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BETTI NUMBERS OF REGIONS OF ATTRACTION

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Summary: it is shown that in a dynamical system, under certain conditions, the Betti numbers of a limit set coincide with those of regions of uniform attraction.

Let P be a topological space; a (global) <u>dynamical system</u> on P (cf. [3, chap. V], this journal, p. 121) is a map τ with the following properties:

1° $\tau : P \times E^1 \rightarrow P$ is continuous (the value of τ at (x, θ) will be denoted by $x \tau \theta$),

2° $x \tau 0 = x$,

3° $(x \tau \theta_1) \tau \theta_2 = x \tau (\theta_1 + \theta_2)$.

For fixed $\theta \in E^1$, define continuous maps $t_\theta : P \rightarrow P$ by

$$t_\theta x = x \tau \theta .$$

From 2°, t_0 is the identity map; from 3° ,

$$t_\theta t_\sigma = t_{\theta + \sigma} ;$$

in fact, it is obvious that $\{t_\theta\}_{\theta \in E^1}$ form a continuous abelian group of homeomorphisms $P \approx P$.

Further terminology: A <u>trajectory</u> (through $x \in P$) is a subset of P of the form

$$\{x \tau \theta : \theta \in E^1\} ;$$

a <u>critical point</u> is a common fixed point of all t_θ , i.e. a singleton trajectory; a <u>cycle</u> is a non-singleton trajectory through a fixed point of some t_θ with $\theta \neq 0$ (θ is then termed a period of the cycle); a subset $X \subset P$ is (+)-invariant if

$$t_\theta X \subset X \quad \text{for all } \theta \geq 0 .$$

Proposition 1. Each t_θ is homotopic to the identity map, via the homotopy $h : P \times \langle 0, 1 \rangle \to P$,

$$h(x, \lambda) = x \top \lambda \theta .$$

This result was exploited in [this journal, pp. 123-4] to obtain conditions for existence of critical points. It may be noticed that 3^0 is not used at all, so that stronger results may be expected.

A second proposition we shall reproduce here was obtained in [2, theorem 6]; the j_q-characteristic is defined there; π_q will denote the q-th Betti number, χ the Euler characteristic.

Proposition 2. Let $f : X \to X$ be a continuous map of a triangulable space X, let the iterates of f converge,

$$f^n \to f^\infty \text{ uniformly as } n \to \infty .$$

Then, if $Y = f^\infty(X)$ is triangulable,

$$j_q(f^m) = \frac{\pi_q(Y)}{1 - \lambda}$$

for all q and $1 \leq m \leq \infty$.

Theorem. Given a dynamical system on a topological space P. Let $\theta > 0$, let $X \subset P$ be (+)-invariant and such $t_\theta, t_{2\theta}, t_{3\theta}, \ldots$ converge uniformly on X. Then, setting $X_\infty = \bigcap_{n \geq 1} t_{n\theta}(X)$,

$$\pi_q(X) = \pi_q(X_\infty) \quad \text{for all } q$$

if both X, X_∞ are triangulable.

Proof. Note first that $t_{n\theta} = t_\theta^n$, the n-th iterate of t_θ ; and that X_∞ is the image of X under $\lim t_{n\theta}$. From proposition 2,

$$\frac{\pi_q(Y)}{1 - \lambda} = j_q(t_\theta)$$

for all q .

Homotopic maps have coinciding homologues, and hence coinciding j_q-characteristics; thus from proposition 1,

$$j_q(t_\theta) = j_q\,(id_X) = \frac{\pi_q(X)}{1-\lambda}$$

for all q . The asserted formula now follows from the two exhibited relations.

Corollary 1. With the assumptions of the theorem, $\chi(X) = \chi(X_\infty)$. (This follows from $\chi = \sum_q (-1)^q \pi_q$.)

Corollary 2. Let x_0 be a critical point of a dynamical system on P . Then, for any triangulable (+)-invariant set $X \subset P$ such that

$$x\,\tau\,\theta \to x_0 \quad \text{for} \quad \theta \to +\infty \quad , \text{uniformly for} \quad x \in X,$$

there hold

$$\pi_0(X) = 1, \qquad \pi_q(X) = 0 \quad \text{for} \quad q \neq 1 .$$

In particular, if x_0 is uniformly asymptotically stable, then this holds for each triangulable (+)-invariant set sufficiently near to x_0 .

Corollary 3. Let C be a cycle with period $\theta > 0$ of a dynamical system on P . Then, for any triangulable (+)-invariant set $X \subset P$ such that

$$\lim_{n\to\infty} x\,\tau\,n\theta \in C, \quad \text{uniformly for} \quad x \in X,$$

there hold

$$\pi_0(X) = \pi_1(X) = 1 \; ,$$
$$\pi_0(X) = 0 \quad \text{for} \quad 0 \neq q \neq 1 .$$

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[3] NIEMYCKIJ V.V., STEPANOV V.V., Qualitative theory of differential equations (in Russian), 2nd. ed., Gostechizdat, Moscow-Leningrad.