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CONCERNING A PROOF OF $\aleph_{\alpha+1} \leq 2^{\aleph_\alpha}$ WITHOUT [redacted] AXIOM OF CHOICE

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We denote Σ_0 (Σ^* resp.) the set theory with the axioms of the groups A, B, C (A, B, C, D, E resp.) see [1].

Our considerations are done in the set theory Σ_0. It is well known, that it is possible to define, in this theory, the class L of constructible sets.

Let $c \subseteq L$. We denote $L(c)$ the class constructed analoguously to the construction of L – the only difference is that we define $F'\alpha = c$, where α is the least ordinal number such that $(x)\,[x \in c \rightarrow (\exists \beta)[\beta \in \alpha \,\&\, x = F'\beta]]$. If $c \in L$, we have obviously $L(c) = L$.

Restricting the relation $\in$ on $L(c)$, we obtain a model of the theory Σ^*, which we denote by $\Delta(c)$ (see [2]).

Lemma 1. The ordinal numbers of Σ_0 are the same as the ordinal numbers of $\Delta(c)$.

Lemma 2. Every cardinal number α of Σ_0 is a cardinal number of $\Delta(c)$.

Proof. Let there be a $1-1$-mapping of α onto $\beta \in \alpha$ in $\Delta(c)$. Then the same mapping is a $1-1$- mapping of α onto β in Σ_0.

Lemma 3. **Every regular cardinal number α of Σ_0 is a regular cardinal number of $\Delta(c)$.**

Proof. Let α be confinal to $\beta \in \alpha$ in $\Delta(c)$. Then it is confinal to β in Σ_0.

Lemma 4. Let γ be a cardinal number in Δ such that $\omega_{\alpha+1}$ is the first greater one (in Δ). Then we have $\aleph_{\alpha+1} \leq 2^{\aleph_\alpha}$ in Σ_0.

Proof. Obviously $\omega_\alpha \in \gamma \in \omega_{\alpha+1}$. Hence there exists a set c which is a $1-1$ - mapping of ω_α onto γ. Evidently $c \subseteq L$. $\omega_{\alpha+1}$ is the first cardinal greater than ω_α in $\Delta(c)$. Really, if there is a δ such that $\omega_\alpha \in \delta \in \omega_{\alpha+1}$ and such that δ is a cardinal in $\Delta(c)$, we have $\gamma \in \delta$ and hence δ is cardinal number in Δ - a contradiction. Since the axiom of choice holds true in $\Delta(a)$, we have $\aleph_{\alpha+1} \leq 2^{\aleph_\alpha}$ in $\Delta(a)$ and hence $\aleph_{\alpha+1} \leq 2^{\aleph_\alpha}$ in the theory.

Theorem. Any cardinal number $\omega_{\alpha+1}$ in Σ_0 such that $\aleph_{\alpha+1} \not\leq 2^{\aleph_\alpha}$ is an inaccessible cardinal number in Δ.

Proof. $\omega_{\alpha+1}$ is regular cardinal number in Δ by lemma 3. By lemma 4 $\omega_{\alpha+1}$ is inaccessible cardinal number in Δ.

Corollary 1. If the system of axioms $\Sigma_0 + (\exists \alpha)[\aleph_{\alpha+1} \not\leq 2^{\aleph_\alpha}]$ is consistent, the system $\Sigma^{\aleph}$ + "there exists an inaccessible cardinal number" is consistent too.

Corollary 2. If the existence of an inaccessible cardinal number contradicts with the axioms of the set theory, then $\aleph_{\alpha+1} \leq 2^{\aleph_\alpha}$ is provable without using of the axiom of choice.

L i t e r a t u r e :

[1] K. GÖDEL, The consistency of the axiom of choice and of the generalized continuum hypothesis with the axioms of set theory, Annals of Math. Studies 3, Princeton 1940.

[2] A. LÉVY, A generalization of Gödel's notion of constructibility, The Journ. of Symb.Log. 25(1960), No 2.