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Stanislav Tomášek

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ON TENSOR PRODUCTS OF ABELIAN GROUPS

S. TOMÁŠEK, Liberec

§ 1.

In this paper we shall consider Abelian groups only. The group operation we denote by addition. Z means the ring of all integers. Any Abelian group G is considered as a module with respect to the operation of multiplication $(n, x) \to n x$ for arbitrary n in Z and x in G. A mapping f of a group G into a group K is called Z-linear if $f(x+y) = f(x) + f(y)$ for every x in G and y in G. Similarly we define a Z-bilinear mapping.

If G and K are Abelian groups we denote by $G \otimes K$ their tensor product. Any element z in $G \otimes K$ is of the form (see [1])

$$z = x_1 \otimes y_1 + \dots + x_n \otimes y_n ,$$

where x_i is in $G\,(1 \le i \le n)$, y_i in $K\,(1 \le i \le n)$ and n is an arbitrary integer. Similarly we denote for a subset A of G, B subset of K, by $A \otimes B$ the set of all $x \otimes y \in G \otimes K$, where x is in A, y in B.

For our further discussion we shall assume that G and K are topological Abelian groups, $\{U\}$ and $\{V\}$ mean the systems of all neighborhoods of zero element in G and K.

For any $U \in \{U\}$, $V \in \{V\}$ and for any positive integer n we define:

$$(1)\quad H^n_{U,V} = \{z \in G \otimes K ;\ n z \in \sum_{i=1}^{n} U_i \otimes V_i ,\ V_i = V,\ U_i = U,\ 1 \le i \le n\}$$

(2) $\Omega_{U,V} = \bigcup_{n=1}^{\infty} H^{n}_{U,V}$.

<u>Lemma</u>. For $U \in \{U\}$, $W \in \{U\}$, $V \in \{V\}$ and $U + U \subseteq W$ holds
$$\Omega_{U,V} + \Omega_{U,V} \subseteq \Omega_{W,V} .$$

Proof. If x_1 is in $\Omega_{U,V}$, x_2 in $\Omega_{U,V}$, then there exist two integers n, m satisfying $nx_1 = x_1 \otimes y_1 + \dots + x_n \otimes y_n$, $m.x_2 = x'_1 \otimes y'_1 + \dots + x'_m \otimes y'_m$ for suitable $x_i \in U, y_i \in V (1 \leq i \leq n), x'_i \in U, y'_i \in V (1 \leq i \leq m)$. Making use of the equality

$$2mn(x_1+x_2) = m[(2x_1)\otimes y_1 + \dots + (2x_n)\otimes y_n] + n[(2x'_1)\otimes y'_1 + \dots + (2x'_m)\otimes y'_m]$$

we prove $x_1 + x_2 \in H^{2nm}_{W,V} \subseteq \Omega_{W,V}$.

Hence the collection $\{\Omega_{U,V};\ U \in \{U\},\ V \in \{V\}\}$ satisfies evidently therefore the axioms of a group topology in the tensor product $G \otimes K$.

<u>Definition 1</u>. The topology in $G \otimes K$ defined by $\{\Omega_{U,V};\ U \in \{U\},\ V \in \{V\}\}$ is called the tensor product topology and is denoted by π .

In the tensor product $G \otimes K$ it will be considered throughout this paper the topology π only.

<u>Remark 1</u>. a) Every neighborhood of the form (2) has the following property: $nx \in \Omega_{U,V}$ for a given $x \in G \otimes K$ and some positive integer n implies $x \in \Omega_{U,V}$.

b) If G is a discrete group, $\Omega_{\{0\},V}$ consists exactly of all cyclic elements of $G \otimes K$.

c) The canonical Z -bilinear mapping $(x,y) \to x \otimes y$ of $G \times K$ into $G \otimes K$ is continuous in $(0,0)$. The Z -linear mapping $x \to x \otimes y$ of G into $G \otimes K$ is not continuous in general (e.g. if R is the additive group of real numbers with the natural topology, K a discrete group with a finite basis $\{e_i\}_{i=1}^{n}$, then $x \to x \otimes e_i$ $(1 \leq i \leq n)$ of R into $R \otimes K$ is not continuous).

In the following proposition we shall establish a sufficient condition for the continuity of $(x, y) \to x \otimes y$. An element x in G will be termed bounded if for every $U \in \{U\}$ there exists an integer $n > 0$ satisfying $x \in n\,U$.

Proposition 1. Let G and K be two topological groups, f Z-bilinear mapping of $G \times K$ into a topological group H continuous in $(0, 0)$. Then f is continuous in every point (x_0, y_0), where x_0 is a bounded element in G, y_0 a bounded element in K.

Proof. Let W and W_1 be two neighborhoods of zero element in H, $W_1 + W_1 + W_1 \subseteq W$. There exist neighborhoods U_1, V_1 in G, K such that $f(U_1, V_1) \subseteq W$. For some n, m hold $x_0 \in n\,U_1$, $y_0 \in m\,V_1$ and we choose neighborhoods U, V in G, K satisfying $U + \ldots + U \subseteq U_1$ (m summands), $V + \ldots + V \subseteq V_1$ (n summands). For $(u, v) \in U \times V$ it follows $f(x_0 + u, y_0 + v) - f(x_0, y_0) = f(x_0, v) + f(u, y_0) + f(u, v)$ is in $W_1 + W_1 + W_1 \subseteq W$.

Remark 2. A similar result holds for the Z-linear mapping $x \to f(x, y_0)$ of G into H, where y_0 is bounded in K.

Definition 2. We shall say that a subset A of an Abelian group G is convex in G if $z \in A$ for any $z \in G$ satisfying $k \cdot z \in A + \ldots + A$ (k summands) for some k. A topological group having a fundamental system of convex neighborhoods is called locally convex.

A quotient group G/G_0 of a locally convex group G need not be locally convex (e.g. additive group of real numbers modulo 1 is not locally convex).

Proposition 2. a) A subgroup of a locally convex group is locally convex.

b) If G is a locally convex group, G_0 a disivible subgroup of G (i.e. for any $y \in G_0$ and $n \in Z$ there exists $y' \in G_0$ with $ny' = y$), then the quotient group G/G_0 is locally convex.

c) If G_i $(1 \leq i \leq n)$ are locally convex groups, then the direct product $G = \prod_{i=1}^{n} G_i$ is locally convex.

Proof. The statements a) and c) are evident. In order to prove b), we take an arbitrary neighborhood $\varphi(U)$ of zero element in G/G_0 , where φ is the canonical mapping $G \to G/G_0$. If $n\varphi(x) \in \varphi(U) + \ldots + \varphi(U)$ (n summands), then $nx = x_1 + \ldots + x_n + y$ for some $x_i \in U \, (1 \leq i \leq n)$, $y \in G_0$. For $x_0 \in G_0$, $n \cdot x_0 = y$, from the equality $n(x - x_0) = x_1 + \ldots + x_n$ it follows $x - x_0 \in U$ and $\varphi(x) = \varphi(x - x_0)$ is in $\varphi(U)$.

<u>Theorem 1</u>. The topology π in $G \otimes K$ is locally convex. The proof is evident.

<u>Remark 3</u>. If G is a topological group with the topology τ, then there exists a finest locally convex topology τ^* which is coarser than τ. The fundamental system of neighborhoods for τ^* can be defined by $co(\mathcal{U}) = \bigcup_{n=1}^{\infty} K_{\mathcal{U}}^{n}$, where $K_{\mathcal{U}}^{n} = \{x \in G;\, n \cdot x \in U + \ldots + U$ (n summands)$\}$, $n = 1, 2, \ldots$; and $U \in \{U\}$. The proposition 1 of [3] is also true for the topology π.

<u>Examples</u>. 1. Let D be the group of p-adic numbers (see [2];[3], § 3) with the topology τ (see [3], § 3). Then τ^* is clearly the trivial topology, hence the tensor product topology π in $D \otimes D$ is also trivial.

2. Let K be the multiplicative group of complex numbers. For any neighborhood $U_\varepsilon = \{x \in K;\ |x - 1| < \varepsilon\}$ we have

$\omega(U_\varepsilon) = \{x \in K;\ 1-\varepsilon < |x| < 1+\varepsilon\}$. In particular if τ is the usual topology (see [2]) in the additive group R/Z of the real numbers modulo 1, then τ^* is a trivial topology and hence the tensor product topology in $(R/Z) \otimes (R/Z)$ is also trivial.

Theorem 2. Let G and K be two topological Abelian groups. On the tensor product $G \otimes K$ there exists a unique locally convex topology with the properties

(a) The canonical Z-bilinear mapping $(x, y) \to x \otimes y$ is continuous in $(0, 0)$.

(b) If H is a locally convex group, then the canonical isomorphism of the group $\mathcal{L}(G, K; H)$ of all Z-bilinear mappings $G \times K \to H$ onto the group $\mathcal{L}(G \otimes K; H)$ of all Z-linear mappings $G \otimes K \to H$ defines an isomorphism of the group $\mathcal{B}(G, K; H)$ of all continuous in $(0, 0)$ Z-bilinear mappings $G \times K \to H$ onto the group $\mathcal{B}(G \otimes K; H)$ of all continuous Z-linear mappings $G \otimes K \to H$.

Proof. Let the image of $f \in \mathcal{L}(G, K; H)$ in $\mathcal{L}(G \otimes K; H)$ under the canonical isomorphism be denoted by f^*. It suffices to prove that $f \in \mathcal{B}(G, K; H)$ implies $f^* \in \mathcal{B}(G \otimes K; H)$. For any convex neighborhood W of zero element in H there exist neighborhoods U, V in G, K such that $f(U, V) \subseteq W$. For $x \in \Omega_{u,v}$ we have $nx \in U \otimes V + \dots + U \otimes V$ (n summands) for a suitable n; from $nf^*(x) = f^*(nx) \in$ $\in f^*(U \otimes V) + \dots + f^*(U \otimes V)$ it follows that $f^*(x) \in W$. The uniqueness of such a topology is clear.

For the topology π in $G \otimes K$ are true propositions 2 and 4 of [3]. If G and K are Abelian groups, G' and K' subgroups in G and K, then the tensor products $G \otimes K / \Gamma(G', K')$

$(\Gamma(G', K')$ means the subgroup in $G \otimes K$ generated by the set of all $x \otimes y$, x is in G' or y is in K') and $(G/G') \otimes (K/K')$ are Z-isomorphic (see [1]), but the canonical mapping Φ of $G \otimes K / \Gamma(G', K')$ onto $(G/G') \otimes (K/K')$ is not open in general. For example let G be the additive group of real numbers, G' the additive subgroup of integers, K a discrete group with a finite basis. Then $G \otimes K$ and $G \otimes K / \Gamma(G', 0)$ are discrete. The topology of $(G/G') \otimes K$ is not discrete. This proves that propositions 3 and 4 of [3] are false for the topology π.

By an annihilator (see [4]) of the group K in G we mean the set of all elements $x \in G$ such that $x \otimes y = 0$ for every $y \in K$.

Proposition 3. Let G and K be two Abelian groups, G' a subgroup in G contained in the annihilator of the group K in G, K' a subgroup in K contained in the annihilator of the group G in K. Then the canonical Z-isomorphism Φ of $G \otimes K$ onto $(G/G') \otimes (K/K')$ is a topological isomorphism.

Proof. It is evident that $\Phi : x \otimes y \to \varphi(x) \otimes \psi(y)$, where φ and ψ are canonical mappings of $G \to G/G'$ and $K \to K/K'$, is continuous. It suffices to prove that Φ is open.

Let $\sum_{i=1}^{k} \varphi(x_i) \otimes \psi(y_i) \in (G/G') \otimes (K/K')$ be an arbitrary element in $\Omega_{\varphi(U), \psi(V)}$. There exist an integer n and $u_i \in U$, $v_i \in V$ $(1 \leq i \leq n)$ such that $n(\sum_{i=1}^{k} \varphi(x_i) \otimes \psi(y_i)) = \sum_{i=1}^{n} \varphi(u_i) \otimes \psi(v_i)$. We set $x = \sum_{i=1}^{k} x_i \otimes y_i$, $w = nx - \sum_{i=1}^{n} u_i \otimes v_i$. From $\Phi(w) = \Phi(nx) - \Phi(\sum_{i=1}^{n} u_i \otimes v_i) = n(\sum_{i=1}^{k} \varphi(x_i) \otimes \psi(y_i)) - \sum_{i=1}^{n} \varphi(u_i) \otimes \psi(v_i) = 0$ it follows $w = 0$, hence

$$nx = \sum_{i=1}^{m} (u_i \otimes v_i) \in U \otimes V + \dots + U \otimes V \quad (n \text{ summands}).$$

This proves $\Phi(\Omega_{U,V}) \supseteq \Omega_{\varphi(U),\psi(V)}$.

Remark 4. It was mentioned that $G \otimes K$ is not separated in general. If we denote by Γ the closure of zero element in $G \otimes K$, then the quotient group $(G \otimes K)/_\Gamma$ is separated. It can be shown that $(G \otimes K)/_\Gamma$ is locally convex. We can therefore extend some results of this § to the case of $(G \otimes K)/_\Gamma$.

The following statement seems to be interesting: Let G_i, K_i $(i = 1, 2)$ be four Abelian groups, u and v continuous Z-linear open mappings of G_1 onto G_2 and of K_1 onto K_2. We suppose next that G_1 (or K_1) is divisible (i.e. for any $y \in G_1$ and any $n \in Z$ there exists $y' \in G_1$ with $ny' = y$). Then the mapping $(u \otimes v)_{\Gamma_1}^{\Gamma_2}$ of $(G_1 \otimes K_1)/_{\Gamma_1}$ onto $(G_2 \otimes K_2)/_{\Gamma_2}$ obtained by factorization of $u \otimes v$ is open (Γ_i $(i = 1, 2)$ is the closure of zero element in $G_i \otimes K_i$ $(i = 1, 2)$).

The proof of this statement does not present any difficulty.

Remark 5. We can construct the completion $G \hat{\otimes} K$ of $(G \otimes K)/_\Gamma$. It is easy to see that $G \hat{\otimes} K$ is locally convex whenever G or K is divisible.

§ 2.

In this section C means the field of rational, real or complex numbers. The unit element of C will be denoted by 1. We recall that for any $x = \lambda_1 \otimes x_1 + \dots + \lambda_n \otimes x_n$ of $C \otimes G$ a multiplication by a scalar $\lambda \in C$ can be defined in the following manner (see [1]):

$$(3) \qquad \lambda . x = \lambda \lambda_1 \otimes x_1 + \dots + \lambda \lambda_n \otimes x_n .$$

In case C is the field of rational numbers, every element $x \in C \otimes G$ is of the form (see [1]) $x = \kappa \otimes y$, where

$\lambda \in C$, $y \in G$.

Definition 3. Let E be a vector space over C. We shall say that E is a general topological vector space (abbreviated g-space) if E is a topological space and

(a) $(x, y) \to x + y$ is continuous in $E \times E$

(b) $(\lambda, x) \to \lambda x$ is continuous in $(0, 0) \in C \times E$

(c) $x \to \lambda x$ is continuous in $0 \in E$ for every $\lambda \in C$.

It can be shown that a topology of a g-space is described by a basis of a filter $\mathcal{F}$ in E satisfying

(a′) $U \in \mathcal{F}$, $\lambda \in C$, $|\lambda| \leq 1$ imply $\lambda U \subseteq U$,

(b′) for any $U \in \mathcal{F}$ there exists $V \in \mathcal{F}$ such that $V + V \subseteq U$,

(c′) if $U \in \mathcal{F}$, $\lambda \in C$, then $\lambda V \subseteq U$ for some $V \in \mathcal{F}$.

Similarly we define a locally convex g-space.

Proposition 4. Let G be a topological group, C the field of rational, real or complex numbers with the natural topology. Then the tensor product $C \otimes G$ with respect to the topology π is a g-space. If every neighborhood of zero element in G generates G, then $C \otimes G$ is a topological vector space.

Proof. In order to prove that $C \otimes G$ is a g-space if suffices to show (a′) and (c′). If $U = \{\lambda \in C; |\lambda| \leq \varepsilon\}$ then for any neighborhood V of zero element in G holds

$$\lambda \Omega_{u,v} \subseteq \Omega_{u,v} .$$

Similarly $\lambda \Omega_{w,v} \subseteq \Omega_{u,v}$ for $\lambda W \subseteq U$. It remains to prove that, if G is generated by V, for every $x \in C \otimes G$ there exists $\lambda \in C$ satisfying $\lambda \cdot x \in \Omega_{u,v}$.

Obviously we may assume that V is a symmetric neighborhood in G. Let $x = \lambda_1 \otimes y_1 + \dots + \lambda_n \otimes y_n$ be an element of

$C \otimes G$. Every y_i $(1 \leq i \leq n)$ is of the form $y_i = y_i + \dots + y_i^{k_i}$, where $y_i^{\kappa} \in V\,(1 \leq \kappa \leq k_i,\ 1 \leq i \leq n)$. We choose $\lambda \in C$ satisfying $\lambda \lambda_i \in U (1 \leq i \leq n)$ and put $p = k_1 + \dots + k_n$. From $x = \sum_{i=1}^{n} \sum_{\kappa=1}^{k_i} (\lambda \lambda_i \otimes y_i^{\kappa}) \in U \otimes V + \dots + U \otimes V$ (p summands) it follows $\lambda \cdot p^{-1} \cdot x \in H_{u,v}^{p} \subseteq \Omega_{u,v}$, where $H_{u,v}^{p}$ is defined in (1).

Proposition 5. For any topological group G there exists a Z-linear and continuous mapping onto a subgroup of a locally convex q-space. If G is generated by every neighborhood of zero element, we can replace q-space in the first assertion by a locally convex vector space.

Proof. We define a mapping φ of G into $C \otimes G$ by

$$\varphi(x) = 1 \otimes x \tag{4}$$

for any $x \in G$. The mapping φ is clearly Z-linear and continuous. The rest of the proof follows from Proposition 4 and from Theorem 1 (see also § 1 of [3]).

If G is a torsion-free group, C the field of rational numbers, then the mapping (4) is a Z-isomorphism (see [1]).

Theorem 3. Let G be a locally convex torsion-free Abelian group, C the field of rational numbers with the natural topology. Then the mapping (4) is a topological Z-isomorphism of G into $C \otimes G$.

Proof. It suffices to prove that φ is an open mapping. Let $\Omega_{u,v}$ be an arbitrary neighborhood of zero element in $C \otimes G$. We may suppose that U is of the form $U = \{\kappa \in C;\ |\kappa| \leq k^{-1}\}$, where k is an integer, and V is a symmetric convex neighborhood of zero element in G. We shall prove that $\Omega_{u,v} \cap \varphi(G) \subseteq \varphi(V)$. Let $z = 1 \otimes x$ be an arbitrary element in $\Omega_{u,v} \cap \varphi(G)$. There exist $\kappa_i \in U$, $x_i \in V\ (1 \leq i \leq n)$ such that

$$(5)\quad m(1\otimes x) = r_1\otimes x_1 + \dots + r_n\otimes x_n \, .$$

If we put $r_i = n_i/m_i$ $(1 \leq i \leq n)$, where n_i and m_i $(1 \leq i \leq n)$ are integrs, then $|kn_i| \leq |m_i|$ $(1 \leq i \leq n)$. From (5) it follows $1\otimes nx = m_1^{-1}\otimes n_1x_1 + \dots + m_n^{-1}\otimes n_n x_n$.

Putting $p = \prod_{i=1}^{n} m_i$ the equality $1\otimes npx = 1\otimes n_1m_1^{-1}px_1 + \dots + 1\otimes n_n m_n^{-1} p x_n$ implies $npx = n_1m_1^{-1}px_1 + \dots + n_n m_n^{-1} p x_n$, hence, with respect to the relations $|kn_i| \leq |m_i|$ $(1 \leq i \leq n)$, we obtain $knp\cdot x \in V + \dots + V$ (np summands). From the convexity of V it follows $x \in V$. This concludes the proof.

<u>Proposition 6</u>. If G is a separated locally convex group, C the field of rational numbers with the natural topology, then $C\otimes G$ is a separated locally convex group.

Proof. It is evident that G is torsion-free. If $0 \neq z \in C\otimes G$, then we may suppose that $z = r\otimes x$, $0 \neq r \in C$, $0 \neq x \in G$. We define a neighborhood $U = \{\lambda \in C; |\lambda| \leq r\}$ in C and choose a symmetric neighborhood V in G not containing x. We shall prove that z is not contained in $\Omega_{U,V}$. Suppose, to the contrary, that z is in $\Omega_{U,V}$. Then for some $\lambda_i = k_i m_i^{-1}$ $(1 \leq i \leq n)$, $x_i \in V$ $(1 \leq i \leq n)$ holds $n(r\otimes x) = \lambda_1\otimes x_1 + \dots + \lambda_n \otimes x_n$; from $nk\cdot m^{-1}\otimes x = m_1^{-1}\otimes k_1x_1 + \dots + m_n^{-1}\otimes k_n x_n$, where $r = k\cdot m^{-1}$, it follows $nkm_1\cdot\ldots\cdot m_n x = mk_1m_2\cdot\ldots\cdot m_n x_1 + \dots + k_n m m_1\cdot\ldots\cdot m_{n-1}x_n$. Making use of $|k_i m| \leq |m_i k|$, $(1 \leq i \leq n)$, we conclude $nkm_1\cdot\ldots\cdot m_n x \in V + \dots + V$ ($nkm_1\cdot\ldots\cdot m_n$ summands), hence $x \in V$.

If G is an Abelian group, H a vector space over C, f a Z-linear mapping of G into H, then there exists (see [1]) a C-linear mapping g of $C\otimes G$ into H

defined by

$$(6) \qquad g(\lambda \otimes x) = \lambda \cdot f(x).$$

<u>Proposition 7</u>. Let G be a topological Abelian group, H a locally convex $\mathcal{g}$-space over C, f a Z-linear continuous mapping of G into H. Then the mapping g defined by (6) is continuous.

Proof. Let W be a convex neighborhood in H satisfying $\lambda W \subseteq W$ for any $\lambda \in C$, $|\lambda| \leq 1$. There exists a neighborhood V in G, $f(V) \subseteq W$. It is easy to prove that $g(\Omega_{u,v}) \subseteq W$, where $U = \{\lambda \in C;\ |\lambda| \leq 1\}$.

<u>Theorem 4</u>. On the tensor product $C \otimes G$ there exists a unique locally convex topology with the properties

(a) The canonical mapping $x \to \varphi(x) = 1 \otimes x$ of G into $C \otimes G$ is continuous;

(b) For any locally convex $\mathcal{g}$-space H and for any continuous Z-linear mapping f of G into H, the mapping g defined in (6) is a continuous C-linear mapping of $C \otimes G$ into H.

The proof of this statement is evident.

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