Miroslav Hušek

Generalized proximity and uniform spaces. I

Commentationes Mathematicae Universitatis Carolinae, Vol. 5 (1964), No. 4, 247--266

Persistent URL: `http://dml.cz/dmlcz/104981`

Commentationes Mathematicae Universitatis Carolinae
5, 4 (1964)

GENERALIZED PROXIMITY AND UNIFORM SPACES I.

M. HUŠEK, Praha

In this paper, generalized proximity and uniformity are introduced and studied. Filter is a fundamental concept of our generalizations. A semi-uniformity for a set P is a filter in exp $(P \times P)$ intersection of which contains the diagonal Δ_P ; a proximity for a set P is given, if for each $X \subset P$ a filter in exp P is given such that its intersection contains X . Of course, these or similar generalizations and its characterizations occur in various papers e.g. by A. Appert, I. Konishi, D. Tamari, N.C. Jarutkin, W.J. Pervin (see [4]), C.H. Dowker (see [3]) (non-symmetric proximities and uniformities), S. Leader, A. Goetz, V.S. Krishnan, V.A. Efremovič and A.S. Švarc (see [2]) (characterizations of uniform spaces by means of nets). We shall prove that the categories of proximity and semi-uniform spaces and some their subcategories are S-categories over the category of sets with respect to the forgetful functors (for S-categories see [5]). Hence it is easy to characterize subobjects factor-objects, products and sums in these categories by means of theorems in [5].

Next, special properties of functors and subcategories are introduced (e.g. to be projectivity-preserving, hereditary, productive etc.). The purpose of these definitions will be seen in the part II which is in preparation. In that part we shall investigate properties of functors from the introduced categories in other ones (e.g. to preserve sub-objects, sums etc.).

I thank Zdeněk Frolík for help and valuable suggestions.

We use the terminology of [5] and now we shall introduce some symbols and terms more, used in sequel (not necessary in part I).

The category of all sets will be denoted by $\mathcal{M}$. We mark $\exp' P = \exp P - (\emptyset)$ for every set P, $\mathcal{D} M = \{x \mid \langle x, y \rangle \in M$ for some $y\}$, $\mathcal{E} M = \mathcal{D} M^{-1}$ for every <u>relation</u> M and similarly $\mathcal{D} \xi = x$, $\mathcal{E} \xi = y$ for every pair $\xi = \langle x, y \rangle$. A <u>mapping</u> is a single-valued relation unlike a <u>morphism</u> of a category of structs, e.g. of topological spaces, which is a triple $\langle f, \langle P, u \rangle, \langle Q, v \rangle \rangle$ where f is a continuous mapping of $\langle P, u \rangle$ into $\langle Q, v \rangle$ (then f = graph $\langle f, \langle P, u \rangle, \langle Q, v \rangle \rangle$).

By $\mathcal{C}(P)$ we denote the class of all nets ranging in a set P. A <u>subnet</u> of a net M is a net M/A where A is a right-cofinal subset of $\mathcal{D} M$.

We do not define the concept of <u>forgetful functor</u> but it will be clear in every situation. E.g. the forgetful functor from the category of all topological spaces into $\mathcal{M}$ is the covariant functor F for which $F \langle P, u \rangle = P$, graph $F \varphi =$ = graph φ.

We shall write $\lambda \longleftrightarrow u \longleftrightarrow \{\mathcal{U}_x \mid x \in P\}$ if u is a <u>closure</u> for a set P, if every $\mathcal{U}_x$ is the <u>neighborhood system</u> of x in $\langle P, u \rangle$ (i.e. $\mathcal{U}_x = \{U \mid U \subset P,\ x \in P - u(P - U)\}$) and if λ is the <u>convergence class</u> of $\langle P, u \rangle$ (i.e. $\lambda = \{\langle M, x \rangle \mid M \in \mathcal{C}(P),\ x \in P,\ M$ is eventually in each $U \in \mathcal{U}_x\}$).

The following properties are characteristic for u, $\mathcal{U}_x$, λ :

$u\emptyset = \emptyset$, $X \subset uX$ for all $X \subset P$,

$u(X_1 \cup X_2) = uX_1 \cup uX_2$ for all $X_i \subset P$ $(i = 1,2)$;

every $\mathcal{U}_x$ is a filter in $\exp P$ intersection of which contains x ;

$\langle M, x \rangle \in \lambda$ whenever $\mathcal{E} M = (x) \subset P$,

if $\langle M, x \rangle \in \lambda$ and if M' is a subnet of M then $\langle M', x \rangle \in \lambda$, if $\langle M, x \rangle \in \mathcal{C}(P) \times P - \lambda$ then $\lambda \cap (\mathcal{C}(\mathcal{E} M') \times (x)) = \emptyset$ for some subnet M' of M .

A closure u is called topological if $uuX = uX$ for all $X \subset P$. The category of all closure spaces with continuous mappings will be denoted by C . C is an S-category over $\mathcal{M}$ with respect to the forgetful functor.

1. Some special properties of functors in S-categories

In this section, let $\mathcal{K}$ be an S-category over $\mathcal{C}$ with respect to a covariant functor T_i $(i = 1,2)$, F be a covariant functor from $\mathcal{K}_1$ in $\mathcal{K}_2$ such that $T_1 = T_2 \circ F$. The order R_A will be simply denoted by $<$.

Definition 1.1. We shall say that F is projective (more precisely projectivity-preserving), if for any nonvoid family $\{\varphi_i \mid i \in I\}$ of morphisms of $\mathcal{K}_1$ $F[\mathcal{K}_1 - \underleftarrow{\mathrm{Lim}} \{\varphi_i \mid i \in I\}] = \mathcal{K}_2 - \underleftarrow{\mathrm{Lim}} \{F\varphi_i \mid i \in I\}$ provided the left side exists.

We shall say that F is hereditary if for any object X of $\mathcal{K}_1$ $\langle Y', f' \rangle$ is a subobject of FX in $\mathcal{K}_2$ if and only if $FY < Y' < FY$, $T_2 f' = T_1 f$ for some subobject $\langle Y, f \rangle$ of X in $\mathcal{K}_1$.

We shall say that F is productive if for any nonvoid family $\{X_i \mid i \in I\}$ of objects of $\mathcal{K}_1$ $F[\mathcal{K}_1 - \prod \{X_i / i \in I\}] = \mathcal{K}_2 - \prod \{FX_i \mid i \in I\}$ provided the left side exists.

If $\mathcal{K}_1$ is a subcategory of $\mathcal{K}_2$ and if F is the identity functor from $\mathcal{K}_1$ in $\mathcal{K}_2$ then we shall say that $\mathcal{K}_1$ is projective (in $\mathcal{K}_2$), hereditary (in $\mathcal{K}_2$), productive (in $\mathcal{K}_2$) resp., provided F has the same property.

Dually <u>inductive</u> (more precisely <u>inductivity-preserving</u>), <u>cohereditary</u>, <u>coproductive</u> functors and subcategories are defined.

<u>Theorem 1.1</u>. Let F be projective. Then F is hereditary and productive. Dually for inductive functors.

Proof. Let $\langle Y, f \rangle$ be a subobject of X in $\mathcal{K}_1$. Then by theorem 5 in [5] $\langle T_1 Y, T_1 f \rangle$ is a subobject of T_1 X in $\mathcal{C}$ and $Y = \mathcal{K}_1 - \varprojlim f$. It follows that $\langle T_2 F Y, T_2 F f \rangle$ is a subobject of $T_2 F X$ and $F Y = \mathcal{K}_2 - \varprojlim F f$. Hence $\langle FY , Ff \rangle$ is a subobject of FX in $\mathcal{K}_2$.

Let $\langle Y', f' \rangle$ be a subobject of FX in $\mathcal{K}_2$. Then $\langle T_2 Y', T_2 f' \rangle$ is a subobject of T_2 FX in $\mathcal{C}$ and $Y' = \mathcal{K}_2 - \varprojlim f'$. There exists $\langle Y, f \rangle$ such that $Y = \mathcal{K}_1 - \varprojlim f$, $\mathcal{E} f = X$, $T_1 f = T_2 f'$. $\langle Y, f \rangle$ is a subobject of X in $\mathcal{K}_1$ and by the first part of our proof $\langle FY, Ff \rangle$ is a subobject of FX in $\mathcal{K}_2$. By remark 2 in [5] $FY < Y' < FY$. The assertion about products follows at once from a special case of theorem 4 in [5].

<u>Theorem 1.2</u>. Assume that $X = \mathcal{K}_1 - \varprojlim \{ \varphi_i \mid i \in I \}$. Then $FX = \mathcal{K}_2 - \varprojlim \{ F \varphi_i \mid i \in I \}$ if and only if there are morphisms ψ_i in $\mathcal{K}_1$ such that $T_1 \psi_i = T_1 \varphi_i$, $\mathcal{E} \psi_i = \mathcal{E} \varphi_i$ for all $i \in I$ and that $\mathcal{D} \psi_i = \mathcal{D} \psi_j \in F^{-1} [\mathcal{K}_2 - \varprojlim \{ F \varphi_i \mid i \in I \}]$ for all $\langle i, j \rangle \in I \times I$.

Dually for inductive limits.

Proof. The necessity is obvious. We shall prove the sufficiency. Evidently $FX < \mathcal{K}_2 - \varprojlim\{F\varphi_i \mid i \in I\}$. By the assumption there is an object $Y \in F^{-1}[\mathcal{K}_2 - \varprojlim\{F\varphi_i \mid i \in I\}]$ and morphisms $\psi_i \in T_1^{-1}[T_1\varphi_i] \cap \mathrm{Hom}_{\mathcal{K}_1}(Y, \mathcal{E}\varphi_i)$ for each $i \in I$. Hence $Y < X$ and consequently $\mathcal{K}_2 - \varprojlim\{F\varphi_i \mid i \in I\} < FX$.

Corollary. Let $\mathcal{K}_1$ be a full subcategory of $\mathcal{K}_2$.
(a) $\mathcal{K}_1$ is projective in $\mathcal{K}_2$ if and only if for any nonvoid family $\{\varphi_i \mid i \in I\}$ of morphisms of $\mathcal{K}_1$ $\mathcal{K}_2 - \varprojlim\{\varphi_i \mid i \in I\}$ is an object of $\mathcal{K}_1$ provided it exists.
(b) $\mathcal{K}_1$ is projective in $\mathcal{K}_2$ if each object X of $\mathcal{K}_2$ has its upper modification $\langle Y, \varphi\rangle$ in $\mathcal{K}_1$ such that $T_2\varphi = i_{T_2X}$.
Dually for inductive subcategories.

2. Proximity spaces

Theorem 2.1 Let P be a set. Consider the following conditions for $p \subset \exp P \times \exp P$, $\wp \subset \mathcal{C}(P) \times \exp' P$, $\mathcal{N} \subset \exp P \times \exp\exp P$:

(1) $\langle X, \emptyset\rangle \in p \cup p^{-1}$ for no $X \subset P$;
(2) $\langle X, Y\rangle \in p$ whenever $X \cup Y \subset P$, $X \cap Y \neq \emptyset$;
(3) if $Y_1 \cup Y_2 \subset P$ then $\langle X, Y_1 \cup Y_2\rangle \in p$ if and only if either $\langle X, Y_1\rangle \in p$ or $\langle X, Y_2\rangle \in p$;

(α) $\langle M, X\rangle \in \wp$ whenever $\mathcal{E}M = (x) \subset X \subset P$;
(β) if $\langle M, X\rangle \in \wp$ and if M' is a subnet of M then $\langle M', X\rangle \in \wp$
(γ) if $\langle M, X\rangle \in \mathcal{C}(P) \times \exp P - \wp$ then $\wp \cap (\mathcal{C}(\mathcal{E}M') \times (X)) = \emptyset$ for some subnet M' of M;

(a) $\mathcal{N}$ is a single-valued relation $\{\mathcal{N}_X \mid X \in \exp P\}$;
(b) $\bigcap \mathcal{N}_X \supset X$ for each $X \subset P$, $\mathcal{N}_\emptyset = \exp P$;
(c) $\mathcal{N}_X$ is a filter in $\exp P$ for each $X \in \exp P$.

Then:

$R_1 = \{\langle p, \wp\rangle \mid p = \{\langle X, Y\rangle \mid Y \subset P$ and $\wp \cap (\mathcal{C}(Y) \times (X)) \neq \emptyset\} = \{\langle p, \wp\rangle \mid \wp = \{\langle M, X\rangle \mid M \in \mathcal{C}(P)\ \langle X, \mathcal{E}M'\rangle \in p$ for each subnet M' of $M\}$ is a one-to-one relation for the class of all relations p on $\exp P$ satisfying (1),(2),(3) and the class of all relations $\wp \subset \mathcal{C}(P) \times \exp' P$ satisfying (α), (β), (γ);

$R_2 = \{\langle p, \mathcal{N}\rangle \mid p = \{\langle X, Y\rangle \mid Y \subset P$ and $P - Y \notin \mathcal{N}_X\} = \{\langle p, \mathcal{N}\rangle \mid \mathcal{N}_X = \{Y \mid Y \subset P$ and $\langle X, P - Y\rangle \notin p\}$ for each $X \subset P\}$
is a one-to-one relation for the class of all relations p on $\exp P$ satisfying (1),(2),(3) and the class of all relations $\mathcal{N} \subset \exp P \times \exp\exp P$ satisfying (a), (b), (c);

$R_3 = \{\langle \wp, \mathcal{N}\rangle \mid \wp = \{\langle M, X\rangle \mid M \in \mathcal{C}(P)$, M is eventually in each $Y \in \mathcal{N}_X\}\} = \{\langle \wp, \mathcal{N}\rangle \mid \mathcal{N}_X = \{Y \mid Y \subset P$ and $\wp \cap (\mathcal{C}(P - Y) \times (X)) = \emptyset\}$ for each $X \subset P\}$
is a one-to-one relation for the class of all relations $\wp \subset \mathcal{C}(P) \times \exp' P$ satisfying (α), (β), (γ) and the class of all relations $\mathcal{N} \subset \exp P \times \exp\exp P$ satisfying (a), (b), (c).

We write $\wp \longleftrightarrow p \longleftrightarrow \mathcal{N}$ provided $\wp$, p, $\mathcal{N}$ fulfil the above conditions and $\langle p, \wp\rangle \in R_1$, $\langle p, \mathcal{N}\rangle \in R_2$ (then $\langle \wp, \mathcal{N}\rangle \in R_3$, because $R_3 = R_2 \circ R_1^{-1}$).

Definition 2.1. Let $\wp \longleftrightarrow p \longleftrightarrow \{\mathcal{N}_X \mid X \in \exp P\}$. Then p is called a proximity for P and the pair $\langle P, p\rangle$ is called a proximity space. The relation $\wp$ is the convergence class of $\langle P, p\rangle$ and every $\mathcal{N}_X$ is the neighborhood system of X in $\langle P, p\rangle$.

Definition 2.2. A proximity p, a proximity space $\langle P, p\rangle$ resp., is called monotone if one of the following equivalent conditions is fulfilled ($\wp \longleftrightarrow p \longleftrightarrow \{\mathcal{N}_X \mid X \in P\}$):

1) $\langle X, Y \rangle \in p$, $P \supset Z \supset X$ implies $\langle Z, Y \rangle \in p$;

2) $\langle M, X \rangle \in \rho$, $P \supset Z \supset X$ implies $\langle M, Z \rangle \in \rho$;

3) $P \supset X \supset Y$ implies $\mathcal{N}_X \subset \mathcal{N}_Y$.

Definition 2.3. Let r be a relation in $\exp' P$. Then $p = \{\langle X, Y \rangle \mid$ either $X \cap Y \neq \emptyset$ or for each finite cover $\mathcal{O}\!\iota$ of Y there is a $Z \in \mathcal{O}\!\iota$ such that $\langle X, Z' \rangle \in r$ whenever $P \supset Z' \supset Z\} \cap (\exp' P \times \exp' P)$ is a proximity which is called generated by r .

We shall say that a relation $\sigma \subset \mathcal{C}(P) \times \exp' P$ generates a convergence class ρ if $P \subset \mathcal{C}(P) \times \exp' P$ and ρ is the smallest relation greater than σ , satisfying the conditions (α), (β), (γ) of theorem 2.1.

$(\rho = \{\langle M, X \rangle \mid M \in \mathcal{C}(P)$ and $\sigma' \cap (\mathcal{C}(\mathcal{E} M') \times (X)) \neq \emptyset$ for each subnet M' of $M\}$ where $\sigma' = \{\langle M, X \rangle \mid M \in \mathcal{C}(P)$, $X \subset P$ and either $\mathcal{E} M = (x) \subset X$ or M is a subnet of a net N for which $\langle N, X \rangle \in \sigma\})$.

Remark 2.1. (a) If r from definition 2.3 fulfils the condition (2) of theorem 2.1 then p is the greatest proximity smaller than r . The generating relations occurring in this paper always satisfy (1),(2) and the part "if" of (3),(α),(β) resp. (i.e. $\sigma = \sigma'$).

(b) In generating $\mathcal{N}$ we restrict ourselves on the well-known concepts of sub-bases or bases of filters.

(c) We shall write usually $X\ p\ Y$ instead of $\langle X, Y \rangle \in p$ and X non p Y instead of $\langle X, Y \rangle \notin p$, $X \cup Y \subset P$.

Definition 2.4. Let f be a mapping of a proximity space $\langle P, p \rangle$ into another one $\langle Q, q \rangle$ and let $\rho \longleftrightarrow p \longleftrightarrow \{\mathcal{U}_X \mid X \subset P\}$, $\sigma \longleftrightarrow q \longleftrightarrow \{\mathcal{V}_X \mid X \subset P\}$.

(A) We say that f is upper proximally continuous if one of the following equivalent conditions is fulfilled:

(a) if $X\,p\,Y$ then $f[X]\,q\,f[Y]$;

(b) if $\langle M, X\rangle \in \wp$ then $\langle f \circ M, f[X]\rangle \in \delta$;

(c) if $X \subset P$, $V \in \mathcal{V}_{f[X]}$ then $f^{-1}[V] \in \mathcal{U}_X$.

(B) We say that f is <u>lower proximally continuous</u> if one of the following equivalent conditions is fulfilled:

(a) if $X \cup Y \subset Q$, $f^{-1}[X]\,p\,f^{-1}[Y]$ then $X\,q\,Y$;

(b) if $X \subset Q$, $\langle M, f^{-1}[X]\rangle \in \wp$ then $\langle f \circ M, X\rangle \in \delta$;

(c) if $V \in \mathcal{V}_X$ then $f^{-1}[V] \in \mathcal{U}_{f^{-1}[X]}$.

(C) We say that f is <u>proximally continuous</u> if f is both upper and lower proximally continuous.

<u>Remark 2.2</u>. Evidently, the class of all proximity spaces with upper proximally continuous mappings, lower proximally continuous mappings, proximally continuous mappings resp., forms a category $\mathcal{P}^U$, $\mathcal{P}^L$, $\mathcal{P}$ resp.

<u>Theorem 2.2</u>. Let f be a mapping of a proximity space $\langle P, p\rangle$ into another one $\langle Q, q\rangle$.

(a) Let $f[P] = Q$ or let q be monotone. Then if f is upper proximally continuous it is also lower proximally continuous and hence proximally continuous.

(b) Let f be one-to-one or let p be monotone. Then if f is lower proximally continuous it is also upper proximally continuous and hence proximally continuous.

<u>Example 2.1.</u> Let $P = (a, b, c)$, $Q = (\alpha, \beta)$,

$p = \{\langle A, B\rangle \mid A \cap B \neq \emptyset$ or $A \neq (a, b)\} \cap (\exp' P \times \exp' P)$

$q = \{\langle X, Y\rangle \mid X \cap Y \neq \emptyset$ or $\alpha \notin X\} \cap (\exp' Q \times \exp' Q)$.

If we put $f\,a = f\,b = \alpha$, $f\,c = \beta$, $g\,\alpha = c$, $g\,\beta = b$ then f is a lower proximally continuous mapping of the proximity space $\langle P, p\rangle$ onto the monotone proximity space $\langle Q, q\rangle$ which is not upper proximally continuous and g is an upper proximally continuous one-to-one mapping of the monotone proximity space $\langle Q, q \rangle$ into the proximity space $\langle P, p \rangle$ which is not lower proximally continuous.

Definition 2.5. We say that a proximity p is finer than a proximity q or that a proximity q is coarser than a proximity p (sign $p < q$) if p, q are proximities for the same set P and the identity mapping $\Delta_p : \langle P, p \rangle \to \langle P, q \rangle$ is proximally continuous (i.e. $U\mathfrak{D}p = U\mathfrak{D}q$ and $p \subset q$).

Theorem 2.3. The set of all proximities for a set P is complete in the order $<$. Let $A \neq \emptyset$ and for each $\alpha \in A$ p_α be a proximity for a set P and $\wp_\alpha \longleftrightarrow p_\alpha \longleftrightarrow \{\mathcal{U}_X^\alpha \mid X \subset P\}$. Let

$$\sigma_1 \longleftrightarrow q_1 = \sup \{p_\alpha \mid \alpha \in A\} \longleftrightarrow \mathcal{V}_X^1 \mid X \subset P\},$$
$$\sigma_2 \longleftrightarrow q_2 = \inf \{p_\alpha \mid \alpha \in A\} \longleftrightarrow \mathcal{V}_X^2 \mid X \subset P\}.$$

Then

1) $q_1 = \bigcup \{p_\alpha \mid \alpha \in A\}$;
2) $\mathcal{V}_X^1 = \bigcap \{\mathcal{U}_X^\alpha \mid \alpha \in A\}$ for each $X \subset P$;
3) $\bigcup \{\wp_\alpha \mid \alpha \in A\}$ generates σ_1;
4) $\bigcap \{p_\alpha \mid \alpha \in A\}$ generates q_2;
5) $\bigcup \{\mathcal{U}_X^\alpha \mid \alpha \in A\}$ is a subbase of $\mathcal{V}_X^2$ for each $X \subset P$;
6) $\sigma_2 = \bigcap \{\wp_\alpha \mid \alpha \in A\}$.

If $\{p_\alpha \mid \alpha \in A\}$ is left-directed then $q_2 = \bigcap \{p_\alpha \mid \alpha \in A\}$ and $\mathcal{V}_X^2 = \bigcup \{\mathcal{U}_X^\alpha \mid \alpha \in A\}$ for all $X \subset P$.

Theorem 2.4. The categories $\mathcal{P}^U$, $\mathcal{P}^L$ are S-categories over $\mathcal{M}$ with respect to the forgetful functors.

Proof. The proof of the conditions (1),(2),(5) of definition 1 in [5] is easy (notice that $\{\langle X, Y \rangle \mid X \cup Y \subset P,\ X \cap Y \neq \emptyset\}$, $\exp' P \times \exp' P$ resp. is the finest, the coarsest resp., proximity for a set P). (4) was proved in theorem 2.3. It remains to prove (3). Let $\langle f, \langle P, p \rangle, \langle Q, q \rangle\rangle$ be a morphism of $\mathcal{P}^U$, $\mathcal{P}^L$ resp. Let R be a set and $\varphi : P \to R$, $\psi : R \to Q$ mappings with the composition $\psi \circ \varphi = f$. We want to define a proximity r for R such that φ, ψ are upper proximally continuous, lower proximally continuous resp. It is sufficient to put

$r = \{\langle X, Y\rangle \mid X \cup Y \subset R,\ \psi[X]\ q\ \psi[Y]\}$ in the first case and

$r = \{\langle X, Y\rangle \mid X \cup Y \subset R$ and either $X \cap Y \neq \emptyset$ or $\varphi^{-1}[X]\ p\ \varphi^{-1}[Y]\}$ in the second case.

Remark 2.3. The category $\mathcal{P}$ fulfils all the conditions of definition 1 in [5] except (3) as follows from the following proposition.

Let $\langle P, p\rangle$ be a non-monotone proximity space. Then there is a proximity q for P and a proximally continuous mapping $f : \langle P, p\rangle \rightarrow \langle P, q\rangle$ such that the mappings

$$\varphi = f : \langle P, p\rangle \longrightarrow \langle f[P], r\rangle,$$

$$\psi = \Delta_{f[P]} : \langle f[P], r\rangle \longrightarrow \langle P, q\rangle$$

are proximally continuous for no proximity r for $f[P]$.

We shall give a short proof of this proposition. There are subsets K, M, N of P such that $M\ p\ N$, K non $p\ N$, $K \supset M$. Let $f = \Delta_{P-(K-M)} \cup (K - M) \times (m)$ where $m \in M$. Then f is a proximally continuous mapping of $\langle P, p\rangle$ into $\langle P, q\rangle$ where $q = \{\langle X, Y\rangle \mid X \cup Y \subset P$ and either $X \cap Y \neq \emptyset$ or $f^{-1}[X]\ p\ f^{-1}[Y]$ or $f[A] = X$, $A\ p\ f^{-1}[Y]$ for some $A \subset P\}$. Now, let r be a proximity for $f[P]$. The proximal continuity of φ implies $M\ r\ N$ and the proximal continuity of ψ implies M non $r\ N$.

Remark 2.4. Let us denote by $\mathcal{P}_M$ the full subcategory of $\mathcal{P}$ generated by all monotone proximity spaces. It follows from theorem 2.2 that $\mathcal{P}_M$ is a full subcategory both of $\mathcal{P}^U$ and $\mathcal{P}^L$.

Lemma 2.1. For each proximity p_o there exists a coarsest monotone proximity p_1 finer than p_o and a finest monotone proximity p_2 coarser than p_o . If $\wp_i \longleftrightarrow p_i \longleftrightarrow \{ \mathcal{U}_X^i \mid X \subset P \}$, $(i = 0,1,2)$ then

1) $\{ \langle X, Y \rangle \mid Z\, p_o\, Y$ whenever $Z \supset X \}$ generates p_1 ;

2) $\bigcup \{ \mathcal{U}_Y^o \mid P \supset Y \supset X \}$ is a subbase of $\mathcal{U}_X^1$ for each $X \subset P$;

3) $\wp_1 = \{ \langle M, X \rangle \mid X \in \exp' P$ and $\langle M, Y \rangle \in \wp_o$ whenever $X \subset Y \subset P \}$;

4) $p_2 = \{ \langle X, Y \rangle \mid P \supset X$ and $Z\, p_o\, Y$ for some $Z \subset X \}$;

5) $\mathcal{U}_X^2 = \bigcap \{ \mathcal{U}_Y^o \mid Y \subset X \}$ for each $X \subset P$;

6) $\{ \langle M, X \rangle \mid P \supset X$ and $\langle M, Z \rangle \in \wp_o$ for some $Z \subset X \}$ generates $\wp_2$.

Theorem 2.5. Each object $\langle P, p_o \rangle$ of $\mathcal{P}^L$ has its lower modification $\langle \langle P, p_1 \rangle , \langle \Delta_P , \langle P, p_1 \rangle , \langle P, p_o \rangle \rangle \rangle$ in $\mathcal{P}_M$ and each object $\langle P, p_o \rangle$ of $\mathcal{P}^U$ has its upper modification $\langle \langle P, p_2 \rangle , \langle \Delta_P , \langle P, p_o \rangle , \langle P, p_2 \rangle \rangle \rangle$ in $\mathcal{P}_M$. Hence each object of $\mathcal{P}$ has its upper and lower modifications in $\mathcal{P}_M$.

Corollary 1. $\mathcal{P}_M$ is an S-category over $\mathcal{M}$ with respect to the forgetful functor.

Proof. See theorem 1 in [5].

Corollary 2. $\mathcal{P}_M$ is projective in $\mathcal{P}^U$ and inductive in $\mathcal{P}^L$.

Proof. See corollary (b) of theorem 1.2.

Remark 2.5. It follows from theorems 2.2, 2.4 and the foregoing corollary that

$$\mathcal{P}^U - \varinjlim \{ \varphi_i \mid i \in I \} = \mathcal{P}_M - \varinjlim \{ \varphi_i \mid i \in I \}$$

if $\{ \varphi_i \mid i \in I \}$ is a nonvoid family of epimorphisms of $\mathcal{P}_M$ and that

$$\mathcal{P}^L - \varprojlim \{ \varphi_i \mid i \in I \} = \mathcal{P}_M - \varprojlim \{ \varphi_i \mid i \in I \}$$

if $\{\varphi_i \mid i \in I\}$ is a nonvoid family of monomorphisms of $\mathcal{P}_M$. Hence $\mathcal{P}_M^{\cdot}$ is cohereditary in $\mathcal{P}^U$ and hereditary in $\mathcal{P}^L$

Lemma 2.2. Let $\mathcal{K}$ be an S-category over a category $\mathcal{L}$ with respect to a covariant functor T, $\mathcal{K}'$ be a subcategory of $\mathcal{K}$ and an S-category over $\mathcal{L}$ with respect to $T/\mathcal{K}'$ and let $\mathcal{L}$ have inversion property. Suppose that an object X of $\mathcal{K}$ has an upper modification in $\mathcal{K}'$. If there is an object Y' of $\mathcal{K}'$ such that $\langle X, Y' \rangle \in R_{TX}$ (for R_A see definition 1 in [5]) then there is a smallest object Y of $\mathcal{K}'$ greater than X in the order R_{TX} (then $T\varphi = i_{TX}$ for some $\varphi \in \mathrm{Hom}_{\mathcal{K}}(X, Y)$) and $\langle Y, \varphi \rangle$ is an upper modification of X in $\mathcal{K}'$.

Proof. Let $\langle Z, \psi \rangle$ be an upper modification of X in $\mathcal{K}'$. Evidently $\varphi = \chi \circ \psi$ for some $\chi \in \mathrm{Hom}_{\mathcal{K}'}(Z, Y')$. Hence ψ is a monomorphism. It is easy to see that ψ is also an epimorphism and hence a bimorphism. Indeed, otherwise $\psi' \circ \psi = \psi'' \circ \psi$ for some different morphisms ψ', ψ'' of $\mathcal{K}'$ with $\mathcal{E}\psi' = \mathcal{E}\psi''$ and this contradicts our assumption that $\langle Z, \psi \rangle$ is an upper modification. As $T\psi$ is invertible, there are isomorphisms χ', ψ' in $\mathcal{K}'$ such that $T\psi' = T\psi$, $\psi' \circ \chi' = i_Z$. It follows from the equalities

$$T(\chi \circ \psi') = T\chi \circ T\psi' = T\chi \circ T\psi = T(\chi \circ \psi) = T\varphi = i_{TX},$$

$$T(\chi' \circ \psi) = T\chi' \circ T\psi = T\chi' \circ T\psi' = i_{TX}$$

that $\mathcal{D}\psi'$ is an object of $\mathcal{K}'$ greater than X and smaller than Y'. Now, it is sufficient to put $Y = \mathcal{D}\psi'$.

Theorem 2.6. (a) The class of all objects of $\mathcal{P}^L$ having the upper modifications in $\mathcal{P}_M$ is precisely the class of objects of $\mathcal{P}_M$. (b) Let us put for a moment

$\tau = \{\langle p, q\rangle \mid q$ is the coarsest monotone proximity finer than $p\}$ and $\mu_Q = \{\langle p, q\rangle \mid \langle\langle Q, q\rangle, \Delta_Q\rangle$ is a subobject of $\langle \bigcup \mathfrak{D}\, p, p\rangle$ in $\mathcal{P}^U\}$. A proximity space $\langle P, p\rangle$, as an object of $\mathcal{P}^U$, has its lower modification in $\mathcal{P}_M$ if and only if $\tau\mu_Q\, p = \mu_Q\tau\, p$ for all $Q \subset P$ (i.e. if $Q \subset P$, q is the coarsest monotone proximity finer than $p \cap (\exp Q \times \exp Q)$ then $K\, p\, N$ provided $M\, q\, N$, $P \supset K \supset M$).

Proof. (a) Let $\langle P, p\rangle$ be a non-monotone proximity space. There are subsets K, M, N of P such that $M\, p\, N$, K non $p\, N$, $K \supset M$. Put $Q = (a,b,c)$, $q = \exp' Q \times \exp' Q - (\langle (a),(b)\rangle)$, $f = (K \times (a)) \cup (N \times (b)) \cup ((P - (K \cup N)) \times (c))$. Then $\langle Q, q\rangle$ is a monotone proximity space, f is a lower proximally continuous mapping $\langle P, p\rangle \longrightarrow \langle Q, q\rangle$ but f is not lower proximally continuous of $\langle P, p'\rangle$ into $\langle Q, q\rangle$ where p' is the finest monotone proximity coarser than p. Hence $\langle P, p\rangle$ has no upper modification in $\mathcal{P}_M$ (see the foregoing lemma).

(b) Our assertion follows from the characterization of subobjects expressed in theorem 5 of [5] and from the fact that a mapping f of a monotone proximity space $\langle Q', q'\rangle$ into $\langle P, p\rangle$ is upper proximally continuous if and only if the monotone proximity of $\mathcal{P}^U - \underrightarrow{\mathrm{Lim}}\, \langle f, \langle Q', q'\rangle, \langle f[Q'], \mu_{f[Q']}\, p\rangle\rangle$ is finer than $\mu_{f[Q']}\, p$ (see theorem 2 of [5] and remark 2.4.).

<u>Example 2.2</u>. Let P be a set, $P \supset X_0 \neq \emptyset$, card $(P - X_0) \geqq 2$, $p = \{\langle X, Y\rangle \mid X \cup Y \subset P$, either $X \cap Y \neq \emptyset$ or $X = X_0$, $Y \neq \emptyset\}$. $\langle P, p\rangle$ is a non-monotone proximity space fulfilling the condition of theorem 2.6(b).

<u>Remark 2.6</u>. It was said in the introduction that we can

construct products, subobjects etc. in $\mathcal{P}^U$, $\mathcal{P}^L$, $\mathcal{P}_M$ resp., from those in $\mathcal{M}$. It follows from theorems in [5] that for this construction it is sufficient to know characterizations of sup, inf in R_A and of objects in $\mathcal{P}^U$, $\mathcal{P}^L$, $\mathcal{P}_M$ resp., projectively (inductively) generated by one morphism. Characterizations of sup, inf are described in theorem 2.3 ; characterizations of generated objects are left to the reader.

3. Semi-uniform spaces

Theorem 3.1. Let P be a set. Consider the following conditions for $\mathcal{U} \subset \exp (P \times P)$, $\mathcal{C} \subset \mathcal{C}(P \times P)$:

(a) $\mathcal{U}$ is a filter in $\exp(P \times P)$;

(b) $\bigcap \mathcal{U} \supset \Delta_P$;

(α) $M \in \mathcal{C}$ whenever $\mathcal{E} M = (\langle x, x \rangle \cdot) \subset P \times P$;

(β) if $M \in \mathcal{C}$ and if M' is a subnet of M then $M' \in \mathcal{C}$;

(γ) if $M \in \mathcal{C}(P \times P) - \mathcal{C}$ then $\mathcal{C} \cap \mathcal{C}(\mathcal{E} M') = \emptyset$ for some subnet M' of M .

Then $R = \{\langle \mathcal{U}, \mathcal{C} \rangle \mid \mathcal{C} = \{M \mid M \in \mathcal{C}(P \times P)$, M is eventually in each $U \in \mathcal{U}\}\} = \{\langle \mathcal{U}, \mathcal{C} \rangle \mid \mathcal{U} = \{U \mid U \subset P \times P$, each $M \in \mathcal{C}$ is eventually in $U\}\}$ is a one-to-one relation, for the class of all $\mathcal{U} \subset \exp(P \times P)$ satisfying (a),(b) and the class of all $\mathcal{C} \subset \mathcal{C}(P \times P)$ satisfying (α),(β),(γ). We write $\mathcal{U} \longleftrightarrow \mathcal{C}$ provided $\mathcal{U}$, $\mathcal{C}$ fulfil the above conditions and $\langle \mathcal{U}, \mathcal{C} \rangle \in R$.

Definition 3.1. Let $\mathcal{U} \longleftrightarrow \mathcal{C}$, $P = \mathfrak{D} \bigcup \mathcal{U}$. Then we call $\mathcal{U}$ a semi-uniformity for P , $\langle P, \mathcal{U} \rangle$ a semi-uniform space and $\mathcal{C}$ the convergence class of $\langle P, \mathcal{U} \rangle$.

Definition 3.2. Let $\mathcal{U} \longleftrightarrow \mathcal{C}$. The semi-uniformity $\mathcal{U}$, the semi-uniform space $\langle P, \mathcal{U} \rangle$ resp. is called <u>symmetric</u> if one of the following equivalent conditions is fulfilled:

1) if $U \in \mathcal{U}$ then $U^{-1} \in \mathcal{U}$;

2) if $M \in \mathcal{C}$ then $\{\langle a, \langle \mathcal{E} M_a , \mathcal{D} M_a \rangle\rangle \mid a \in \mathcal{D} M\} \in \mathcal{C}$;

The semi-uniformity $\mathcal{U}$, the semi-uniform space $\langle P, \mathcal{U} \rangle$ resp., is called <u>uniformity</u>, <u>uniform space</u> resp., if one of the following equivalent conditions is fulfilled:

1) each $U \in \mathcal{U}$ contains $V \circ V$ for some $V \in \mathcal{U}$;

2) if $M \in \mathcal{C}$, $N \in \mathcal{C}$, $\mathcal{D} M = \mathcal{D} N$, $\mathcal{E} M_a = \mathcal{D} N_a$ for all $a \in \mathcal{D} M$, then $\{\langle a, \langle \mathcal{D} M_a , \mathcal{E} N_a \rangle\rangle \mid a \in \mathcal{D} M\} \in \mathcal{C}$.

<u>Remark 3.1</u>. We shall use this notation:
if $M \in \mathcal{C}(P \times P)$ then $\alpha M = \{\langle a, \mathcal{D} M_a \rangle \mid a \in \mathcal{D} M\}$, $\beta M = \{\langle a, \mathcal{E} M_a \rangle \mid a \in \mathcal{D} M\}$. Hence we can assign in one-to-one way to each $\mathcal{C} \subset \mathcal{C}(P \times P)$ the relation $\rho_{\mathcal{C}} = \{\langle \alpha M , \beta M \rangle \mid M \in \mathcal{C}\}$ on $\mathcal{C}(P)$. If $\mathcal{U} \leftrightarrow \mathcal{C}$ then $\rho_{\mathcal{C}}$ is reflexive. By definition 3.2 $\mathcal{U}$ is symmetric if and only if $\rho_{\mathcal{C}}$ is symmetric, $\mathcal{U}$ is a uniformity if and only if $\rho_{\mathcal{C}}$ is transitive. So $\mathcal{U}$ is a symmetric uniformity if and only if $\rho_{\mathcal{C}}$ is an equivalence.

<u>Remark 3.2</u>. Similarly as in definition 2.3 we shall say that $\mathfrak{D} \subset \mathcal{C}(P \times P)$ <u>generates</u> a convergence class $\mathcal{C}$ if $\mathcal{C} \subset \mathcal{C}(P \times P)$ is the smallest class containing $\mathfrak{D}$ and satisfying the conditions (α),(β),(γ) of theorem 3.1 . ($\mathcal{C} = \{M \mid M \in \mathcal{C}(P \times P)$, $\mathfrak{D}' \cap \mathcal{C}(\mathcal{E} M') \neq \emptyset$ for each subnet M' of $M\}$, where $\mathfrak{D}' = \mathfrak{D} \cup \{M \mid M \in \mathcal{C}(P \times P)$, $\mathcal{E} M = (\langle x, x \rangle)$ for some $x\}$. As a rule $\mathfrak{D}' = \mathfrak{D}$.)

<u>Definition 3.3</u> . Let f be a mapping of a semi-uniform space $\langle P, \mathcal{U} \rangle$ into another one $\langle Q, \mathcal{V} \rangle$ and let $\mathcal{U} \leftrightarrow \mathcal{C}$, $\mathcal{V} \leftrightarrow \mathfrak{D}$. We say that f is <u>uniformly continuous</u> if $(f \times f)^{-1}[V] \in \mathcal{U}$ for each $V \in \mathcal{V}$ or equivalently if $(f \times f) \circ M \in \mathfrak{D}$ for each $M \in \mathcal{C}$.

<u>Remark 3.3</u>. Evidently, the semi-uniform spaces with uni-

formly continuous mappings form a category $\mathcal{U}$. We denote by $\mathcal{U}_S$, $\mathcal{U}_U$, $\mathcal{U}_{SU}$ resp., the full subcategory of $\mathcal{U}$ generated by symmetric semi-uniform spaces, uniform spaces, symmetric uniform spaces resp.

Definition 3.4. We say that a semi-uniformity $\mathcal{U}$ is finer than another one $\mathcal{V}$ or that $\mathcal{V}$ is coarser than $\mathcal{U}$ (sign $\mathcal{U} < \mathcal{V}$) if $\mathcal{U}$, $\mathcal{V}$ are semi-uniformities for the same set P and if the identity mapping $\Delta_P : \langle P, \mathcal{U} \rangle \to \langle P, \mathcal{V} \rangle$ is uniformly continuous (i.e. $\mathfrak{D} \cup \mathcal{U} = \mathfrak{D} \cup \mathcal{V}$, $\mathcal{U} \supset \mathcal{V}$).

Theorem 3.2. The set of all semi-uniformities for a set P is complete in the order $<$. Let for each $\alpha \in A$ $(A \neq \emptyset)$ $\mathcal{U}_\alpha$ be a semi-uniformity for P and $\mathcal{U}_\alpha \longleftrightarrow \mathcal{L}_\alpha$. Suppose that $\mathcal{V}_1 = \sup \{ \mathcal{U}_\alpha \mid \alpha \in A \} \longleftrightarrow \mathfrak{D}_1$, $\mathcal{V}_2 = \inf \{ \mathcal{U}_\alpha \mid \alpha \in A \} \longleftrightarrow$ $\longleftrightarrow \mathfrak{D}_2$.

Then

1) $\mathcal{V}_1 = \bigcap \{ \mathcal{U}_\alpha \mid \alpha \in A \}$;

2) $\bigcup \{ \mathcal{L}_\alpha \mid \alpha \in A \}$ generates $\mathfrak{D}_1$;

3) $\bigcup \{ \mathcal{U}_\alpha \mid \alpha \in A \}$ is a subbase of $\mathcal{V}_2$;

4) $\mathfrak{D}_2 = \bigcap \{ \mathcal{L}_\alpha \mid \alpha \in A \}$.

If $\{ \mathcal{U}_\alpha \mid \alpha \in A \}$ is left-directed then $\mathcal{V}_2 = \bigcup \{ \mathcal{U}_\alpha \mid \alpha \in A \}$.

Theorem 3.3. $\mathcal{U}$ is an S-category over $\mathcal{M}$ with respect to the forgetful functor.

Proof. We shall prove only the condition (3) of definition 1 in [5]. ((4) was proved in theorem 3.2 and the remaining conditions are trivially fulfilled; notice that $(P \times P)$, $\{ U \mid P \times P \supset U \supset \Delta_P \}$ resp., is the coarsest, the finest resp., semi-uniformity for P .) Let f be a uniformly continuous mapping of $\langle P, \mathcal{U} \rangle$ into $\langle Q, \mathcal{V} \rangle$, $f = \psi \circ \varphi$, $\mathfrak{D} \psi = R$. If we put $\mathcal{W} = \{ W \mid R \times R \supset W \supset (\psi \times \psi)^{-1} [V]$ for some $V \in \mathcal{V} \}$, then the mappings $\varphi : \langle P, \mathcal{U} \rangle \to \langle R, \mathcal{W} \rangle$,

$\psi : \langle R, \mathcal{W} \rangle \to \langle Q, \mathcal{V} \rangle$ are uniformly continuous.

Lemma 3.1. For every semi-uniformity $\mathcal{U}_0$ there exists a coarsest symmetric semi-uniformity $\mathcal{U}_1$ finer than $\mathcal{U}_0$, a finest symmetric semi-uniformity $\mathcal{U}_2$ coarser than $\mathcal{U}_0$, a finest uniformity $\mathcal{U}_3$ coarser than $\mathcal{U}_0$ and a finest symmetric uniformity $\mathcal{U}_4$ coarser than $\mathcal{U}_0$. If $\mathcal{U}_i \longleftrightarrow \mathcal{C}_i$ for $i \in (0,1,2,3,4)$ then

1) $\{U \mid U \in \mathcal{U}_0$ or $U^{-1} \in \mathcal{U}_0\}$ is a subbase for $\mathcal{U}_1$; $\{M \mid M \in \mathcal{C}_0$ and $\{\langle a, \langle \mathcal{E} M_a, \mathcal{D} M_a \rangle\rangle \mid a \in \mathcal{D} M\} \in \mathcal{C}_0\} = \mathcal{C}_1$;

2) $\{U \mid U \in \mathcal{U}_0$ and $U^{-1} \in \mathcal{U}_0\} = \mathcal{U}_2$; $\{M \mid M \in \mathcal{C}_0$ or $\{\langle a, (M_a)^{-1}\rangle \mid a \in \mathcal{D} M\} \in \mathcal{C}_0\}$ generates $\mathcal{C}_2$;

3) $\{U \mid$ there is a sequence $\{U_n\}$ in $\mathcal{U}_0$ such that $U_0 \subset U$ and $U_{n+1} \circ U_{n+1} \subset U_n$ for each $n\} = \mathcal{U}_3$;

4) $\{U \mid$ there is a sequence $\{U_n\}$ in $\mathcal{U}_0$ such that $U_0 \subset U$ and $U_{n+1} \circ U_{n+1} \subset U_n$, $U_n = U_n^{-1}$ for each $n\} = \mathcal{U}_4$.

Theorem 3.4. Each object $\langle P, \mathcal{U}_0 \rangle$ of $\mathcal{U}$ has its lower modification $\langle\langle P, \mathcal{U}_1 \rangle, \langle \Delta_P, \langle P, \mathcal{U}_1 \rangle, \langle P, \mathcal{U}_0 \rangle\rangle\rangle$ in $\mathcal{U}_S$, its upper modification $\langle\langle P, \mathcal{U}_2 \rangle, \langle \Delta_P, \langle P, \mathcal{U}_0 \rangle, \langle P, \mathcal{U}_2 \rangle\rangle\rangle$ in $\mathcal{U}_S$, its upper modification $\langle\langle P, \mathcal{U}_3 \rangle, \langle \Delta_P, \langle P, \mathcal{U}_0 \rangle, \langle P, \mathcal{U}_3 \rangle\rangle\rangle$ in $\mathcal{U}_U$, its upper modification $\langle\langle P, \mathcal{U}_4 \rangle, \langle \Delta_P, \langle P, \mathcal{U}_0 \rangle, \langle P, \mathcal{U}_4 \rangle\rangle\rangle$ in $\mathcal{U}_{SU}$ and each object $\langle P, \mathcal{U}_0 \rangle$ of $\mathcal{U}_U$ has its lower modification $\langle\langle P, \mathcal{U}_1 \rangle \langle \Delta_P, \langle P, \mathcal{U}_1 \rangle, \langle P, \mathcal{U}_0 \rangle\rangle\rangle$ in $\mathcal{U}_{SU}$.

Proof follows from the foregoing lemma and for the upper modifications from the fact that if $f : \langle P, \mathcal{U} \rangle \to \langle Q, \mathcal{W} \rangle$ is a uniformly continuous mapping then the semi-uniformity $\{V \mid P \times P \supset V \supset (f \times f)^{-1}[W]$ for some $W \in \mathcal{W}\}$ is coarser than $\mathcal{U}$ and it is symmetric semi-uniformity, uniformity, symmetric uniformity resp., provided $\mathcal{W}$ has the same property.

The last assertion is the consequence of the fact that $\mathcal{U}_1$ is a uniformity provided $\mathcal{U}_o$ is a uniformity.

Corollary. $\mathcal{U}_S$, $\mathcal{U}_U$, $\mathcal{U}_{SU}$ are S-categories over $\mathcal{M}$ with respect to the forgetful functors.

Proof. See theorem 1 in [5].

Corollary. $\mathcal{U}_S$ is projective and inductive in $\mathcal{U}$. $\mathcal{U}_U$ is projective in $\mathcal{U}$. $\mathcal{U}_{SU}$ is projective in $\mathcal{U}$, $\mathcal{U}_S$, $\mathcal{U}_U$ and inductive in $\mathcal{U}_U$.

Proof. See corollary (b) of theorem 1.2.

Remark 3.4. The upper modification of an object ξ of $\mathcal{U}$ in $\mathcal{U}_{SU}$ is the upper modification in $\mathcal{U}_U$ of the upper modification of ξ in $\mathcal{U}_S$.

Example 3.1. a) Let $\mathcal{U} = \{ U \mid P \times P \supset U \supset (X \times X) \cup \cup (Y \times Y) \}$ where $X \cup Y = P$, $X \cap Y \neq \emptyset$, $X \neq P$, $Y \neq P$. The symmetric semi-uniformity $\mathcal{U}$ has neither the coarsest symmetric uniformity finer than $\mathcal{U}$ nor the coarsest uniformity finer than $\mathcal{U}$. Indeed, $\sup \{ \mathcal{V} \mid \mathcal{V} < \mathcal{U}$, $\mathcal{V}$ is a symmetric uniformity $\} = \mathcal{U}$ because

$\mathcal{U} = \{ U \mid P \times P \supset U \supset (X \times X) \cup \Delta_P \} \cap \{ U \mid P \times P \supset U \supset (Y \times Y) \cup \cup \Delta_P \}$.

(b) The finest symmetric semi-uniformity coarser than the uniformity $\{ U \mid P \times P \supset U \supset (P \times (a)) \cup \Delta_P \}$, where a is an element of at least three-point set P , is not a uniformity. Hence, the upper modification of an object ξ of $\mathcal{U}$ in $\mathcal{U}_{SU}$ need not be the upper modification in $\mathcal{U}_S$ of the upper modification of ξ in $\mathcal{U}_U$.

Theorem 3.5. Let $\langle f, \langle P, \mathcal{U}_o \rangle , \langle Q, \mathcal{V}_o \rangle \rangle = f'$ be a morphism of $\mathcal{U}$. Put $\langle P, \mathcal{U}_1 \rangle = \mathcal{U} - \underleftarrow{\mathrm{Lim}}\, f'$, $\langle Q, \mathcal{V}_1 \rangle = \mathcal{U} - \underrightarrow{\mathrm{Lim}}\, f'$, $\mathcal{U}_i \longleftrightarrow \mathcal{C}_i$, $\mathcal{V}_i \longleftrightarrow \mathcal{D}_i$ $(i = 0,1)$. Then

1) $\mathcal{U}_1 = \{ U \mid P \times P \supset U \supset (f \times f)^{-1} [V]$ for some $V \in \mathcal{V}_o \}$;

2) $\mathscr{C}_1 = \{M \mid M \in \mathscr{C}(P \times P),\ (f \times f) \circ M \in \mathscr{D}_0\}$;

3) $\mathscr{V}_1 = \{V \mid V \supset \Delta_Q,\ (f \times f)^{-1}[V] \in \mathscr{U}_0\} = \{V \mid Q \times Q \supset$ $\supset V \supset \Delta_Q \cup (f \times f)[U]$ for some $U \in \mathscr{U}_0\}$;

4) $\{M \mid M \in \mathscr{C}(Q \times Q),\ (f \times f) \circ N = M$ for some $M \in \mathscr{C}_0\}$ generates $\mathscr{D}_1$.

Corollary. $\mathscr{U}_U$ is coproductive in $\mathscr{U}$ and $\mathscr{U}_{SU}$ is coproductive in $\mathscr{U}$, $\mathscr{U}_S$.

Proof. $\mathscr{V}_1$ in theorem 3.5 is a uniformity provided $\mathscr{U}_0$ is a uniformity and f is one-to-one. Our corollary now follows from the following obvious statement:
if $\{\mathscr{V}_\alpha \mid \alpha \in A\}$ is a nonvoid set of uniformities for a set P with the property
$\langle \alpha, \alpha' \rangle \in A \times A - \Delta_A$, $V_\alpha \in \mathscr{V}_\alpha$, $V_{\alpha'} \in \mathscr{V}_{\alpha'}$, implies $\{x \mid \text{card } V_\alpha^{-1}[x] > 1\} \cap \{x \mid \text{card } V_{\alpha'}\ [x] > 1\} = \emptyset$ then $\sup \{\mathscr{V}_\alpha \mid \alpha \in A\}$ is a uniformity.

Example 3.2. Let $P = (a,b,c,d)$, $Q = (\alpha, \beta, \gamma)$, $U_0 = (a, b) \times (a, b) \cup (c, d) \times (c, d)$, $V_0 = (\alpha, \beta) \times (\alpha, \beta) \cup (\beta, \gamma) \times (\beta, \gamma)$.
$\mathscr{U} = \{U \mid P \times P \supset U \supset U_0\}$ is a symmetric uniformity for P , $\mathscr{V} = \{V \mid Q \times Q \supset V \supset V_0\}$ is a symmetric semi-uniformity for Q which is not a uniformity, $\langle Q, \mathscr{V} \rangle = \mathscr{U} - \underrightarrow{\mathrm{Lim}}\, f$ where $f = \langle (\langle a, \alpha \rangle, \langle b, \beta \rangle, \langle c, \beta \rangle, \langle d, \gamma \rangle), \langle P, \mathscr{U} \rangle, \langle Q, \mathscr{V} \rangle \rangle$.
It follows that $\mathscr{U}_{SU}$ is not cohereditary in $\mathscr{U}$, $\mathscr{U}_S$ and hence $\mathscr{U}_U$ is not cohereditary in $\mathscr{U}$.

References:

[1] E. ČECH, Topological space (2nd edition, Prague, in preparation).

[2] V.A. EFREMOVIČ, A.S. ŠVARC, A new definition of uniform

spaces. Metrization of proximity spaces. Doklady AN 89(1953),393-396.

[3] C.H. DOWKER, Mappings of proximity structures. Gen.Top. and its Rel.to Mod.Anal.and Alg.(1962, Prague) 139-141.

[4] W.J. PERVIN, Quasi-uniformization of topological spaces. Math.Ann.147(1962),316-317.

[5] M. HUŠEK, S-categories, CMUC 5,1(1964),37-46.