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ALGEBRAIC DEPENDENCE STRUCTURES

(Preliminary communication)

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The present results - representing a generalization of some ideas of the papers [1] and [5] - were, together with several applications to (non-commutative) groups, lattices and modules, a subject of the author's lecture read in the Conference on General Algebra in Warsaw, September 7-11, 1964.

Let S be a given set, $\mathcal{P}S$ its powerset, $\mathcal{F} \subseteq \mathcal{P}S$ the subfamily of all its finite subsets. x and X denote always an element and a subset of S, respectively.

By a <u>relation</u> ρ <u>on</u> S we understand a subset ρ of the cartesian product $S \times \mathcal{P}S$. For a relation ρ on S, define the subfamily $\mathcal{I}_\rho \subseteq \mathcal{P}S$ of ρ-<u>independent subsets</u> by

$(\rho \to \mathcal{I}_\rho)$ $\qquad I \in \mathcal{I}_\rho \longleftrightarrow \forall x (x \in I \to [x, I \setminus (x)] \notin \rho)$.

Further, define two mappings $\mathcal{D}_\rho$ and $\mathcal{D}_\rho^R$ of S into $\mathcal{P}S$ by

$(\mathcal{D}_\rho)$ $\qquad X \in \mathcal{D}_\rho(x) \longleftrightarrow [x, X] \in \rho$

and

$(\mathcal{D}_\rho^R)$ $\qquad X \in \mathcal{D}_\rho^R(x) \longleftrightarrow \exists I (I \subseteq X \wedge I \in \mathcal{I}_\rho \wedge x \notin I \wedge$
$\wedge [x, I] \in \rho)$.

Two relations ρ_1 and ρ_2 on S are said to be <u>associated</u> or <u>similar</u> if

$$\mathcal{I}_{\rho_1} = \mathcal{I}_{\rho_2}$$

or

$$x \notin X \to ([x, X] \in \rho_1 \longleftrightarrow [x, X] \in \rho_2),$$

respectively.

A relation ρ on S satisfying the following two conditions

(F/M) $[x, X] \in \rho \leftrightarrow \exists F\ (F \subseteq X \wedge F \in \mathcal{F} \wedge [x, F] \in \rho\ )$,

(E_r) $I \in \mathcal{J}_\rho \wedge [x_1, I] \notin \rho \wedge [x_1, I \cup (x_2)] \in \rho \rightarrow$
$\rightarrow [x_2, I \cup (x_1)] \in \rho$,

is said to be an <u>A-dependence relation on</u> S . It is said to be <u>proper</u>, or <u>regular</u> if, moreover,

(I) $$x \in X \rightarrow [x, X] \in \rho$$

or

(R) $$x \notin X \wedge [x, X] \in \rho \rightarrow \exists I\ (I \subseteq X \wedge I \in \mathcal{J}_\rho \wedge [x, I] \in \rho\ )$$

is satisfied, respectively.

If ρ is an A-dep. relation on S , $I \in \mathcal{J}_\rho$ and $x \notin I$, then

$$[x, X] \in \rho \leftrightarrow I \cup (x) \notin \mathcal{J}_\rho \quad .$$

For a mapping C of $\mathcal{P}$ S into $\mathcal{P}$ S , define the subfamily $\mathcal{J}_C \subseteq \mathcal{P}$ S of C-independent subsets by

($C \rightarrow \mathcal{J}_C$) $$I \in \mathcal{J}_C \leftrightarrow \forall X\ (X \subseteq I \wedge I \subseteq C(X) \rightarrow X = I) .$$

If the conditions

(φ/μ) $$C(X) = \bigcup_{\substack{F \subseteq X \\ F \in \mathcal{F}}} C(F) ,$$

($\mathcal{E}_k$) $$I \in \mathcal{J}_C \wedge x_1 \in C(I \cup (x_2)) \setminus C(I) \rightarrow x_2 \in C(I \cup (x_1)) ,$$

(ι) $$X \subseteq C(X) ,$$

are fulfilled, then C is called an <u>A-dependence closure operation in</u> S . For such a closure operation

$$C(I) = \bigcup_{I \cup (x) \notin \mathcal{J}_C} I \cup (x)$$

holds for every $I \in \mathcal{J}_C$.

A subfamily $\mathcal{J}$ of $\mathcal{P}$ S satisfying the condition

(f/m) $$I \in \mathcal{J} \leftrightarrow \forall F\ (F \subseteq I \wedge F \in \mathcal{F} \rightarrow F \in \mathcal{J}\ )$$

is said to be an <u>A-independence net of</u> S .

The following theorem describes the relation between any two of the following concepts of an A-dependence structure (S, ρ), (S, C) and $(S, \mathcal{J})$, where ρ, C and $\mathcal{J}$ are A-dep. relation on S, A-dep. closure operation in S and A-indep. net of S, respectively.

Theorem. To any A-dep. relation ρ on S there corresponds a well-defined A-indep. net $\mathcal{J}_\rho$ of S. On the other hand, to any A-indep. net of S there corresponds a set of (associated) A-dep. relations on S which form, under the natural operations of join and meet, a lattice $\mathbb{L}$ with infinite joins and $\mathbf{O}$. The lattice $\mathbb{L}$ splits into convex sublattices of similar relations, the greatest element of each of these sublattices being the corresponding proper relation. The correspondence in which every element of such sublattice is mapped into the corresponding greatest element is a lattice-homomorphism of $\mathbb{L}$ onto the sublattice $\mathbb{L}_p$ of all proper relations with the ideal of all regular relations. Denoting by $\mathbf{1}$, $\mathbf{O}_p$ and $\mathbf{O}$ the greatest element of $\mathbb{L}$, the least element of $\mathbb{L}_p$ and $\mathbb{L}$, respect., we have

$$\mathcal{D}_{\mathbf{1}}(x) = \mathcal{D}^R(x) \cup (\mathcal{P} S \setminus \mathcal{J}) \cup \mathcal{S}(x),$$

$$\mathcal{D}_{\mathbf{O}_p}(x) = \mathcal{D}^R(x) \cup \mathcal{S}(x),$$

$$\mathcal{D}_{\mathbf{O}}(x) = \mathcal{D}^R(x),$$

where $\mathcal{S}(x)$ is the subfamily of all subsets X such that $x \in X$.

As a consequence, for any A-indep. net of S, there is a uniquely determined proper regular A-dep. relation on S.

To any A-dep. closure operation C in S there corresponds a well-defined A-indep. net $\mathcal{J}_C$ of S. On the other hand, to any A-indep. net of S there corresponds a lattice of A-dep.

closure operations in S which is isomorphic to the corresponding lattice $\mathbb{L}$ of all proper A-dep. relations. The least element of this lattice is the corresponding Schmidt's "mehrstufige Austauschstrukture" (see [5]).

In what follows we consider a (fixed) A-indep. net $\mathcal{J}$ of S (with the closure operation $C : C(I) = \bigcup_{I \cup (x) \notin \mathcal{J}} I \cup (x)$).

For the purpose of establishing an invariant (rank or dimension) of certain A-dep. structures, let us introduce the following concept of a <u>canonic subset of</u> S . The family $\mathcal{C} \subseteq \mathcal{J}$ of all canonic subsets is defined by

$$(\mathcal{C}) \quad I \in \mathcal{C} \leftrightarrow I \in \mathcal{J} \wedge \forall X [X \in \mathcal{J} \wedge X \subseteq C(I) \wedge I \subseteq C(X) \rightarrow \rightarrow C(I) \subseteq C(X)].$$

Also, define the family $\mathcal{J}^*$ of all <u>maximal</u> subsets of S by

$$(\mathcal{J}^*) \qquad I \in \mathcal{J}^* \leftrightarrow I \in \mathcal{J} \wedge C(I) = S ,$$

and the family $\mathcal{B}$ of all <u>bases</u> of S by

$$(\mathcal{B}) \qquad \mathcal{B} = \mathcal{C} \cap \mathcal{J}^* .$$

A <u>GA-indep. net of</u> S is an A-indep. net $\mathcal{J}$ of S such that $\mathcal{B} \neq \emptyset$ and

$$I_1 \subseteq I_2 \wedge I_2 \in \mathcal{C} \rightarrow I_1 \in \mathcal{C} .$$

If, moreover, $\mathcal{B} = \mathcal{J}^*$, $\mathcal{J}$ is called a <u>LA-indep. net of</u> S .

Through the following generalization of the Steinitz's Exchange Theorem

$$X \in \mathcal{J} \wedge I \in \mathcal{C} \wedge X \subseteq C(I) \wedge I \subseteq C(X) \rightarrow$$
$$\rightarrow \forall x [x \in X \setminus I \rightarrow \exists I_0 (\emptyset \neq I_0 \subseteq I \setminus X \wedge X \setminus (x) \cup I_0 \in \mathcal{J} \wedge$$
$$\wedge I \subseteq C(X \setminus (x) \cup I_0) ,$$

one can prove the fundamental

<u>Theorem.</u> $X \in \mathcal{J} \wedge I \in \mathcal{C} \wedge X \subseteq C(I) \rightarrow \mathrm{card}\,(X) \leq \mathrm{card}\,(I)$.

Then, the implication

$$X \in \mathcal{J}^* \wedge I_1 \in \mathcal{B} \wedge I_2 \in \mathcal{B} \rightarrow \mathrm{card}(X) \leq \mathrm{card}\,(I_1) = \mathrm{card}\,(I_2)$$

is a simple corollary enabling us to define the rank of any GA-dependence structure (i.e. any structure with a GA-indep.net).

The following theorem shows the relation with the results of [2], [3], [4] and [6]:

Theorem. For a given A-indep. net $\mathcal{J}$, the following conditions are equivalent:

($\mathcal{FC}$) $\mathcal{J} \cap \mathcal{F} \subseteq \mathcal{C}$;

($\mathcal{C}$) $\mathcal{J} = \mathcal{C}$;

($\mathcal{FN}$) $I \in \mathcal{J} \cap \mathcal{F} \wedge I \cup (x) \notin \mathcal{J} \wedge I \cup (y) \notin \mathcal{J} \wedge x \neq y \rightarrow$
$\rightarrow \forall z (z \in I \rightarrow I \setminus (z) \cup (x) \cup (y) \notin \mathcal{J})$;

($\mathcal{N}$) $I \in \mathcal{J} \wedge I \cup (x) \notin \mathcal{J} \wedge I \cup (y) \notin \mathcal{J} \wedge x \neq y \rightarrow$
$\rightarrow \forall z (z \in I \rightarrow I \setminus (z) \cup (x) \cup (y) \notin \mathcal{J})$;

($\mathcal{FW}$) $I_1 \in \mathcal{J} \cap \mathcal{F} \wedge I_2 \in \mathcal{J} \cap \mathcal{F} \wedge \text{card}(I_1) < \text{card}(I_2) \rightarrow$
$\rightarrow \exists x (x \in I_2 \wedge x \notin I_1 \wedge I_1 \cup (x) \in \mathcal{J})$;

($\mathcal{W}$) $I_1 \in \mathcal{J} \wedge I_2 \in \mathcal{J} \wedge \text{card}(I_1) < \text{card}(I_2) \rightarrow$
$\rightarrow \exists x (x \in I_2 \wedge x \notin I_1 \wedge I_1 \cup (x) \in \mathcal{J})$;

($\mathcal{FB}$) $I_1 \in \mathcal{J} \cap \mathcal{F} \wedge I_2 \in \mathcal{J} \cap \mathcal{F} \wedge I_1 \subseteq C(I_2) \wedge I_2 \subseteq C(I_1) \rightarrow$
$\rightarrow \forall x [x \in I_1 \setminus I_2 \rightarrow \exists y (y \in I_2 \setminus I_1 \wedge I_1 \setminus (x) \cup (y) \in \mathcal{J})]$;

($\mathcal{B}$) $I_1 \in \mathcal{J} \wedge I_2 \in \mathcal{J} \wedge I_1 \subseteq C(I_2) \wedge I_2 \subseteq C(I_1) \rightarrow$
$\rightarrow \forall x [x \in I_1 \setminus I_2 \rightarrow \exists y (y \in I_2 \setminus I_1 \wedge I_1 \setminus (x) \cup (y) \in$
$\in \mathcal{J})]$.

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