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Miroslav Hušek

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GENERALIZED PROXIMITY AND UNIFORM SPACES II.

M. HUŠEK, Praha

This paper is an immediate continuation of [4]. Therefore all symbols, terms and definitions introduced there are used without any references. In this last part II we investigate relations between proximity spaces and semi-uniform spaces (§ 4), proximity spaces and closure spaces (§ 5) and finally between semi-uniform spaces and closure spaces (§ 6). The most of our results, referring to preservation of projective or inductive properties (e.g. to be a subspace, a limit of a presheaf etc.) by competent functors, are consequences of more general propositions from [3] and [4], § 1. The last § 7 is devoted to the brief account of other relations (embedding of $\mathcal{P}^U$ into $\mathcal{C}$, $\mathcal{U}$ into $\mathcal{P}_{S-U}$).

4. Relations between proximity and semi-uniformity.

Definition 4.1. Let $\wp \leftrightarrow p \leftrightarrow \{\mathcal{U}_X \mid X \subset P\}$, $\mathcal{U} \leftrightarrow \mathcal{L}$, $\mathfrak{D} \cup \mathcal{U} = P$. We shall say that p is induced by $\mathcal{U}$ (sign $\mathcal{U} \rightarrow p$) if one of the following equivalent conditions is fulfilled:

1) $p = \{\langle X, Y\rangle \mid X \cup Y \subset P,\ \mathcal{L} \cap \mathcal{L}(X \times Y) \neq \emptyset\}$;

2) $\mathcal{U}_X = \{U[X] \mid U \in \mathcal{U}\}$ for all $X \in \exp' P$;

3) $\{\langle \beta M, X\rangle \mid M \in \mathcal{L} \cap \mathcal{L}(X \times P)\}$ generates $\wp$.

Definition 4.2. Let p be a monotone proximity, $\wp \leftrightarrow p \leftrightarrow \{\mathcal{U}_X \mid X \subset P\}$, $\mathcal{U} \leftrightarrow \mathcal{L}$. We shall say that $\mathcal{U}$ is induced by p (sign $p \rightarrow \mathcal{U}$) if one of the following equivalent conditions is fulfilled:

1) $\mathcal{C} = \{M \mid M \in \mathcal{C}(P \times P),\ \mathcal{E}\alpha M' p\, \mathcal{E}\beta M'$ for any subnet M' of $M\}$;

2) $\mathcal{C} = \{M \mid M \in \mathcal{C}(P \times P),\ \langle \beta M', \mathcal{E}\alpha M' \rangle \in p$ for any subnet M' of $M\}$;

3) $\{(X \times Y) \cup ((P-X) \times P) \mid X \subset P,\ Y \in \mathcal{U}_X\}$ is a subbase for $\mathcal{U}$. (Hence $\{\bigcup\{X_i \times Y_i \mid i \in I\} \mid \{X_i \mid i \in I\}$ is a finite disjoint cover of P and $Y_i \in \mathcal{U}_{X_i}$ for all $i \in I\}$ is a base for $\mathcal{U}$).

Remark 4.1. (a) The operations $\longrightarrow$ in the foregoing definitions determine covariant functors $\Phi : \mathcal{P}_M \to \mathcal{U}$, $\psi : \mathcal{U} \to \mathcal{P}_M$ (it is almost self-evident that each proximity induced by a semi-uniformity is monotone).

(b) The case of non-monotone proximity spaces is rather difficult. The following example shows that conditions (1), (2),(3) of definition 4.2 are not equivalent in this case and that there are no covariant functors $\Phi^U : \mathcal{P}^U \to \mathcal{U}$, $\Phi^L : \mathcal{P}^L \to \mathcal{U}$ such that $\Phi^U \xi = \Phi^L \xi$ for every proximity space ξ, $\mathit{graph}\ \Phi^U f = \mathit{graph}\ f$ ($\mathit{graph}\ \Phi^L f = \mathit{graph}\ f$) for each morphism f of $\mathcal{P}^U$ ($\mathcal{P}^L$), $\Phi^U / \mathcal{P}_M = \Phi^L / \mathcal{P}_M = \Phi$. (But clearly we have covariant functors $\Phi \circ I_1 : \mathcal{P}^U \to \mathcal{U}$, $\Phi \circ I_2 : \mathcal{P}^L \to \mathcal{U}$, where I_1, I_2 are covariant functors from $\mathcal{P}^U$, $\mathcal{P}^L$ resp., in $\mathcal{P}_M$ assigning to each object its modification - see theorem 2.5.)

Example 4.1. (a) We shall demonstrate that the conditions in definition 4.2 are not equivalent provided p is a non-monotone proximity. In every case $\{(X \times Y) \cup ((P-X) \times P) \mid X \cup Y \subset \bigcup \mathcal{D}_p,\ X \mathrel{non\,p} (P-Y)\}$ is a subbase for the semi-uniformity induced by the coarsest monotone proximity finer than p.

Let P be the set of real numbers, $p = \{\langle X, Y \rangle \mid X \cup Y \subset P,$

either $X \cap Y \neq \emptyset$ or $\operatorname{card} X \geqq \aleph_0$, $X \subset \{-\frac{1}{n+1} \mid n \in N\}$, $Y \ni -\min X$ or $\operatorname{card} X = 1$, $X \subset]1, \to[$, $Y \neq \emptyset\}$. Denote by p_1 the coarsest monotone proximity finer than p and by p_2 the finest monotone proximity coarser than p (p_1 is the finest proximity for P and $p_2 \cap (]1,\to[\times]1,\to[)$ is the coarsest proximity for $]1,\to[$). Let $p_1 \to \mathcal{U}_1$, $p_2 \to \mathcal{U}_2$, $\mathcal{C}'$ be generated by $\tilde{\mathcal{C}}' = \{M \mid M \in \mathcal{C}(P \times P), \mathcal{E}\alpha M' p \mathcal{E}\beta M'$ for any subnet M' of $M\}$, $\mathcal{C}''$ be generated by $\tilde{\mathcal{C}}'' = \{M \mid M \in \mathcal{C}(P \times P)$, $\mathcal{E}\alpha M' p\, \mathcal{E}\beta M''$ whenever M'' is a subnet of M' and M' is a subnet of $M\}$, $\mathcal{U}' \leftrightarrow \mathcal{C}'$, $\mathcal{U}'' \leftrightarrow \mathcal{C}''$. Then $\mathcal{U}_1 \supsetneqq \mathcal{U}'' \supsetneqq \mathcal{U}' \supsetneqq \mathcal{U}_2$ (notice that the sequence $\{\langle n, \langle -\frac{1}{n+1}, \frac{1}{n+1}\rangle\rangle \mid n \in N\} \in \mathcal{C}' - \mathcal{C}''$) and $\tilde{\mathcal{C}}' \neq \mathcal{C}'$, $\tilde{\mathcal{C}}'' \neq \mathcal{C}''$.

(b) Let us have covariant functors $\Phi^U : \mathcal{P}^U \to \mathcal{U}$, $\Phi^L : \mathcal{P}^L \to \mathcal{U}$ described in remark 4.1(b). Let P be at least a four-point set, Q_1, Q_2, Q_3 be subsets of P such that $\emptyset \neq Q_i \neq P$ $(i = 1, 2, 3)$, $\operatorname{card}(P - Q_1) \geqq 2$, $P \times P = (Q_1 \times Q_2) \cup (Q_2 \times Q_2) \cup (Q_3 \times Q_3)$, $q_i = \exp' Q_i \times \exp' Q_i$, $p = \{\langle X, Y\rangle \mid X \cup Y \subset P$, either $X \cap Y \neq \emptyset$ or $X \in \exp' Q_i$ and $Y \cap Q_i \neq \emptyset$ for some $i\}$. If $\langle P, \mathcal{U}\rangle = \Phi^U \langle P, p\rangle = \Phi^L \langle P, p\rangle$, then $\mathcal{U} = (P \times P)$ because the identity mappings of $\langle Q_i, q_i\rangle$ into $\langle P, p\rangle$ are upper proximally continuous and q_i are monotone. But this leads to a contradiction because there is a lower proximally continuous mapping f of $\langle P, p\rangle$ onto a monotone proximity space $\langle R, \kappa\rangle$, where κ does not induce $(R \times R)$. (Put $R = (a, b)$, $\kappa = \{\langle X, Y\rangle \mid X \cup Y = R$, either $X \cap Y \neq \emptyset$ or $X = (b)$, $Y = (a)\}$, $f = ((Q_1 \cup (x)) \times (a)) \cup ((P - (Q_1 \cup (x))) \times (b))$ where $x \in P - Q_1$.)

<u>Definition 4.3</u>. We shall say that a proximity is <u>semi-uniformizable, uniformizable, symmetric semi-uniformizable, symmetric uniformizable</u> resp., if it is induced by a semi-uniformity,

uniformity etc. Similarly for proximity spaces. We shall denote by $\mathcal{P}_{S-U}$, $\mathcal{P}_U$, $\mathcal{P}_S$, $\mathcal{P}_{SU}$, resp., the full subcategory of $\mathcal{P}_M$ generated by the class of all semi-uniformizable proximity spaces, uniformizable proximity spaces etc.

<u>Theorem 4.1.</u> There is a coarsest semi-uniformity $\mathcal{U}$ in the set of all semi-uniformities inducing a given proximity p. $\mathcal{U}$ is induced by p.

Proof. Let $\mathcal{V} \rightarrow p \rightarrow \mathcal{U} \rightarrow q$. Evidently $\mathcal{V} < \mathcal{U}$, $p = q$.

<u>Theorem 4.2.</u> There is a finest monotone proximity p in the set of all monotone proximities inducing a given semi-uniformity $\mathcal{U}$. p is induced by $\mathcal{U}$.

Proof. Let $q \rightarrow \mathcal{U} \rightarrow p \rightarrow \mathcal{V}$ evidently $p < q$, $\mathcal{V} = \mathcal{U}$.

<u>Remark 4.2.</u> $\psi[\mathcal{U}] = \mathcal{P}_{S-U}$, $\psi[\mathcal{U}_S] = \mathcal{P}_S$, $\psi[\mathcal{U}_U] = \mathcal{P}_U$, $\psi[\mathcal{U}_{SU}] = \mathcal{P}_{SU}$.

<u>Theorem 4.3.</u> Let $\rho \leftrightarrow p \leftrightarrow \{\mathcal{U}_X \mid X \subset P\}$. The following conditions are equivalent:

1) p is semi-uniformizable ;

2) if $X_1 \cup X_2 \subset P$ then $(X_1 \cup X_2)\, p\, Y$ if and only if either $X_1\, p\, Y$ or $X_2\, p\, Y$;

3) if $X_1 \cup X_2 \subset P$ then $\mathcal{U}_{X_1 \cup X_2} = \mathcal{U}_{X_1} \cap \mathcal{U}_{X_2}$;

4) ρ is generated by the class $\rho' = \{\langle M, X\rangle \mid M \in \mathcal{C}(P)$, $X \subset P$, there is $N \in \mathcal{C}(X \times P)$ such that $\beta N = M$ and $\langle \beta N', \varepsilon(\alpha N')\rangle \in \rho$ for each subnet N' of $N\}$.

Proof. Implications $1 \Leftrightarrow 2 \Leftrightarrow 3$, $1 \Rightarrow 4$ follow from definitions 4.1 and 4.2. From the remaining implications it is easy to prove $4 \Rightarrow 2$.

<u>Theorem 4.4.</u> Let $\rho \leftrightarrow p \leftrightarrow \{\mathcal{U}_X \mid X \subset P\}$. Then the

following conditions are equivalent:

1) p is symmetric semi-uniformizable;

2) p is symmetric;

3) $Y \in \mathcal{U}_X$ implies $P - X \in \mathcal{U}_{P-Y}$;

4) $\rho = \{\langle M, X\rangle \mid X \subset P$ and $\rho \cap (\mathcal{C}(x) \times (\mathcal{E} M')) \neq \emptyset$ for each subnet M' of $M\}$.

Proof. It is obvious that $1 \Rightarrow 2 \Leftrightarrow 3$, $2 \Leftrightarrow 4$. If $W = (X \times Y) \cup (P - X) \times P$ then $W^{-1} = (P - Y) \times (P - X) \cup (Y \times P)$. It follows $3 \Rightarrow 1$.

<u>Theorem 4.5</u>. Let $\rho \longleftrightarrow p \longleftrightarrow \{\mathcal{U}_X \mid X \subset P\}$, p be semi-uniformizable. Then the following conditions are equivalent:

1) p is uniformizable;

2) if X *non* p Y then X *non* p $(P - Z)$, Z *non* p Y for some Z ;

3) if $Z \, p \, Y$ for each $Z \in \mathcal{U}_X$ then $X \, p \, Y$;

4) if $Y \in \mathcal{U}_X$ then $Y \in \mathcal{U}_Z$ for some $Z \in \mathcal{U}_X$;

5) if $\langle M, Z\rangle \in \rho$ for each $Z \in \mathcal{U}_X$ then $\langle M, X\rangle \in \rho$;

6) let A be a right-directed set, for each $\alpha \in A$ be $\langle M_\alpha, X_\alpha\rangle \in \rho$ and for each net N with $\mathcal{D} N = A$, $N_\alpha \in X_\alpha$ for all $\alpha \in A$ be $\langle N, X\rangle \in \rho$ (i.e. for any $Y \in \mathcal{U}_X$ there is $\alpha \in A$ such that $X_\beta \subset Y$ for each $\beta \in A$, $\beta > \alpha$), then $\langle M, X\rangle \in \rho$ where $\mathcal{D} M = A \times \Pi\{\mathcal{D} M_\alpha \mid \alpha \in A\}$, $M_{\langle \alpha, \{a_\alpha \mid \alpha \in A\}\rangle} = (M_\alpha)_{a_\alpha}$.

Proof. It is easy to verify that $1 \Rightarrow 2 \Leftrightarrow 3 \Leftrightarrow 4 \Rightarrow 5 \Rightarrow 6 \Rightarrow 3$. Suppose that 4 is true, $p \rightarrow \mathcal{U} \rightarrow p$. Then $\mathcal{U}$ is a uniformity. Indeed, for any $W = (X \times Y) \cup (P - X) \times P \in \mathcal{U}$ there is a $V \in \mathcal{U}$ such that $V \circ V \subset W$ (Put $V = (X \times Z) \cup ((Z - X) \times Y) \cup ((P - Z) \times P)$ where $Z \in \mathcal{U}_X$, $Y \in \mathcal{U}_Z$.) Hence $4 \Rightarrow 1$.

Corollary. Let $\rho \leftrightarrow p \leftrightarrow \{\mathcal{U}_X \mid X \subset P\}$. Then the following conditions are equivalent:

1) p is symmetric uniformizable;

2) if $X \operatorname{non} p\, Y$ then $X \operatorname{non} p\,(P-Z)$, $Y \operatorname{non} p\, Z$ for some Z;

3) if $X \operatorname{non} p\, Y$ then $Z_1 \cap Z_2 = \emptyset$ for some $Z_1 \in \mathcal{U}_X$, $Z_2 \in \mathcal{U}_Y$;

4) if $Y \in \mathcal{U}_X$ then $P - Z \in \mathcal{U}_{P-Y}$ for some $Z \in \mathcal{U}_X$.

<u>Theorem 4.6</u>. Let $\mathcal{U} \leftrightarrow \mathcal{C}$, $P = \mathcal{D} \cup \mathcal{U}$. Then the following conditions are equivalent:

1) $\mathcal{U}$ is induced by a monotone proximity;

2) $\{U \mid U \in \mathcal{U}, \{U[x] \mid x \in P\}$ is a finite set$\}$ is a base for $\mathcal{U}$;

3) $\mathcal{C} = \{M \mid M \in \mathcal{C}(P \times P), \mathcal{C} \cap \mathcal{C}(\mathcal{E}(\alpha M') \times \mathcal{E}(\beta M')) \neq \emptyset$ for each subnet M' of M.

<u>Theorem 4.7</u>. Let $p \to \mathcal{U}$. If p is uniformizable, symmetric, symmetric uniformizable resp., then $\mathcal{U}$ is a uniformity, symmetric semi-uniformity, symmetric uniformity resp.

Proof follows from the proofs of theorems 4.4, 4.5.

<u>Remark 4.3.</u> If a symmetric semi-uniformity $\mathcal{U}$ is induced by a monotone proximity then $\{V \mid V \in \mathcal{U},\ V = \bigcup\{X_i \times X_i \mid i \in I\},\ \operatorname{card} I < \aleph_0\}$ is a base for $\mathcal{U}$. (If $U = \bigcup\{X_i \times Y_i \mid i \in I\}$ where $\{X_i \mid i \in I\}$ is a disjoint family and if $U = U^{-1}$ then $U = \bigcup\{(X_i \cup X_j) \times (X_i \cup X_j) \mid \langle i, j\rangle \in I \times I, U \cap (X_i \times X_j) \neq \emptyset\}$.)

<u>Definition 4.4</u>. A semi-uniformity is called <u>proximally coarse</u> if it is induced by a proximity. We denote by $\mathcal{U}^c$, $\mathcal{U}^c_s$, $\mathcal{U}^c_u$, $\mathcal{U}^c_{su}$ resp., the full subcategory of $\mathcal{U}$ determined by the proximally coarse semi-uniformities, etc.

<u>Theorem 4.8</u>. The categories $\mathcal{P}_{s-u}$ and $\mathcal{U}^c$, $\mathcal{P}_s$ and

$\mathcal{U}^c_s$, $\mathcal{P}_U$ and $\mathcal{U}^c_U$, $\mathcal{P}_{SU}$ and $\mathcal{U}^c_{SU}$ resp., are isomorphic. The competent isomorphism is

$\Phi/\mathcal{P}_{S-U} = (\Psi/\mathcal{U}^c)^{-1}$, $\Phi/\mathcal{P}_S = (\Psi/\mathcal{U}^c_S)^{-1}$, $\Phi/\mathcal{P}_U = (\Psi/\mathcal{U}^c_U)^{-1}$, $\Phi/\mathcal{P}_{SU} = (\Psi/\mathcal{U}^c_{SU})^{-1}$ resp.

__Theorem 4.9__. Each object of $\mathcal{P}_M$ has its lower modification in $\mathcal{P}_{S-U}$ and each object of $\mathcal{U}$ has its upper modification in $\mathcal{U}^c$. Moreover, the upper modification in $\mathcal{U}^c$ of an object of $\mathcal{U}_S$, $\mathcal{U}_U$, $\mathcal{U}_{SU}$ resp., is an object of $\mathcal{U}^c_S$, $\mathcal{U}^c_U$, $\mathcal{U}^c_{SU}$ resp. Hence each object of $\mathcal{U}^c$ has its upper modifications in $\mathcal{U}^c_S$, $\mathcal{U}^c_U$, $\mathcal{U}^c_{SU}$ and each object of $\mathcal{P}_{S-U}$ has its upper modifications in $\mathcal{P}_S$, $\mathcal{P}_U$, $\mathcal{P}_{SU}$. Each object of $\mathcal{U}^c$, $\mathcal{U}^c_U$ resp., has its lower modification in $\mathcal{U}^c_S$, $\mathcal{U}^c_{SU}$ resp. and so each object of $\mathcal{P}_{S-U}$, $\mathcal{P}_U$ resp., has its lower modification in $\mathcal{P}_S$, $\mathcal{P}_{SU}$ resp.

__Corollary 1__. $\mathcal{P}_{S-U}$, $\mathcal{P}_S$, $\mathcal{P}_U$, $\mathcal{P}_{SU}$, $\mathcal{U}^c$, $\mathcal{U}^c_S$, $\mathcal{U}^c_U$, $\mathcal{U}^c_{SU}$ are S-categories over $\mathcal{M}$ with respect to the forgetful functors.

__Corollary 2__. $\mathcal{P}_{S-U}$ is inductive in $\mathcal{P}_M$, $\mathcal{P}_S$ is projective and inductive in $\mathcal{P}_{S-U}$, $\mathcal{P}_U$ is projective in $\mathcal{P}_{S-U}$, $\mathcal{P}_{SU}$ is projective in $\mathcal{P}_{S-U}$, $\mathcal{P}_S$, $\mathcal{P}_U$ and inductive in $\mathcal{P}_U$. (It is easy to prove that $\mathcal{P}_{S-U}$ is hereditary in $\mathcal{P}_M$, $\mathcal{P}_U$ is coproductive in $\mathcal{P}_{S-U}$ and hence that $\mathcal{P}_{SU}$ is coproductive in $\mathcal{P}_{S-U}$, $\mathcal{P}_S$.) $\mathcal{U}^c$ is projective and cohereditary in $\mathcal{U}$. Other assertions about $\mathcal{U}^c_i$ follow from theorem 4.8 and from the first part of this corollary.

__Remark 4.4__. It is almost self-evident that $\mathcal{P}_{S-U}$ is not productive in $\mathcal{P}_M$ and that $\mathcal{U}^c_{SU}$ is not coproductive in $\mathcal{U}_{SU}$. The fact that $\mathcal{U}^c_{SU}$ is not cohereditary in $\mathcal{U}^c_S$ (and hence that $\mathcal{P}_{SU}$ is not cohereditary in $\mathcal{P}_S$) follows from example 3.2.

__Theorem 4.10__. Let $\langle P, \mathcal{V} \rangle$ be the upper modification in $\mathcal{U}^c$ of a semi-uniform space $\langle P, \mathcal{U} \rangle$. If $\mathcal{U} \leftrightarrow \mathcal{C}$, $\mathcal{V} \leftrightarrow \mathcal{D}$

then

1) $\{U \mid U \in \mathcal{U}, \{U[x] \mid x \in P\}$ is a finite set$\}$ is a base for $\mathcal{V}$;

2) $\mathcal{D} = \{M \mid M \in \mathcal{C}(P \times P), \mathcal{C} \cap \mathcal{C}(E\alpha M' \times E\beta M') \neq \emptyset$ for each subnet M' of $M\}$.

Theorem 4.11. Assume that $\langle P, p_0 \rangle$ is a monotone proximity space, $\langle P, p_1 \rangle$ the lower modification of $\langle P, p_0 \rangle$ in $\mathcal{P}_{s-u}$, $\langle P, p_2 \rangle$ the upper modification of $\langle P, p_1 \rangle$ in $\mathcal{P}_s$, $\langle P, p_3 \rangle$ the lower modification of $\langle P, p_1 \rangle$ in $\mathcal{P}_s$, $\rho_i \leftrightarrow p_i \leftrightarrow \{\mathcal{U}_X^i \mid X \subset P\}$. Then

1) p_0^{-1} generates p_1^{-1};

2) $\mathcal{U}_X^1 = \bigcup\{\bigcap\{\mathcal{U}_Z^0 \mid Z \in \mathcal{O}\} \mid \mathcal{O}$ is a finite cover of $X\}$ for all $X \subset P$;

3) $\{\langle M, X\rangle \mid M \in \mathcal{C}(P), X \subset P$, there is an $N \in \mathcal{C}(X)$ such that $\mathcal{D}N = \mathcal{D}M$ and $\langle M', EN'\rangle \in \rho_0$ for any subnet M', N' of M, N resp., with $\mathcal{D}M' = \mathcal{D}N'\}$ generates ρ_1;

4) $p_2 = p_1 \cup p_1^{-1}$;

5) $\mathcal{U}_X^2 = \{Y \mid Y \in \mathcal{U}_X^1$ and $P - X \in \mathcal{U}_{P-Y}^1\}$ for all $X \subset P$;

6) $\rho_1 \cup \{\langle M, X\rangle \mid M \in \mathcal{C}(P), X \subset P, \rho_1 \cap (\mathcal{C}(X) \times (EM')) \neq \emptyset$ for any subnet M' of $M\}$ generates ρ_2;

7) p_3 is generated by q^{-1} where q is generated by $p_1 \cap p_1^{-1}$;

8) $\{Y \mid Y \subset P$, there is a finite cover $\mathcal{O}$ of X such that for any $Z \in \mathcal{O}$ either $Y \in \mathcal{U}_Z^1$ or $P - Z \in \mathcal{U}_{P-Y}^1\}$ is a subbase for $\mathcal{U}_X^3$ for all $X \subset P$;

9) $\rho_3 = \{\langle M, X\rangle \mid M \in \mathcal{C}(P), X \subset P$, there is an $N \in \mathcal{C}(X)$ such that $\langle M', EN'\rangle \in \rho_1$, $\langle N', EM'\rangle \in \rho_1$ for any subnets M', N' of M, N resp., with $\mathcal{D}M' = \mathcal{D}N'\}$.

Remark 4.5. The upper modification in $\mathcal{P}_U$ of an object $\langle P, p \rangle$ of $\mathcal{P}_{S-U}$ may be constructed by transfinite induction:

$$p_0 = p$$

$$p_\xi = \exp P \times \exp P - \{\langle X, Y \rangle \mid (\langle X, P-Z \rangle, \langle Z, Y \rangle) \cap p_{\xi'} = \emptyset$$ for some $Z \subset P \}$ if $\xi = \xi' + 1$

$$p_\xi = \cup \{ p_{\xi'} \mid \xi' < \xi \}$$ in the remaining case.

All p_ξ are semi-uniformizable proximities for P and $p_{\xi'} < p_\xi$ provided $\xi' < \xi$. Hence there is an ordinal number ξ such that $p_\xi = p_{\xi+1}$. The proximity space $\langle P, p_\xi \rangle$ is the upper modification of $\langle P, p \rangle$ in $\mathcal{P}_U$.

If $p_\xi \leftrightarrow \{\mathcal{U}_X^\xi \mid X \subset P\}$ then $\mathcal{U}_X^\xi = \{ U \mid$ there is a monotonely densely ordered set A with the first element α_0, with the last element $\alpha_1 \neq \alpha_0$, and there is $\{ U_\alpha \mid \alpha \in A \}$ such that $U_{\alpha_0} = U$, $U_{\alpha_1} = X$ and $U_\alpha \in \mathcal{U}^0_{U_{\alpha'}}$ if $\alpha < \alpha'$ (take for A dyadic rational numbers in $[0, 1]$).

By remark 3.4 the upper modification in $\mathcal{P}_U$ of an upper modification of $\langle P, p \rangle$ in $\mathcal{P}_S$ is the upper modification of $\langle P, p \rangle$ in $\mathcal{P}_{SU}$. We cannot commute $\mathcal{P}_S$ and $\mathcal{P}_U$ in this method as it is shown in example 3.1 (put P finite).

It follows from example 3.1 that a symmetric proximity space need not have the lower modification in $\mathcal{P}_U$ or in $\mathcal{P}_{SU}$.

Example 4.1. We shall show that an object of $\mathcal{P}_M$ need not have the upper modifications in $\mathcal{P}_{S-U}$, $\mathcal{P}_U$, $\mathcal{P}_S$, $\mathcal{P}_{SU}$. Let $\{ X_\alpha^i \mid \alpha \in A_i \}$, $(i = 1, 2)$, be two disjoint covers of a set P (i.e. $R_i = \cup \{ X_\alpha^i \times X_\alpha^i \mid \alpha \in A_i \}$ are equivalences on P). Hence

$p_i = \{\langle X, Y\rangle \mid X \cup Y \subset P,\ Y \cap \bigcup\{X^i_\alpha \mid \alpha \in A_i,\ X \cap X^i_\alpha \neq \emptyset\} \neq \emptyset\}$
are symmetric uniformizable proximities for P. If $R_1 - R_2 \neq \emptyset$, $R_2 - R_1 \neq \emptyset$, then $p_1 \cap p_2$ is not a proximity; $p_1 \cap p_2$ generates

$p = \{\langle X, Y\rangle \mid X \cap X^i_{\alpha_i} \neq \emptyset,\ i \in (1,2),\ Y \cap X^1_{\alpha_1} \cap X^2_{\alpha_2} \neq \emptyset$ for some $\alpha_i \in A_i\}$.

Let there exist $\alpha_1 \in A_1$, $\alpha_2 \in A_2$ such that $X^1_{\alpha_1} - X^2_{\alpha_2} \neq \emptyset$, $X^2_{\alpha_2} - X^1_{\alpha_1} \neq \emptyset$ and that $X^1_{\alpha_1} \cap X^2_{\alpha_2} \neq \emptyset$. Then $(X^1_{\alpha_1} \dotplus X^2_{\alpha_2})\, p\, (X^1_{\alpha_1} \cap X^2_{\alpha_2})$, $(X^1_{\alpha_1} - X^2_{\alpha_2})\, \mathit{non}\, p\, (X^1_{\alpha_1} \cap X^2_{\alpha_2})$, $(X^2_{\alpha_2} - X^1_{\alpha_1})\, \mathit{non}\, p\, (X^1_{\alpha_1} \cap X^2_{\alpha_2})$ and consequently p is a monotone non semi-uniformizable proximity. Hence $\langle P, p\rangle$ has no upper modifications in $\mathcal{P}_{S-U}$, $\mathcal{P}_S$, $\mathcal{P}_U$, $\mathcal{P}_{SU}$.

Theorem 4.12. Suppose that $\mathcal{U}$ is a given proximally coarse semi-uniformity for a set P, $q = \sup\{p \mid p \to \mathcal{U}\}$. If $q \leftrightarrow \{\mathcal{U}_X \mid X \subset P\}$ then $\mathcal{U}_{(x)} = \{U[x] \mid U \in \mathcal{U}\}$ for all $x \in P$ and $\mathcal{U}_X = (P)$ provided $X \subset P$, $\operatorname{card} X > 1$.

Proof. Assume that $\operatorname{card} P > 1$. Then $q = \sup\{p_{\langle a, b\rangle} \mid \langle a, b\rangle \in P \times P - \Delta_P\}$ where $p_{\langle a,b\rangle} \leftrightarrow \{\mathcal{V}_X \mid X \subset P\}$, $\mathcal{V}_X = \{U[X] \mid U \in \mathcal{U}\}$ if $(a, b) - X \neq \emptyset$, $\mathcal{V}_X = (P)$ otherwise.

Lemma 4.1. (a) Let $\mathcal{U}_\alpha$ ($\alpha \in A \neq \emptyset$) be semi-uniformities for a set P. If $\mathcal{U}_\alpha \to p_\alpha$ and if all $\mathcal{U}_\alpha$ except one are proximally coarse then $\inf \mathcal{U}_\alpha \to \inf_{\mathcal{P}_{S-U}} p_\alpha$.

(b) Let p_α ($\alpha \in A \neq \emptyset$) be monotone proximities for a set P. If $p_\alpha \to \mathcal{U}_\alpha$ and if all p_α except finite number are semi-uniformizable then $\sup p_\alpha \to \sup_{\mathcal{U}C} \mathcal{U}_\alpha$.

Proof. It is sufficient to suppose that $A = (1, 2)$.

(a) Let $\mathcal{U}_1$ be proximally coarse, $\inf(\mathcal{U}_1, \mathcal{U}_2) \to p'$, $\inf_{\mathcal{P}_{S-U}}(p_1, p_2) = p$. Evidently $p' < p$. If $U_1 = \bigcup\{X_i \times Y_i \mid i \in I\} \in \mathcal{U}_1$, $\{X_i \mid i \in I\}$ is a finite disjoint family, $U_2 \in \mathcal{U}_2$ then

$$(U_1 \cap U_2)[X] = \bigcup\{(U_1 \cap U_2)[X \cap X_i] \mid i \in I\} = \bigcup\{U_2[X \cap X_i] \cap Y_i \mid i \in I\}.$$

Hence each neighborhood of X in $\langle P, p \rangle$ is a neighborhood of X in $\langle P, p \rangle$. Consequently $p < p'$.

(b) Let $\sup(p_1, p_2) \to \mathcal{U}'$, $\sup(\mathcal{U}_1, \mathcal{U}_2) = \mathcal{U}$. Evidently $\mathcal{U} < \mathcal{U}'$. If $U \in \mathcal{U}$ then $U \supset \bigcup\{X_i^1 \times Y_i^1 \mid i \in I\}$, $U \supset \bigcup\{X_j^2 \times Y_j^2 \mid j \in J\}$ where $\{X_i^1 \mid i \in I\}$, $\{X_j^2 \mid j \in J\}$ are finite disjoint covers of P,
$X_i^1 \text{ non } p_1 (P - Y_i^1)$ for all $i \in I$, $X_j^2 \text{ non } p_2 (P - Y_j^2)$ for all $j \in J$.
So U contains $\bigcup\{(X_i^1 \cap X_j^2) \times (Y_i^1 \cup Y_j^2) \mid \langle i, j \rangle \in I \times J\}$ which is an element of $\mathcal{U}'$. Consequently $\mathcal{U}' < \mathcal{U}$.

__Theorem 4.13__. Let $\mathcal{U}$, $\mathcal{V}$ be semi-uniformities, p, q be monotone proximities. (a) If $\mathcal{U} \to p$, $\mathcal{V} \to q$, $f: \langle P, p \rangle \to \langle Q, q \rangle$ is a proximally continuous mapping then there is a semi-uniformity $\mathcal{V}' \to q$ such that $f: \langle P, \mathcal{U} \rangle \to \langle Q, \mathcal{V}' \rangle$ is uniformly continuous; there is a semi-uniformity $\mathcal{U}' \to p$ such that $f: \langle P, \mathcal{U}' \rangle \to \langle Q, \mathcal{V} \rangle$ is uniformly continuous if and only if p is finer than p_1 which is induced by $\{U \mid P \times P \supset U \supset (f \times f)^{-1}[V]$ for some $V \in \mathcal{V}\}$.

(b) If $p \to \mathcal{U}$, $q \to \mathcal{V}$, $f: \langle P, \mathcal{U} \rangle \to \langle Q, \mathcal{V} \rangle$ is a uniformly continuous mapping then there is a proximity $p' \to \mathcal{U}$ such that $f: \langle P, p' \rangle \to \langle Q, q \rangle$ is proximally continuous; there is a proximity $q' \to \mathcal{V}$ such that $f: \langle P, p \rangle \to \langle Q, q' \rangle$ is proximally continuous if and only if $\mathcal{V}$ is coarser than $\mathcal{V}_1$ which is induced by $\{\langle X, Y \rangle \mid X \cup Y \subset Q$, either $X \cap Y \neq \emptyset$ or $f^{-1}[X] \, p \, f^{-1}[Y]\}$.

Proof. We shall prove only (a) because the proof of (b) is similar. The first assertion is obvious. Let $\{(f \times f)^{-1}[V] \mid V \in \mathcal{V}\}$ be a base for $\mathcal{U}_1$. There exists $\mathcal{U}' \to p$ such that $f: \langle P, \mathcal{U}' \rangle \to \langle Q, \mathcal{V} \rangle$ is uniformly continuous if and only if there exists $\mathcal{U}' \to p$, $\mathcal{U}' < \mathcal{U}_1$. But this is equivalent with the condition in our theorem (put $\mathcal{U}' = \inf(\mathcal{U}_p, \mathcal{U}_1)$ where $p \to \mathcal{U}_p$ and use lemma 4.1).

<u>Corollary</u>. ψ and ψ_1, where $\psi_1 : \mathcal{P}_M \to \mathcal{U}^c$, $\mathrm{graph}\,\psi_1 = \mathrm{graph}\,\psi$, are projective and cohereditary functors.

Φ, Φ_1, Φ_2, Φ_3 and Φ_4 are inductive and hereditary functors

($\Phi_1 : \mathcal{U} \to \mathcal{P}_{s-u}$, $\mathrm{graph}\,\Phi_1 = \mathrm{graph}\,\Phi$; $\Phi_2 : \mathcal{U}_s \to \mathcal{P}_s$, $\mathrm{graph}\,\Phi_2 = \mathrm{graph}\,\Phi/\mathcal{U}_s$;

$\Phi_3 : \mathcal{U}_u \to \mathcal{P}_u$, $\mathrm{graph}\,\Phi_3 = \mathrm{graph}\,\Phi/\mathcal{U}_u$; $\Phi_4 : \mathcal{U}_{su} \to \mathcal{P}_{su}$, $\mathrm{graph}\,\Phi_4 = \mathrm{graph}\,\Phi/\mathcal{U}_{su}$.)

Functors $\Phi/\mathcal{U}_s$, $\Phi_1/\mathcal{U}_s$, $\Phi_2/\mathcal{U}_{su}$ are inductive and hereditary.

Functors $\Phi/\mathcal{U}_u$, $\Phi_1/\mathcal{U}_u$, $\Phi/\mathcal{U}_{su}$, $\Phi_1/\mathcal{U}_{su}$, $\Phi_2/\mathcal{U}_{su}$ are coproductive and hereditary.

<u>Remark 4.6</u>. (a) The fact that functors $\Phi_i/\mathcal{U}_u$, $\Phi_i/\mathcal{U}_{su}$ $(i \neq 3, 4)$ are not cohereditary follows from that about imbedding $\mathcal{P}_{su} \to \mathcal{P}_s$ (see remark 4.4). It is known that functors Φ_i and $\Phi_i/\mathcal{U}_j$ are not productive (see e.g.[2]).

(b) The assertion about inductivity of Φ_1 means that the operations taking the upper modification in $\mathcal{U}^c$ of objects of $\mathcal{U}$ and taking inductive limits commute. But we can turn put this assertion directly from the following general proposition which is a corollary of theorem 1.2:

Let $\mathfrak{k}_i$ be an S-category over $\mathcal{C}$ with respect to a covariant functor T_i, $\mathfrak{k}_2$ a subcategory of $\mathfrak{k}_1$, F a covariant functor from $\mathfrak{k}_1$ in $\mathfrak{k}_2$ assigning to each object of $\mathfrak{k}_1$ its upper modification in $\mathfrak{k}_2$ such that $T_1 = T_2 \circ F$. Then F is inductive. (Dually for lower modifications.)

<u>5. Relations between proximity and closure</u>.

Definition 5.1. Let $\wp \leftrightarrow p \leftrightarrow \{\mathcal{U}_X \mid X \subset P\}$. We say that a closure u is induced by p (sign $p \rightarrow u$) if one of the following equivalent conditions is fulfilled ($\lambda \leftrightarrow u \leftrightarrow \{\mathcal{V}_x \mid x \in P\}$):

1) $u = \{\langle X, \{x \mid (x)\, p\, X\}\rangle \mid X \subset P\}$;

2) $\mathcal{V}_x = \mathcal{U}_{(x)}$ for all $x \in P$;

3) $\lambda = \{\langle M, x\rangle \mid \langle M, (x)\rangle \in \wp\}$.

Remark 5.1. We can define a covariant functor $F : \mathcal{P}^U \rightarrow \mathcal{C}$ in the obvious way. Evidently, $F[\mathcal{P}^U] = F[\mathcal{P}_M] = F[\mathcal{P}_{S-U}] = \mathcal{C}$ (put $\mathcal{U}_X = \cap\{\mathcal{V}_x \mid x \in X\}$ for all $X \in exp' P$).

If $f : \langle P, p\rangle \rightarrow \langle Q, q\rangle$ is lower proximally continuous, then $f : F\langle P, p\rangle \rightarrow F\langle Q, q\rangle$ need not be continuous (see example 2.1).

Definition 5.2. We shall say that a closure is symmetric semi-uniformizable, uniformizable, symmetric uniformizable resp., if it is induced by a symmetric proximity, uniformizable proximity, symmetric uniformizable proximity resp. Similarly for closure spaces. We shall denote by $\mathcal{C}_S$, $\mathcal{C}_U$, $\mathcal{C}_{SU}$ resp. the full subcategory of $\mathcal{C}$ determined by the class of all symmetric semi-uniformizable closure spaces, etc.

Theorem 5.1. For every closure u ($\lambda \leftrightarrow u \leftrightarrow \{\mathcal{V}_x \mid x \in P\}$) there is a finest proximity p_1 and a coarsest proximity p_2 inducing u ($\wp_i \leftrightarrow p_i \leftrightarrow \{\mathcal{U}^i_X \mid X \subset P\}$):

1) $p_1 = \{\langle X, Y\rangle \mid X \cup Y \subset P$, either $X \cap Y \neq \emptyset$ or $card\, X = 1$, $X \subset u\, Y\}$;

2) $\mathcal{U}^1_X \ni X$ if $card\, X > 1$, $\mathcal{U}^1_{(x)} = \mathcal{V}_x$ for $x \in P$;

3) $\wp_1 = \{\langle M, X\rangle \mid M \in \mathcal{C}(P)$, $X \subset P$, either M is eventually in X or $X = (x)$, $\langle M, x\rangle \in \lambda\}$;

4) $p_2 = \{\langle X, Y\rangle \mid X \cup Y \subset P$, either $X \cap u\, Y \neq \emptyset$ or $card\, X > 1$, $Y \neq \emptyset\}$;

5) $\mathcal{U}_X^2 = (P)$ if $card\ X > 1$, $\mathcal{U}_{(x)}^2 = \mathcal{V}_x$ for $x \in P$;

6) $p_2 = \{\langle M, X\rangle \mid M \in \mathcal{C}(P),\ X \subset P,$ either $card\ X > 1$ or $X = (x),\ \langle M, x\rangle \in \lambda\}$.

p_2 is a monotone proximity. The finest monotone proximity coarser than p_1 is semi-uniformizable and induces u. (Hence for every closure u there exist a finest and a coarsest monotone proximities induces u, a finest and a coarsest semi-uniformizable proximities inducing u.)

Corollary 1. Let $p \to u$, $q \to v$, $f: \langle P, u\rangle \to \langle Q, v\rangle$ be a continuous mapping. Then there exists p' such that $p' \to u$ and $f: \langle P, p'\rangle \to \langle Q, q\rangle$ is upper proximally continuous. $f: \langle P, p\rangle \to \langle Q, q'\rangle$ is upper proximally continuous for some q' for which $q' \to v$ if and only if $f^{-1}[V] \in \mathcal{U}_Y$ for all $V \in \mathcal{V}_x$ provided $f[Y] = (x)$, where $v \leftrightarrow \{\mathcal{V}_x \mid x \in Q\}$, $p \leftrightarrow \{\mathcal{U}_Y \mid Y \subset P\}$.

Corollary 2. F, $F/\mathcal{P}_M$, $F/\mathcal{P}_{S-U}$ are projective and coproductive functors.

Theorem 5.2. If a closure u is symmetric semi-uniformizable then there are a finest and a coarsest symmetric proximities inducing u. Let u_0 be a closure. There exists a finest symmetric semi-uniformizable closure u_1 coarser than u_0 and a coarsest symmetric semi-uniformizable closure u_2 finer than u_0; u_1 is induced by the upper modification in $\mathcal{P}_S$ of the finest semi-uniformizable proximity inducing u_0, u_2 is induced by the lower modification in $\mathcal{P}_S$ of the coarsest semi-uniformizable proximity inducing u_0. Each object of $\mathcal{C}$ has its upper modification in $\mathcal{C}_S$.

The closures mentioned above may be described like this: if $u_i \leftrightarrow \{\mathcal{U}_x^i \mid x \in P\}$, $i = 0, 1, 2$, then

1) $u_1 X = u_0 X \cup \{x \mid X \cap u_0(x) \neq \emptyset\}$ for all $X \subset P$;

2) $\mathcal{U}_x^1 = \{U \mid U \in \mathcal{U}_x^0,\ U \supset u_0(x)\}$ for all $x \in P$;

3) $u_1 X = \{x \mid$ if $X = \bigcup\{X_i \mid i \in I\}$, I is a finite set, then there is $i \in I$ such that $x \in u_0 X_i$ and either $X_i \cap u_0(x) \neq \emptyset$ or $\operatorname{card} X_i \geq \aleph_0$;

4) $\mathcal{U}_x^2 = \{U - F \mid U \in \mathcal{U}_x^0,\ P \supset F,\ F \cap u_0(x) = \emptyset, \operatorname{card} F < \aleph_0\}$ for all $x \in P$.

Corollary 1. A closure $u \longleftrightarrow \{\mathcal{U}_x \mid x \in P\}$ is symmetric semi-uniformizable if and only if one of the following equivalent conditions is fulfilled:

(a) the relation $\{\langle x, y\rangle \mid x \in u(y)\}$ is symmetric;

(b) $u(x) \subset \bigcap \mathcal{U}x$ for all $x \in P$;

(c) $u(x) = \bigcap \mathcal{U}_x$ for all $x \in P$;

(d) if $\langle x, y\rangle \in P \times P$ and if $y \notin U$ for some $U \in \mathcal{U}_x$ then $x \notin V$ for some $V \in \mathcal{U}_y$.

Corollary 2. $\mathcal{C}_s$ is an S-category over $\mathcal{M}$ with respect to the forgetful functor. $\mathcal{C}_s$ is projective and coproductive in $\mathcal{C}$.

Remark 5.2. The class of all objects of $\mathcal{C}$ having the lower modification in $\mathcal{C}_s$ is precisely the class of objects of $\mathcal{C}_s$.

First we shall prove the following lemma:
Let $\langle P, u\rangle$ be an object of $\mathcal{C}_s$, $\langle Q, v\rangle$ an object of $\mathcal{C}$, v' be the coarsest symmetric semi-uniformizable closure finer than v, $f : \langle P, u\rangle \longrightarrow \langle Q, v\rangle$ be a continuous mapping. Then $f : \langle P, u\rangle \longrightarrow \langle Q, v'\rangle$ is a continuous mapping if and only if $f^{-1}[V] \supset u f^{-1}[x]$ for all $x \in Q$ and for all neighborhoods V of x in $\langle Q, v\rangle$. (The proof follows from the fact that $f : \langle P, u\rangle \rightarrow \langle Q, v'\rangle$ is a continuous mapping if and only if

$f : \langle P, p \rangle \to \langle Q, q \rangle$, where p is the finest symmetric proximity inducing u and q is the coarsest proximity inducing v, *is upper proximally continuous*.)

If $\langle Q, v \rangle$ is not symmetric semi-uniformizable closure space, then there is $x \in P$ and a neighborhood V of x in $\langle Q, v \rangle$ such that $v(x) - V \neq \emptyset$. Let $\langle P, u \rangle$ be the space of real numbers, $y \in v(x) - V$, $f = (] \leftarrow, 0 [\![\times (x)) \cup ([\![0, \rightarrow [\![\times (y))$. Then f is a continuous mapping $\langle P, u \rangle \to \langle Q, v \rangle$ and by the foregoing lemma the mapping $f : \langle P, u \rangle \to \langle Q, v' \rangle$ is not continuous. Hence by lemma 2.2 $\langle Q, v \rangle$ has no lower modification in $\mathcal{C}_s$.

Theorem 5.3. Functors $F/\mathcal{P}_s$, F' are projective and coproductive ($F' : \mathcal{P}_s \to \mathcal{C}_s$, *graph* $F' =$ *graph* $F/\mathcal{P}_s$).

Theorem 5.4. A closure space $\langle P, u \rangle$ is uniformizable if and only if it is topological.

Proof. If u is topological then the finest monotone proximity inducing u is uniformizable. The converse is clear.

Corollary 1. Every uniformizable closure u has the finest uniformizable proximity p inducing u (p is the finest monotone proximity inducing u). Every object of $\mathcal{C}$ has its upper modification in $\mathcal{C}_U$ and hence $\mathcal{C}_U$ is an S-category over $\mathcal{M}$ with respect to the forgetful functor.

Corollary 2. Functors $F/\mathcal{P}_U$, F'' are projective ($F'' : \mathcal{P}_U \to \mathcal{C}_U$, *graph* $F'' =$ *graph* $F/\mathcal{P}_U$). $\mathcal{C}_U$ is projective in $\mathcal{C}$.

Remark 5.3. It is easy to prove that $F/\mathcal{P}_U$, F'' are coproductive functors and that $\mathcal{C}_U$ is coproductive in $\mathcal{C}$.

Theorem 5.5. Every symmetric uniformizable closure u has the finest symmetric uniformizable proximity p inducing u.

Each object of $\mathcal{C}$ has its upper modification in $\mathcal{C}_{su}$ and hence $\mathcal{C}_{su}$ is an S-category over $\mathcal{M}$ with respect to the forgetful functor.

Proof. p is the finest symmetric uniformizable proximity coarser than the finest monotone proximity inducing u.

Corolary. Functors $F/\mathcal{P}_{su}$, F''' ($F''' : \mathcal{P}_{su} \to \mathcal{C}_{su}$, $\text{graph } F''' = \text{graph } F/\mathcal{P}_{su}$), $F'/\mathcal{P}_{su}$, $F''/\mathcal{P}_{su}$ are projective. $\mathcal{C}_{su}$ is projective in $\mathcal{C}$, $\mathcal{C}_s$, $\mathcal{C}_u$.

Theorem 5.6. A closure space $\langle P, u \rangle$ is symmetric uniformizable if and only if the following condition is fulfilled:

U is a neighborhood of x in $\langle P, u \rangle$ if and only if there is a monotonely densely ordered set A with the first element α_0, with the last element $\alpha_1 \neq \alpha_0$ and a family $\{U_\alpha \mid \alpha \in A\}$ such that $U = U_{\alpha_0}$, $U_{\alpha_1} = (x)$ and $U_\alpha \supset \supset u\, U_{\alpha'}$, U_α is a neighborhood of $U_{\alpha'}$ in $\langle P, u \rangle$ provided $\alpha < \alpha'$.

Proof follows from remark 4.5.

Remark 5.4. Now, it is easy to prove that F''', $F/\mathcal{P}_{su}$, $F'/\mathcal{P}_{su}$, $F''/\mathcal{P}_{su}$ are coproductive functors and that $\mathcal{C}_{su}$ is coproductive in $\mathcal{C}$, $\mathcal{C}_s$, $\mathcal{C}_u$.

Theorem 5.7. Every locally compact topological space $\langle P, u \rangle$ (i.e. each $x \in P$ has a base of compact neighborhoods) has the coarsest uniformizable proximity inducing u.

Proof. It is known that if a uniformizable proximity $p \leftrightarrow \{\mathcal{U}_X \mid X \subset P\}$ induces u then $\mathcal{U}_X = \cap \{\mathcal{U}_{(x)} \mid x \in X\}$ provided X is compact. There is a coarsest semi-uniformizable proximity p with this property ($p = \{\langle X, Y \rangle \mid$ if $\mathcal{O}\!\mathcal{L}$ is a finite cover of Y then there is $Z \in \mathcal{O}\!\mathcal{L}$ such that $X' \cap \overline{Z} \neq \emptyset$ for each compact $X' \supset X\}$). This proximity is uniformizable provided u

is locally compact topological.

Corollary. Every locally compact symmetric uniformizable space $\langle P, u \rangle$ has the coarsest symmetric uniformizable proximity inducing u .

Proof follows from the fact that for every uniformizable proximity p there is a coarsest symmetric uniformizable proximity finer than p .

Remark 5.5. It is shown in [1] that the converse of the foregoing corollary is true, too.

Theorem 5.8. The coarsest uniformizable proximity p inducing a given closure u is the coarsest uniformizable proximity finer than the coarsest semi-uniformizable proximity q inducing u if and only if $p = q$.

Proof. Suppose that $p \neq q$. Then there is an infinite set X in $P = \bigcup \mathcal{D} u$ and a set $U \neq P$ such that $X \text{ non } p \,(P - U)$. The coarsest uniformizable proximity r inducing the discrete closure for P is finer than q but it is not finer than p , because $X \text{ non } r \,(P - U)$ if and only if $U = P$.

Remark 5.6. It can be proved that no non-uniformizable semi-uniformizable proximity p has the coarsest uniformizable proximity finer than p .

6. Relations between semi-uniformity and closure.

Definition 6.1 Let $\mathcal{U} \leftrightarrow \ell, \lambda \leftrightarrow u \leftrightarrow \{\mathcal{V}x \mid x \in P\}$. We say that u is induced by a semi-uniformity $\mathcal{U}$ (sign $\mathcal{U} \longrightarrow u$) if $\mathcal{D} \bigcup \mathcal{U} = P$ and if one of the following equivalent conditions is fulfilled:

1) $u = \{\langle X, \bigcap\{U^{-1}[X] \mid U \in \mathcal{U}\}\rangle \mid X \subseteq P\}$;

2) $\mathcal{V}_x = \{U[x] \mid U \in \mathcal{U}\}$ for all $x \in P$;

3) $\lambda = \{ \langle \beta M, x \rangle \mid M \in \mathcal{C}, \ \mathcal{E} \propto M = (x) \}$.

Remark 6.1. We can define a covariant functor $G : \mathcal{U} \to C$ in an obvious way.

Theorem 6.1. Let $\mathcal{U}$ be a semi-uniformity, u be a closure, p be a proximity.

If $\mathcal{U} \to p$ then $\mathcal{U} \to u$ if and only if $p \to u$.

If $p \to \mathcal{U}$ then $p \to u$ if and only if $\mathcal{U} \to u$.

Corollary. $G = F/_{\mathcal{P}_M} \circ \psi$, $F/_{\mathcal{P}_M} = G \circ \Phi$, $G[\mathcal{U}] = C$, $G[\mathcal{U}_S] = C_S$, $G[\mathcal{U}_U] = C_U$, $G[\mathcal{U}_{SU}] = C_{SU}$.,

Theorem 6.2. For every closure u $(\lambda \leftrightarrow u \leftrightarrow \{ \mathcal{V}_x \mid x \in P \})$ there is a finest semi-uniformity $\mathcal{U}_1 \leftrightarrow \mathcal{C}_1$ inducing u and a coarsest semi-uniformity $\mathcal{U}_2 \leftrightarrow \mathcal{C}_2$ inducing u :

1) $\mathcal{U}_1 = \{ \sum \{ V_x \mid x \in P \} \mid V_x \in \mathcal{V}_x \}$;

2) $\mathcal{C}_1$ is generated by $\{ M \mid M \in \mathcal{C}(P \times P), \ \mathcal{E} \propto M = (x)$ for some $x \in P, \ \langle \beta M, x \rangle \in \lambda \}$;

3) $\mathcal{U}_2 = \{ U \mid U \subset P \times P, \ U_{[x]} \in \mathcal{V}_x$ for all $x \in P$, $\operatorname{card} \{ x \mid U_{[x]} \neq P \} < \aleph_0 \}$;

4) $\mathcal{C}_2 = \{ M \mid M \in \mathcal{C}(P \times P), \langle \beta M', x \rangle \in \lambda$ for each subnet M' of M such that $\mathcal{E} \propto M' = (x)$ for some $x \in P \}$.

Corollary. Every symmetric semi-uniformizable closure u has the finest and the coarsest symmetric semi-uniformity inducing u. Every (symmetric) uniformizable closure u has the finest (symmetric) uniformity inducing u.

Proof. Take the competent modifications of $\mathcal{U}_1$ and $\mathcal{U}_2$.

Theorem 6.3. If $\langle P, u \rangle$ is a locally compact topological (symmetric uniformizable) space then there is a coarsest (symmetric) uniformity inducing u.

Proof follows firectly from theorem 5.7.

Remark 6.2. It is shown in [1] that the converse of theorem 6.3 is true in the case of symmetric uniformizable spaces.

Theorem 6.4. Let $\mathcal{U} \to u$, $\mathcal{V} \to v$, $f: \langle P, u \rangle \to \langle Q, v \rangle$ be a continuous mapping. There is a semi-uniformity $\mathcal{U}'$ inducing u such that $f: \langle P, \mathcal{U}' \rangle \to \langle Q, \mathcal{V} \rangle$ is uniformly continuous. There is a semi-uniformity $\mathcal{V}'$ inducing v such that $f: \langle P, \mathcal{U} \rangle \to \langle Q, \mathcal{V}' \rangle$ is uniformly continuous if and only if

$((P - f^{-1}[x]) \times P) \cup (f^{-1}[x] \times f^{-1}[V_x]) \in \mathcal{U}$ for all $x \in P$ and for all neighborhoods V_x of x in $\langle Q, v \rangle$.

Proof. Put the finest semi-uniformity inducing u for $\mathcal{U}'$ and the coarsest semi-uniformity inducing v for $\mathcal{V}'$.

Corollary. Functors G, $G/\mathcal{U}_S$, $G/\mathcal{U}_U$, $G/\mathcal{U}_{SU}$, G', $G'/\mathcal{U}_{SU}$, G'', $G''/\mathcal{U}_{SU}$, G''' are projective and coproductive.

$(G': \mathcal{U}_S \to C_S$, $G'': \mathcal{U}_U \to C_U$, $G''': \mathcal{U}_{SU} \to C_{SU}$, $\mathrm{graph}\, G' = \mathrm{graph}\, G/\mathcal{U}_S$ etc.

Proof. Only the assertion that $G/\mathcal{U}_U$ is coproductive is necessary to prove directly.

7. Conclusion.

7.1. Every proximity $p \leftrightarrow \{\mathcal{U}_X \mid X \subset P \times P\}$ determines a semi-uniformity $\mathcal{U} = \mathcal{U}_{\Delta_P}$ for P.
On the other hand every semi-uniformity $\mathcal{U}$ for P determines a proximity $p \leftrightarrow \{\mathcal{U}_X \mid X \subset P \times P\}$ such that $\mathcal{U}_{\Delta_P} = \mathcal{U}$. It is sufficient to put $\{U \circ X \mid U \in \mathcal{U}\}$ for a base of each $\mathcal{U}_X$ (i.e. $\langle P \times P, p \rangle = \psi[\langle P, \mathcal{V} \rangle \times \langle P, \mathcal{U} \rangle]$ where $\mathcal{V}$ is the finest semi-uniformity for P).

It follows that a covariant functor $\tilde{\psi}: \mathcal{U} \to \mathcal{P}_{S-U}$ defined in this way

$\tilde{\psi}\langle P, \mathcal{U} \rangle = \psi[\langle P, \mathcal{V} \rangle \times \langle P, \mathcal{U} \rangle]$,

$\mathrm{graph}\, \tilde{\psi} f = \mathrm{graph}\, f \times \mathrm{graph}\, f$ (relational product) is an isofunctor.

If we restrict our attention on $\mathcal{U}_U$ we can put $\tilde{\psi}\langle P, \mathcal{U}\rangle = \psi[\langle P, \mathcal{U}_{-1}\rangle \times \langle P, \mathcal{U}\rangle]$ where $\mathcal{U}_{-1} = \{U^{-1} \mid U \in \mathcal{U}\}$.

7.2. Every closure $u \leftrightarrow \{\mathcal{U}_X \mid X \in \exp P\}$ determines a proximity $p_u \leftrightarrow \{\{U\{Y \mid Y \in U\} \mid U \in \mathcal{U}_X\} \mid X \subset P\}$.

Every proximity $p \leftrightarrow \{\mathcal{V}_X \mid X \subset P\}$ determines a closure u for $\exp P$ such that $p = p_u$. It is sufficient to put

$u \leftrightarrow \{\{U \mid U \supset \exp V$ for some $V \in \mathcal{V}_X\} \mid X \in \exp P\}$.

Hence the covariant functor $\tilde{G} : \mathcal{P}_U \to C$ (graph $\tilde{G}f = \{\langle X, U\{fx \mid x \in X\}\rangle \mid X \in \exp P\}$) is an isofunctor.

R e f e r e n c e s:

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