) is trivially fulfilled by compact spaces. Combining Theorems 8 and 6 we obtain, for example, this result: If a uniform space X admits a uniformly finite-dimensional mapping into a product of spaces with finite Δd and a compact space, then $\Delta d X = \delta d X$.

Suppose that for every metric space P there exists a uniformly zero-dimensional mapping of P into a compact space. Consider a uniform space X with $\sigma d X < \Delta d X$ (see [1]). The space X can be embedded into a product of metric spaces. This product has a uniformly zero-dimensional mapping into some compact space (by Theorem 6). But then we obtain $\Delta d X = \sigma d X$, a contradiction.

R e f e r e n c e s :

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