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ON LENZ'S PROBLEM ON THE INDEPENDENCE OF THE AXIOMS OF AFFINE SPACE

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§ 1. An affine space (of dimension ≥ 3) is defined as a set of points (denoted by capitals) in which some subsets called lines (denoted by lower-case letters) and planes (denoted by lower-case Greek letters) are distinguished in such a manner that the axioms (1) - (10) hold (see [6], p. 138). By parallel lines are meant lines which either coincide or lie in a common plane but have no common point.

(1) For every $A, B \neq A$ there exists exactly one line containing A, B.

(2) Every line contains two distinct points.

(3) There exist three points not on the same line.

(4) For any three points not on the same line there exists exactly one plane containing these points.

(5) Every plane contains three points not on the same line.

(6) If $A, B \neq A$ are on a plane α and $C \in \overline{AB}$ [1] then $C \in \alpha$.

(7) There exist four points not on the same plane.

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1) the bar serves to denote the line or plane containing the mentioned objects.

(8) If $a \parallel b$ [2] and $\alpha \supset a$, $\beta \supset b$ are distinct planes with a common point C, then α and β also have another common point $D \neq C$.

(9) Two distinct lines parallel to a given line do not intersect.

(10) For every line ℓ there exists at least one line $\ell' \neq \ell$, $\ell' \parallel \ell$.

H. Lenz, loc.cit., poses the question of the independence of the axiom (8) relative to the others. In the present paper we replace (9) and (10) by the usual axiom of parallelity

(P) Through a given point there is precisely one line parallel to a given line,

and we solve the question of independence of (8) on (1) - (7) and (P). In the present author's opinion, this solves the kernel of Lenz's problem, because (9) and (10) as well as (P) have a two-dimensional character, while (8) has a three-dimensional one.

In § 3 we show that (8) may be deduced from (1) - (7), (P) and from the following assumption

(F) Every line contains at least four distinct points.

In § 4 we show that (8) cannot be deduced from (1) - (7) and (P) only.

§ 2. In this section we consider a structure with axioms (1) - (7), (P) and the following

(T) Every line contains at least three distinct points.

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2) $a \parallel b$ denotes that a and b are parallel lines.

__Lemma 1.__ Let A_i, $i = 1,2,3$, be points not on the same line, $S \notin A_1A_2A_3$, $A'_i \in \overline{SA_i}$, $A'_i \neq S, A_i$, $\overline{A_iA_j} \parallel \overline{A'_iA'_j}$, $i,j = 1,2,3$, $i \neq j$. Then $\overline{A_1A_2A_3} \cap \overline{A'_1A'_2A'_3} = \emptyset$.

__Proof.__ Denote $\alpha = \overline{A_1A_2A_3}$, $\alpha' = \overline{A'_1A'_2A'_3}$ and suppose there exists a $P \in \alpha \cap \alpha'$. Since $\overline{A_iA_j} \parallel \overline{A'_iA'_j}$, the point P cannot lie in the planes $\overline{SA_iA_j}$, $i,j = 1,2,3$, $i \neq j$, and the intersection $\alpha \cap \alpha'$ cannot be a line; hence $\{P\} = \alpha \cap \alpha'$.

I. Assume there exist $\{Q\} = \overline{PA_3} \cap \overline{A_1A_2}$, $\{Q'\} = \overline{PA'_3} \cap \overline{A'_1A'_2}$. Denote by r the line parallel to $\overline{PA'_3}$ through A_3 and set $\{R\} = r \cap \overline{QQ'}$; hence $R \neq Q$. Further denote by t the line parallel to $\overline{A_1A_2}$ through R and set $\{T_k\} = t \cap \overline{SA_k}$, $k = 1,2$. Then $T_k \neq A_k$ and from $\overline{A_kA_3} \parallel \overline{A'_kA'_3}$ it follows that there exist $\{U_k\} = \overline{T_kA_3} \cap \overline{A'_kA'_3}$. Since $t \parallel \overline{A_1A_2} \parallel \overline{A'_1A'_2}$, we have $t \cap \alpha' = \emptyset$ and $t \parallel \overline{U_1U_2}$. Then r meets $\overline{U_1U_2}$ at a point V. Evidently $\{V\} = \overline{A_3QQ'} \cap \overline{U_1U_2}$, hence the line $\overline{PA'_3}$ also passes through V, which is a contradiction with $r \parallel \overline{PA'_3}$. Thus we have $\alpha \cap \alpha' = \emptyset$.

II. If e.g. $\overline{PA_3} \parallel \overline{A_1A_2}$, we choose $E_1 \in \overline{A_1A_3}$, $E_1 \neq A_1, A_3$. The line $\overline{PE_1}$ meets at least one of the lines $\overline{A_2A_3}$, $\overline{A_1A_2}$, say $\overline{A_2A_3}$ at a point E_2. Set $\{E'_k\} = \overline{SE_k} \cap \overline{A'_kA'_3}$, $k = 1, 2$. Since $\{P\} = \alpha \cap \alpha'$, there is $\overline{E_1A_2} \parallel \overline{E'_1A'_2}$, and by the substitution $\begin{pmatrix} A_1 & A_2 & A_3 & Q \\ A_3 & A_2 & E_1 & E_2 \end{pmatrix}$ we obtain $\alpha \cap \alpha' = \emptyset$ according to I.

__Theorem 1.__ Let $A \in l_1, l_2$, $l_1 \neq l_2$, $A' \notin \overline{l_1 l_2}$, let ℓ'_k be the line parallel to ℓ_k through A', $k = 1, 2$; then $\overline{\ell_1 \ell_2} \cap \overline{\ell'_1 \ell'_2} = \emptyset$.

Proof. With respect to (T) one can choose $M_k \in \ell_k$, $M_k \neq A$, $k = 1, 2$, so that $\overline{M_1 M_2} \cap \overline{\ell'_1 \ell'_2} = \emptyset$. Furthermore choose $S \in \overline{AA'}$, $S \neq A, A'$ and set $\{M'_k\} = \overline{SM_k} \cap \ell'_k$. Then $\overline{M_1 M_2} \parallel \overline{M'_1 M'_2}$, and from lemma 1 by the substitution $\begin{pmatrix} A_1 & A_2 & A_3 \\ A & M_1 & M_2 \end{pmatrix}$ we obtain $\overline{\ell_1 \ell_2} \cap \overline{\ell'_1 \ell'_2} = \emptyset$.

§ 3. Now assume that (F) holds.

Lemma 2. Let $a_1 \parallel a_2$, $a_1 \neq a_2$, $A_k \in a_k$, $k = 1, 2$, let $A'_1 \neq A'_2$, $\overline{A'_1 A'_2} \parallel \overline{A_1 A_2}$, $\overline{A_1 A_2} \neq \overline{A'_1 A'_2}$ and let a'_k be the line parallel to a_k through A'_k. Then $a'_1 \parallel a'_2$.

Proof. I. $\overline{A_1 A'_1}$ meets $\overline{A_2 A'_2}$ at a point B.

Choose $S \in \overline{A_1 A_2}$, $S \neq A_1, A_2$ and set $\{S'\} = \overline{SB} \cap \overline{A'_1 A'_2}$. Also choose $M_1 \in a_1$, $M_1 \neq A_1$, and set $\{M_2\} = \overline{SM_1} \cap a_2$, $\{M'_k\} = \overline{BM_k} \cap a'_k$, $k = 1, 2$. From $a_1 \parallel a'_1$, $\overline{A_1 A_2} \parallel \overline{A'_1 A'_2}$ it follows by theorem 1 that $\overline{S'a'_1} \cap \overline{Sa_1} = \emptyset$, and thus $\overline{S'M'_1} \parallel \overline{M_1 M_2}$. Analogously we obtain $\overline{S'M'_2} \parallel \overline{M_1 M_2}$, hence $\overline{S'M'_1} \parallel \overline{S'M'_2}$ and a'_1 and a'_2 lie in the plane $\overline{S'A'_1 M'_2}$. If a'_1 meet a'_2 in a point C', a_1 would meet a_2 at the point $\overline{BC'} \cap \overline{a_1 a_2}$, which is impossible with respect to $a_1 \parallel a_2$. Thus we have $a'_1 \parallel a'_2$.

II. $\overline{A_1 A'_1} \parallel \overline{A_2 A'_2}$.

Choose $B_1 \in \overline{A_1 A'_1}$, $B_1 \neq A_1, A'_1$, denote by ℓ the line parallel to $\overline{A_1 A_2}$ through B_1, $\{C\} = \ell \cap \overline{A_2 A'_2}$

and choose $B_2 \in \overline{A_2 A'_2}$, $B \neq A_2, A'_2, C$. Set $\{S\} = \overline{B_1 B_2} \cap \overline{A_1 A_2}$, $\{S'\} = \overline{B_1 B_2} \cap \overline{A'_1 A'_2}$. Also choose $M_1 \in a_1$, $M_1 \neq A_1$ and set $\{M_2\} = \overline{SM_1} \cap a_2$, $M'_k = \overline{B_k M_k} \cap a'_k$, $k = 1, 2$. In the same manner as in I. it can be shown that $\overline{S'M'_1} = \overline{S'M'_2}$, hence $a'_1, a'_2 \subset \overline{S'A'_1 M'_2}$. By theorem 1, $\overline{A'a_1} \cap \overline{A'_2 a_2} = \emptyset$, and a'_1 cannot meet a'_2.

<u>Theorem 2</u>. If $a_1 \parallel a_2$, $a_1 \neq a_2$, $C \notin \overline{a_1 a_2}$, then the intersection of the planes $\overline{a_1 C}$ and $\overline{a_2 C}$ is a line.

<u>Proof</u>. Choose $A_k \in a_k$, $k = 1, 2$, denote by ℓ the line parallel to $\overline{A_1 A_2}$ through C and choose $A'_1, A'_2 \in \ell$, $A'_k \neq C$, $A'_1 \neq A'_2$. Furthermore denote by a'_k the line parallel to a_k through A'_k, $k = 1, 2$, and by c_k the line parallel to a_k through C. Now, by lemma 2 we have $a'_1 \parallel a'_2$, $c_1 \parallel a'_2$, $c_2 \parallel a'_1$, therefore all these lines lie in the plane $\overline{\ell a'_1}$, and from $C \in c_1, c_2$ there follows $c_1 = c_2$, q.e.d.

Thus, according to theorem 2, <u>the condition (8) holds in the structure with the axioms (1) - (7), (P), (P̄)</u>.

§ 4. In this section we consider <u>a structure with the axioms (1) - (7), (P) and the following</u>

(B) <u>Every line contains exactly two points</u>.

By <u>a schema</u> will be meant a set $\mathcal{S}$ in which some four-point subsets are distinguished so that

(a) $\mathcal{S}$ contains either exactly four elements and no distinguished subset or at least five elements,

(b) the intersection of any two distinguished subsets contains at most two elements.

To a schema $\mathcal{S}$ we construct <u>a new schema</u> $\mathcal{F}(\mathcal{S})$ in the following recurrent manner (analogous to Hall's construction of free projective planes, see [1],[2],[4]).

1° The elements of the schema $\mathcal{S}$ will be termed points of degree 0, and the distinguished subsets in $\mathcal{S}$ planes of degree 0;

2° With every triple of mutually distinct points of degree at most n which do not lie in the same plane of degree at most n we associate, in a one-to-one manner, a new element $p(M,N,P)$, distinct from all preceding ones. These elements will be termed points of degree $n+1$ and the sets $\{M, N, P, p(M,N,P)\}$ will be termed planes of degree $n+1$.

The set of all points of degrees $n = 0, 1, 2, \ldots$ we denote by $\mathcal{F}(\mathcal{S})$, as the lines in $\mathcal{F}(\mathcal{S})$ we define all two-point subsets, and planes in $\mathcal{F}(\mathcal{S})$ will be all planes of degrees $n = 0, 1, 2, \ldots$.

It is easy to see that, <u>for every schema $\mathcal{S}$, the set $\mathcal{F}(\mathcal{S})$ is a structure satisfying the axioms (1) - (7), (P), (B)</u>.

<u>Example 1</u>. If $\mathcal{S}_1$ is the schema formed by a five-element set $\{A, B, C, D, E\}$ with one distinguished subset $\{A, B, C, D\}$, then <u>(8) does not hold in</u> $\mathcal{F}(\mathcal{S}_1)$, since e.g. $\overline{AB} \parallel \overline{CD}$, $E \notin \{A, B, C, D\}$ and by construction we have $p(A,B,E) \neq p(C,D,E)$, hence $\overline{ABE} \cap \overline{CDE} = \{E\}$.

<u>Example 2</u>. If $\mathcal{S}_2$ is the schema formed by a seven-element set $\{A, B, C, D, E, F, G\}$ with four distin-

guished subsets $\{A, B, C, D\}$, $\{A, B, E, F\}$, $\{A, C, E, G\}$, $\{D, E, F, G\}$, then theorem 1 does not hold in $\mathcal{F}(\mathcal{S}_2)$; on substituting $\begin{pmatrix} A & \ell_1 & \ell_2 & A' & \ell_1' & \ell_2' \\ A & \overline{AB} & \overline{AC} & E & \overline{EF} & \overline{EG} \end{pmatrix}$ the

supposition of theorem 1 is satisfied and the conclusion is not, because $\overline{ABC} \cap \overline{EFG} = \{D\}$.

Some fundamental properties of schemas and their free extensions are studied in [3].

§ 5. If neither (F) nor (B) is true, then according to the following lemma 3 every line contains exactly three points.

Lemma 3. All lines of a structure with the axioms (1) - (7) and (P) have the same cardinal number.

Proof. Let a, b be any lines. Choose $A \in a$, $B \in b$, $A \neq B$. Every plane is an affine plane, hence according to a well known result $\mathrm{card}\ a = \mathrm{card}\ \overline{AB}$, $\mathrm{card}\ \overline{AB} = \mathrm{card}\ b$ and thus $\mathrm{card}\ a = \mathrm{card}\ b$.

In this case, it may be also established that (8) need not hold, but a construction of an example is more difficult and extensive, and will be not exhibited here.

R e f e r e n c e s :

[1] P. DEMBOWSKI, Freie und offene projektive Ebenen, Math.Zeitschr.72(1960),410-438.

[2] M. HALL, Projective Planes, Trans.Amer.Math.Soc.54 (1943),229-277.

[3] V. HAVEL, Freie Erweiterungen von Schemen n-ten Grades, Comm.Math.Univ.Carolinae (to appear).

[4] Л.И. КОПЕЙКИНА, Свободные разложения проективных плоскостей, Известия Ак. наук СССР, 9(1945), 495-526.

[5] H. LENZ, Zur Begründung der analytischen Geometrie, Sitzungsberichte der Bayerischen Akad. der Wiss. 1954, math.-nat. Klasse, 17-72.

[6] H. LENZ, Grundlagen der Elementarmathematik, Berlin 1961.

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