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REMARK TO THE SOLUTION OF NON-LINEAR FUNCTIONAL EQUATIONS IN BANACH SPACES

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In the papers of A. Hammerstein [1] and M. Golomb [2] the non-linear functional eqation

$$x = B\,F(x) \tag{1}$$

is investigated. M. Golomb assumed that B is a linear completely continuous self-adjoint positive operator, $F(x)$ is a continuous strongly potential operator. Later M.M. Vajnberg [3], applying the variational principle on the equation (1), proved the existence of the solution of (1) under weaker assumptions. Further the problem to solve the Hammerstein integral equations has been developed by many mathematicians such as N.N. Nazarov [4], C.L. Dolph [5], M.A. Krasnoselskij [6], Wang-Sheng-Wang [7], G.J. Minty [8] and some other under various conditions.

In the last time the great attention is paid to the questions of the approximate solutions, estimates of the speed of their convergence and to the question of uniqueness of the solutions of non-linear functional equations. L.V. Kantorowitch [9] generalized the Newton method to these equations; further this method has been developed by I.P. Mysovskich [10], M.L. Stein [11], M. Altman [12], J. Schröder [13] and others. The variational theory of solving non-linear equations has been discussed in the papers of A. Langenbach [14], S.G. Michlin [15], D.P. Zeragiya [16] and some other under various conditions. The Galerkin's method is investigated in [6]. The method of steepest descent for finite dimensional spaces was discussed by H.B. Curry [17], A.D. Rooth [18], J.B. Crocket - H. Chernoff [19] and some others. It was Yu.G. Lumiste [20] who first applied this method to the solution of non-linear equation $x + F(x) = 0$, where $F(x)$ is a potential operator in Hilbert space H such that the following inequalities $\| F(x+h) - F(x) \| \leq M \| h\|^2$, $(F(x+h) - F(x), h) \geq m \| h\|^2$; $m > 0$, $M < \sqrt{1+2m+2m^2}$ hold for every $x \in H$ and $h \in H$. The last condition is very restricting one. The approximate solution x_n $(n = 1,2,\dots)$ is given by the equality $x_n = x_{n-1} + \varepsilon_n (x_{n-1} + F(x_{n-1}))$, where the parameters ε_n are determined either as the solutions of certain non-linear algebraic equations, or they are chosen so that inequalities $-1/1+m < \varepsilon_n < -1/1+M$ are fulfilled. Kwan-Chao-Chih [21] solved the non-linear equation $F(x) = 0$ by the method of steepest descent, where the parameters are determined from certain quadratic inequalities, under the assumption that $F(x)$ has on the set $E \subset H$ the Frechet's derivative $F'(x)$, which is a positive definite operator $((F'(x) h, h) \geq m \| h\|^2$; $m > 0$) on E. The question of uniqueness is not solved. L. A. Kivistik [22] assumes that the mapping $F(x)$ has the second Frechet's differential in some neighbourhood $\Omega(x_0,r)$ of $x_0 \in H$; $F'(x_0)$ is positive definite and $F'(x)$, $F''(x)$ are uniformly bounded on $\Omega(x_0,r)$. Further he requested the convergence of certain sequences. In the second paper he generalised the method of M.A. Krasnoselskij - S.G. Krein to non-linear equations under similarly restricting assumptions. Similar method proposed B.M. Fridman [23]. The convergence of his method is proved under the following conditions: $\|x - x_0\|$ is small in some ball $\overline{\Omega(x_0,r)}$, the mapping $F(x)$ has the bounded inverse $[F'(x)]^{-1}$ for every $x \in \Omega(x_0,r)$ and the Frechet's derivative $F''(x)$ is uniformly bounded on $\Omega(x_0,r)$. Still other claim is laid on certain parameters, which should be very troublesome in practical computing. The paper of M.M. Vajnberg [24] advanced greatly the theory of the method of steepest descent by presenting much weaker conditions of convergence than any earlier paper. H. Schaefer [25] and S.V. Simeonov [26] and others gave some modification of the method of successive approximation. S.V. Simeonov solved (1), where $B = I$ in semi-ordered Banach spaces under the assumption that Frechet's derivative $F'(x)$ has the property that $mI \leqq F'(x) \leqq MI$ for every $x \in \langle x_1,x_2 \rangle$, where the elements $x_1, x_2 \in B$ are such that $F(x_1) - x_1$, $F(x_2) - x_2$ have different signs. In the paper [27] the non-linear functional equation

(2) $$F(x) = f$$

was solved under the assumption that the continuous Gateaux's derivative $F'(x)$ exists and is a symmetric and positive definite operator on a closed set $E \subset H$. The iterative process

(3) $$x_{n+1} = x_n - PF(x_n) + Pf\ , \quad x_0 \in E$$

is proposed, P being a suitable linear operator. Its convergence is of order $\|x_n - x^*\| \leqq k\alpha^n$, $\alpha < 1$, where x^* is a unique solution of (2) in some neighbourhood of the element $x_1 \in E$. The operator P can be specified to obtain several iterative methods. For example if we put $P = I$, we obtain Wiarda's process and if $P = [F'(x_0)]^{-1}$ we get the Newton-Kantorowitch process.

We shall now prove a general theorem, concerning the solving of the equation (2), with weaker conditions of convergence than the theorem 2 [27]. From it will follow the conditions of convergence of Wiarda's and Newton-Kantorowitch processes. The proof is based on the following theorem which is a modification of the well known theorem of L. Collatz [29].

Theorem 1. Let $F(x)$ be an arbitrary mapping of Banach space B into B and let P be a linear bounded operator in B such that P^{-1} exists. Let the following conditions be fulfilled:

1) There exists a convex closed set $E \subset B$ and a real number α $(0 < \alpha < 1)$ such that for every $u, v \in E$

$$\| PF(u) - PF(v) - (u-v) \| \leq \alpha \| u-v \| .$$

2) The closed ball $\Omega(x_1,r) = \{x \in B: \| x-x_1 \| \leq r\}$, where x_1 is defined by (3), $r = \frac{\alpha}{1-\alpha} \| x_1-x_0 \|$, lies in E. Then the equation (2) has a unique solution x^* in the ball $\Omega(x_1,r)$. The sequence $\{x_n\}$ defined by (3) converges in the norm of B to the solution x^* of (2) and the error of the approximation x_n satisfies the inequality

$$(4) \qquad \| x_n-x^* \| \leq \frac{\alpha^n}{1-\alpha} \| x_1-x_0 \| .$$

We say that $\{x_n\}$ converges in B to the solution x^* of (2) with the speed of the geometric sequence, if $\| x_n-x^* \| \leq$ $\leq kq^n$ where k is a positive constant. We shall understand by the estimate of the speed of its convergence the last inequality.

Theorem 2. Let P_1 be a linear bounded operator in Hilbert space H such that P_1^{-1} exists. Let $F(x)$ be a mapping of H into H such that there exists the Gateaux's derivative $F'(x)$ for every $x \in E \subset H$, where E is a convex closed set of H, and

(5) $\quad \mathrm{Re}\,(P_1 F'(x)h,h) \geqq m\|h\|^2 \; ; \quad m > 0$

holds for every $h \in H$ and $x \in E$. Let ϑ be the number satisfying the inequality $0 < \vartheta < \frac{2m}{K}$, where

$$K = \sup_{x \in E} \|P_1 F'(x)\|^2 < +\infty .$$

Let us put

(6) $\quad x_{n+1} = x_n - PF(x_n) + Pf ,$

$$\alpha(\vartheta) = \sup_{x \in E} \| I - PF'(x) \| , \quad P = \vartheta P_1, \; r = \frac{\alpha}{1-\alpha} \|x_1 - x_0\| ,$$

where x_0 is an arbitrary element from E. Let $\Omega(x_1,r)$ be a closed ball which is contained in E. Then the equation (2) has a unique solution x^* in the ball $\Omega(x_1,r)$. The sequence $\{x_n\}$ defined by (6) converges in the norm of H to the solution x^* of (2) and the inequality (4), where $\alpha = \alpha(\vartheta)$, holds. The estimate of the speed of the convergence is greatest, when $\vartheta = \vartheta_{opt} = \frac{m}{K}$. Then

$$\alpha(\vartheta_{opt}) \leqq (1 - \vartheta_{opt}\, m)^{\frac{1}{2}} .$$

<u>Proof</u>. First of all it is evident that $K > 0$. For every fixed point $x \in E$ and $\vartheta \in (0, \frac{2m}{K})$ we have

$$\|I - PF'(x)\|^2 = \sup_{\|h\|=1} (h - PF'(x)h,\, h - PF'(x)h) =$$

$$\leqq \sup_{\|h\|=1} (1 - 2\,\mathrm{Re}(PF'(x)h,\, h) + \|PF'(x)\|^2) \leqq 1 - 2m\vartheta + \vartheta^2 K .$$

Let us put $f(\vartheta) = 1 - 2m\vartheta + \vartheta^2 K$; then $\| I - PF'(x)\| \leqq \sqrt{f(\vartheta)}$ and hence $\alpha(\vartheta) \leqq \sqrt{f(\vartheta)}$. Further $f(0) = 1$, $f(\frac{2m}{K}) = 1$ and the function $f(\vartheta)$ catches for $\vartheta = \vartheta_{opt} = \frac{m}{K}$ the minimum value $f(\vartheta_{opt}) = 1 - \frac{m^2}{K} = 1 - \vartheta_{opt}\, m$. The function $f(\vartheta)$ is descending on the interval $(0, \vartheta_{opt}\rangle$ and increasing on $\langle \vartheta_{opt}, \frac{2m}{K})$. Hence $\alpha(\vartheta) < 1$ for every $\vartheta \in (0, \frac{2m}{K})$. Now let e be any linear functional in Hilbert space H such that $\|e\| = 1$. We define a real valued function $R(t)$ on the interval $\langle 0, 1\rangle$ by

$R(t) = e(PF(tu + (1-t)v))$ for $u, v \in E$. Then we have $R(1) = e(PF(u))$, $R(0) = e(PF(v))$. From the mean value theorem we obtain $R(1) - R(0) = e[PF(u) - PF(v)] = e(PF'(\tilde{u})(u-v))$, where $\tilde{u}$ is an element which lies on the line-segment connecting the points $u, v \in E$. Because E is a convex set, $\tilde{u} \in E$. Then $\|e[(PF(u) - PF(v) - I(u-v)]\| = \|e[(PF'(\tilde{u}) - I)(u-v)\| =$
$= \|e\|\, \|PF'(\tilde{u}) - I\|\, \|u-v\| \leq \alpha(\vartheta)\|u-v\|$.
From Hahn-Banach theorem we have $\|PF(u) - PF(v) - I(u-v)\|$
$\leq \alpha(\vartheta)\|u-v\|$; $\alpha(\vartheta) < 1$. Thus, all the conditions of the theorem 1 are fulfilled. This completes the proof.

<u>Corollary</u>. Let P_1 be a linear bounded operator in Hilbert space H such that P_1^{-1} exists. Let $F(x)$, where $F(x_0) = 0$ for some element x_0 of H, be mapping from H into H such that there exists the Gateaux's derivative $F'(x)$ on the closed ball $\Omega(x_0,r) \subset H$ and the inequality (5) holds for every $x \in \Omega(x_0,r)$, $h \in H$. Let ϑ be a number satisfying the inequality $0 < \vartheta < 2\,m/K$, where $K = \sup_{x\in\Omega(x_0,r)} \|P_1 F'(x)\|^2 <$
$< +\infty$.

If

$$\|f\| \leq \frac{(1-\alpha(\vartheta))r}{\vartheta\|P_1\|}, \text{ where } \alpha(\vartheta) = \sup_{x\in\Omega(x_0,r)} \|I - PF'(x)\|,$$

$P = \vartheta P_1$, then the equation (2) has a unique solution of x^*

in the ball $\Omega(x_0,r)$. The sequence $\{x_n\}$ defined by (6) converges in the norm of H to the solution x^* of (2) and the inequality (4) holds, where $\alpha = \alpha(\vartheta)$.

Proof. From the assumptions it follows that the mapping $A(x) = x - PF(x) + Pf$ is Lipschitzian with the constant $\alpha(\vartheta) < 1$. Further for every $x \in \Omega(x_0,r)$

$$\|A(x) - x_0\| = \| x-x_0 - PF(x) + PF(x_0) + Pf\| \leq \|PF(x) - PF(x_0) - (x-x_0)\| + \|Pf\| \leq \alpha(\vartheta)\|x-x_0\| + \vartheta\|P_1\|\,\|f\| \leq r .$$

Thus $A(\Omega(x_0,r)) \subset \Omega(x_0,r)$. From Banach's theorem it follows that the equation $x = A(x)$ has a unique solution x^* in the ball $\Omega(x_0,r)$ and therefore the equation (2) has a unique solution x^* in $\Omega(x_0,r)$. This completes the proof.

The condition $\|f\| \leq \dfrac{(1-\alpha(\vartheta))r}{\vartheta\|P_1\|}$ does not restrict the class of the equations (2), because we can choose as P_1 such a linear operator, which norm $\|P_1\|$ is sufficiently small.

Remark 1. Let the equation $F(x) = x - \lambda\Phi(x) = f$ be given, where $\Phi(x)$ is a mapping of H into H. The operator $\Phi(x)$ has the Gateaux's derivative $\Phi'(x)$ on a closed convex set $E \subset H$ such that for every $x \in E$ and $h \in H$ $\mathrm{Re}(\lambda\Phi'(x)h,h) \geqq 0$. Then $m = 1$, $\vartheta_{opt} = 1/K$, $K = \sup_{x\in E}\|I - \lambda\Phi'(x)\|^2$, $\alpha(\vartheta_{opt}) \leqq k = (1 - 1/K)^{\frac{1}{2}}$. We define $\{x_n\}$ by (6) where $P_1 = I$. If the closed ball $\Omega(x_1,r)$ is a subset of E, where $r = \dfrac{k}{1-k}\|x_1-x_0\|$, then all the statements of the theorem 2 hold.

Remark 2. Let the inequality $\mathrm{Re}(\lambda\Phi'(x)h,h) \leq \ell\|h\|^2$ (ℓ real, $\ell < 1$) hold for every $x \in E$ and $h \in H$. Then $m = 1 - \ell$, $\vartheta_{opt} = (1-\ell)/K$,

$\alpha(\vartheta_{opt}) \leq p = \sqrt{1 - (1-\ell)^2/K}$. If the closed ball $\Omega(x_1,r) \subset E$, where $r = \frac{p}{1-p} \| x_1 - x_0 \|$, then hold all the statesments of our theorem. The case in the remark 1 is a special case of the remark 2 when $\ell = 0$.

Remark 3. If we now set in (6) $P_1 = I$, we obtain the sufficient conditions of the convergence of Wiarda's method. If $P_1 = [F'(x_0)]^{-1}$ we get Newton-Kantorowitch method. When in (2) $f \equiv 0$, then the formula (6) has the simple form: $x_{n+1} = x_n - PF(x_n)$; $x_0 \in E$. In the case that H is a real Hilbert space, the conditions of the theorem 2 are the same, only (5) has the form: $(P_1 F'(x)h,h) \geqq m \|h\|^2$; $m > 0$.

We say that λ is a regular value of a linear operator A if $(I - \lambda A)^{-1}$ exists.

Let us set in (6) $P = I + J$, where J is a linear bounded operator in Banach space B . We get the following result.

Theorem 3. Let $\Phi(x)$ be mapping of B into B such that the Gateaux's derivative $\Phi'(x)$ exists on a convex closed set $E \subset B$. Let λ be a regular value of $\Phi'(x)$ for every $x \in E$. Let $G_\lambda(x)$ be the resolvent operator for the operator $\Phi'(x)$ and let J be a linear bounded operator in B such that

(7) $\| \lambda G_\lambda(x) - J \| \leq \rho < p^{-1}$ for every $x \in E$, where $p = \sup_{x \in E} \| I - \lambda \Phi'(x) \| < +\infty$.

Let us set

(8) $$x_{n+1} = (I+J)f - Jx_n + \lambda (I+J)\Phi(x_n) ,$$

$$\alpha_J = \sup_{x \in E} \| \lambda (I+J)\Phi'(x) - J \| , \quad r = \frac{\alpha_J}{1-\alpha_J} \| x_1 - x_0 \| .$$

Let $\Omega(x_1,r)$ be a closed ball which is contained in E. Then the equation

$$(9) \qquad x - \lambda\Phi(x) = f, \quad f \in B$$

has a unique solution x^* in the ball $\Omega(x_1,r)$. The sequence $\{x_n\}$ defined by (8) converges in the norm of B to the solution x^* of (9) and the inequality (4) holds, where

$$\alpha = \alpha_J \leq \rho . \sup_{x \in E} \| I - \lambda\Phi'(x) \| .$$

<u>Proof</u>. To show that $P^{-1} = (I+J)^{-1}$ exists, we use the following lemma: Let T_0, T_1 be linear mappings of B into B such that T_0^{-1} exists and $\| T_1 \| < \frac{1}{\| T_0^{-1} \|}$. Then $T = T_0 + T_1$ has an inverse T^{-1}.

That P^{-1} exists is now clear. It is sufficient to set

$$T_0(x) = I + \lambda G_\lambda(x) = (I - \lambda\Phi'(x))^{-1},$$

$$T_1(x) = P - (I + \lambda G_\lambda(x)) = J - \lambda G_\lambda(x) .$$

For every fixed $x \in E$ we have that

$$PF'(x) = (I+J)(I - \lambda\Phi'(x)) = [(I + \lambda G_\lambda(x)) - (\lambda G_\lambda(x) - J)]$$
$$(I - \lambda\Phi'(x)) = I - (\lambda G_\lambda(x) - J)(I - \lambda\Phi'(x)) .$$

Hence

$$I - PF'(x) = (\lambda G_\lambda(x) - J)(I - \lambda\Phi'(x)) = (I+J)\lambda\Phi'(x) - J$$

and

$$\| I - PF'(x) \| = \| (I+J)\lambda\Phi'(x) - J \| \leq \| \lambda G_\lambda(x) - J \| \cdot$$
$$\| I - \lambda\Phi'(x) \| \leq (\sup_{x \in E} \| \lambda G_\lambda(x) - J \|) \sup_{x \in E} \| I - \lambda\Phi'(x) \| .$$

Therefore $\alpha_J < 1$. Further, by the same way as in theorem 2 we can prove that the mapping $\psi(x) = x - (I+J)(x - \lambda\Phi(x))$ is Lipschitzian on E with the constant $\alpha_J < 1$. From the theorem 1 follow the statements of our theorem. The proof is complete.

The theorem 3 generalizes P.A. Samuelson's result ([30],[31]) for solving of a linear functional eqation . We may assume the condition (7) in the stronger form:

$\| \lambda G_\lambda(x) - J \| \leq \rho < \ell^{-1}$ for every $x \in E$, where $\ell = 1 + \sup_{x \in E} \| \lambda \Phi'(x) \| < +\infty$. From the assumptions of the theorem 3 it follows that the operator $R(\lambda, \Phi'(x)) = (I - \lambda \Phi'(x))^{-1}$ exists and is bounded for every $x \in E \subset B$:

$\| R(\lambda, \Phi'(x)) \| = \| I + \lambda G_\lambda(x) \| \leq 1 + \| \lambda G_\lambda(x) \| \leq 1 + \rho + \| J \|$.

<u>Corollary</u>. Let $\Phi(x)$ be a mapping from B into B , $\Phi(0) = 0$, such that there exists the Gateaux's derivative $\Phi'(x)$ on a closed ball $\Omega(0,r) \subset B$. Let λ be a regular point of $\Phi'(x)$ for every $x \in \Omega(0,r)$. Let $G_\lambda(x)$ be the resolvent operator for $\Phi'(x)$ and let J be a linear bounded operator in B such that the inequality (7) holds for every $x \in \Omega(0,r)$. If $\| f \| \leq \frac{(1 - \alpha_J) r}{1 + \| J \|}$, $\alpha_J \leq \rho \cdot \sup_{x \in \Omega(0,r)} \| I - \lambda \Phi'(x) \|$, then the equation (9) has a unique solution x^* in the ball $\Omega(0,r)$. The sequence $\{x_n\}$ defined by (8) converges in the norm of B to the solution x^* of (9) and the inequality (4) holds, where $\alpha = \alpha_J$.

Let the equation

$$(10) \qquad x - AV(x) = f$$

be given, where A is a linear symmetric completely continuous mapping of H (real or complex) into H , V(x) is in general non-linear operator, $f \in H$. Under these general assumptions we introduce (see [32]) some conditions that $R(\lambda, AV'(x))$, where $V'(x)$ is the *Gateaux's derivative of V(x), exists as the* linear mapping of H into H for every $x \in E \subset H$. This is equivalent to the condition that the equation $x - AV'(u)x = 0$ has only the trivial solution $x \equiv 0$.

We define the characteristic values by: $x - \lambda Ax = 0$, $x \neq 0$. The sequence of characteristic values may be arranged as follows:

$$\leqq \lambda_{-2} \leqq \lambda_{-1} < 0 < \lambda_1 \leqq \lambda_2 \leqq \dots ,$$

where $\lambda_n (\lambda_{-n})$, $n \geqq 1$ are positive (negative). One of the two sequences may be empty. Let $M(E)$ be the set of mappings G_x, $x \in E \subset H$, which have the following properties:

1) All $G_x \in M(E)$ are mappings of H into H ,

2) All $G_x \in M(E)$ are linear bounded and symmetric. From (2) it follows that $(G_x u, u)$ is real for every $u \in H$ and $G_x \in M(E)$. If α is a real number we write $G_x < \alpha I$, $G_x \leqq \alpha I$, $G_x > \alpha I$, $G_x \geqq \alpha I$ when the corresponding product $(G_x u, u)$ is $<$, $\leqq$, $>$, $\geqq$ $\alpha (u,u)$ respectively for every $u \in H$ and $u \not\equiv 0$. If $G_x \in M(E)$ and $G_x \geqq 0$ then there exists a unique linear symmetric and bounded operator $G_x^{1/2}$ such that $G_x^{1/2} \geqq 0$. If $G_x > 0$, then $G_x^{1/2} > 0$. The following two lemmas due to H. Ehrmann [32].

Lemma 1. Let A be a linear completely continuous symmetric mapping of Hilbert space H into H , let λ_i $(i=1,2,\dots)$ be its characteristic values and let $G_x \in M(E)$. Then the equation

$$u - A G_x u = 0$$

has only the solution $u \equiv 0$, i.e. $\mu = 1$ is not an eigenvalue of $A G_x$, if one of the following conditions holds:

a) λ_n and λ_{n+1} (λ_{-n} and $\lambda_{-(n+1)}$), $n \geqq 1$ exist and $\lambda_n I < G_x < \lambda_{n+1} I$ ($\lambda_{-n} I > G_x > \lambda_{-(n+1)} I$) .

b) λ_n (λ_{-n}) exists as the largest positive (smallest negative) characteristic value and $G_x > \lambda_n I$ ($G_x < \lambda_{-n} I$) .

c) There is no positive (negative) characteristic value and $G_x \geq 0$ ($G_x \leq 0$).

d) λ_1 (λ_{-1}) exists and $0 \leq G_x < \lambda_1 I$ ($\lambda_{-1} I < G_x \leq 0$).

e) $\| G_x \| < \min_i (\lambda_i)$.

f) $\alpha I \leq G_x \leq \beta I$, $\langle \alpha, \beta \rangle$ does not contain the characteristic values of A.

We now assume (see [32]) the conditions a) through e) in the stronger form:

$\bar{a}$) λ_n and λ_{n+1} (λ_{-n} and $\lambda_{-(n+1)}$), $n \geq 1$ exist and $\lambda_n I < \alpha_n I \leq G_x \leq \alpha_{n+1} I < \lambda_{n+1} I$ ($\lambda_{-n} I > \alpha_{-n} I \geq G_x \geq \geq \alpha_{-(n+1)} I > \lambda_{-(n+1)} I$).

$\bar{b}$) λ_n (λ_{-n}) exists as the largest positive (smallest negative) characteristic value and $G_x \geq \alpha_n I > \lambda_n I$ ($G_x \leq \alpha_{-n} I < < \lambda_{-n} I$).

$\bar{c}$) There is no positive(negative) characteristic value and $G_x \geq 0$ ($G_x \leq 0$).

$\bar{d}$) λ_1 (λ_{-1}) exists and $0 \leq G_x \leq \alpha_1 I < \lambda_1 I$ ($\lambda_{-1} I < \alpha_{-1} I \leq G_x \leq 0$).

$\bar{e}$) $\| G_x \| \leq \alpha < \min_i |\lambda_i|$.

<u>Lemma 2</u>. Let A be a linear completely continuous symmetric mapping of H into H, let λ_i be its characteristic values and let $G_x \in M(E)$. Finally, let one of the above conditions $\bar{a}$) through $\bar{e}$) and f) be satisfied. Then the inequality $|\mu_i - 1| \geq m > 0$ holds for the eigenvalues μ_i of $A G_x$ where m is a constant which does not depend on G_x but only on the interval $\langle \alpha_i I, \alpha_j I \rangle$ in which G_x is assumed to lie according to the conditions $\bar{a}$) ... $\bar{e}$) and f.

Remark 4. From lemma 2 it follows under its assumptions that the operator $(I - A G_x)^{-1}$ exists and is bounded.

Theorem 4. Let A be a linear completely continuous symmetric mapping of H into H, let λ_i be its characteristic values. Let $V(x)$ be a mapping from H into H such that the Gateaux's derivative $V'(x)$ exists on a convex closed set $E \subset H$ and let $V'(x) = G_x$, $G_x \in M(E)$, satisfy one of the conditions $\bar{a})$ through $\bar{e})$ and f) (as defined for the lemma 2 and 1) for every $x \in E$. Let J be a linear operator in H such that $\| G(x) - J \| \leq \rho < p^{-1}$ holds for every $x \in E$, where $G(x)$ is the resolvent operator for $A G_x$ and $p = \sup_{x \in E} \| I - A G_x \| < +\infty$. Let us set

$$x_{n+1} = (I+J)f - Jx_n + (I+J)\, AV(x_n), \tag{11}$$

$$\alpha_J = \sup_{x \in E} \| (I+J)\, A\, G_x - J \|, \quad r = \frac{\alpha_J}{1-\alpha_J} \| x_1 - x_0 \| .$$

Let $\Omega(x_1, r)$ be a closed ball which is contained in E. Then the equation (10) has a unique solution x^* in the ball $\Omega(x_1, r)$. The sequence $\{x_n\}$ defined by (11) converges in the norm of H to the solution x^* of (10) and the inequality (4) holds, where $\alpha = \alpha_J \leq \rho \cdot \sup_{x \in E} \| I - A G_x \|$.

Proof. To prove the theorem we use the theorem 3 and lemma 2. According to lemma 2 the mapping $(I - A G_x)^{-1}$ exists and is bounded for every $x \in E$. Hence the resolvent operator $G(x)$ is bounded and from the inequality $\| G(x) - J \| \leq \rho < p^{-1}$ it follows that J is bounded operator in H. From theorem 3 follow the statements of the theorem 4. This completes the proof.

Theorem 5. Let $F(x)$ be a weakly continuous mapping from real H into H such that it has on a convex closed bounded set $E \subset H$ the Gateaux's derivative $F'(x)$. Let $PF'(x)$ be a symmetric operator for every $x \in E$ such that $(PF'(x)h,h) \geqq 0$ for every $x \in E$ and $h \in H$, where P is a linear operator in H having the property that P^{-1} exists and $0 < \| P \| < \frac{1}{D}$; $D = \sup_{x \in E} \| F'(x) \| < +\infty$. Let us set $A(x) = x - PF(x) + Pf$ for every $x \in E$. Let $A(E) \subset E$. Then the equation (2) has at least one solution x^* in E. The sequence $\{x_n\}$ defined by

$$x_{n+1} = x_n - \beta PF(x_n) + \beta Pf, \quad 0 < \beta < 1 , \tag{12}$$

where x_0 is an arbitrary element from E , weakly converges in the norm of H to any solution x^* of (2).

Proof. The equation (2) is equivalent to the equation

$$x = A(x) \tag{13}$$

Considering that P is a linear bounded operator, it is clear that $A(x)$ is weakly continuous mapping in H and $A(x) \in E$ for every $x \in E$. Because every bounded set in H is weakly compact and every convex closed set in H is weakly closed, all the assumptions of Schauder's theorem III [33] are fulfilled. Hence there exists at least one solution x^* of (13) and therefore at least one solution x^* of (2). We show that $A(x)$ is Lipschitzian mapping with constant one.

For every $x \in E$ we have $A'(x) = I - PF'(x)$ and

$$\| I - PF'(x) \| = \sup_{\|h\|=1} |(h - PF'(x)h,h)| = \sup_{\|h\|=1} |1 - (PF'(x)h,h)| =$$

$$= \sup_{\|h\|=1} (1 - (PF'(x)h,h)) \leqq 1 ,$$

because $0 \leqq (PF'(x)h,h) \leqq \| P \| . \sup_{x \in E} \| F'(x) \| < 1$ for every $x \in E$ and $h \in H$ with $\|h\| = 1$.

Therefore $\alpha_P = \sup_{x \in E} \| I - PF'(x) \| \leqq 1$. Further, by the similar way as in theorem 1 we can prove that $A(x)$ is Lipschitzian with constant one. Thus all the assumptions of theorem 3 [25] are fulfilled and we get that the sequence $\{x_n\}$ defined by

$$x_{n+1} = (1 - \beta)\, x_n + \beta\, A(x_n) =$$
$$= (1 - \beta)\, x_n + \beta\, x_n - \beta\, PF(x_n) + \beta\, Pf =$$
$$= x_n - \beta\, PF(x_n) + \beta\, Pf \;;\quad 0 < \beta < 1,\; x_0 \in E$$

weakly converges in H to any solution of (2). This completes the proof.

Remark 5 . Equally with the theorem 2 we may put $P = \beta I$, $P = \gamma F'(x_0)^{-1}$, where β , γ are such that the inequality $0 < \| P \| < \frac{1}{D}$ holds.

Example. Let the equation

$$F(x) = x(s) - \int_0^1 st \operatorname{arctg} x(t)dt = f(s)$$

be given, where $f(s) \in L_2(0,1)$. We suppose that $L_2(0,1)$ is real. Let us denote $\Phi(x) = \int_0^1 st \operatorname{arctg} x(t)dt$. Then $\Phi(x)$ is Hammerstein's operator with the symmetrical kernel $K(s,t) = st$. Then Caratheodory's conditions for the function $g(u,t) = \operatorname{arctg} u$ are evidently fulfilled and hence the operator $hu = \operatorname{arctg} u$ is of Nemyckij type in the space $L_2(0,1)$. The function $g(u,t)$ has the continuous derivative $g'_u(u,t)$ for $u \in (-\infty, +\infty)$. Hence the operator hu has a linear Gateaux's differential $Dh(u,v) = \frac{1}{1+u^2(x)} v(x)$ in $L_2(0,1)$ which is bounded and continuous in u, v . Therefore $\Phi(x)$ has the completely continuous Gateaux's differential

$$D\Phi(x,h) = \int_0^1 st \frac{1}{1+x^2(t)} h(t)dt$$

and we may write $D\Phi(x,h) = \Phi'(x)h$. Further

$$(\Phi'(x)h,h) = \int_0^1\int_0^1 \frac{st}{1+x^2(t)} h(t)h(s)dt\,ds \leq$$

$$\leq \int_0^1\int_0^1 st\, h(t)\, h(s)\, dt\, ds = \int_0^1 sh(s)ds. \int_0^1 t\, h(t)dt =$$

$$= (\int_0^1 sh(s)ds)^2 \leq \int_0^1 s^2 ds \int_0^1 h^2(s)ds = \frac{1}{3}\|h\|^2 .$$

Thus

$$(F'(x)h,h) = (h,h) - (\Phi'(x)h,h) \geq \|h\|^2 - \frac{1}{3}\|h\|^2 = \frac{2}{3}\|h\|^2 .$$

Hence the condition (5) of the theorem 2 is fulfilled and therefore we may solve the equation by (6), where $P_1 = I$.

Remark 6 . When this paper was written I acquainted by means of [34] with the result of E.H. Zarantonello [35]. He considers the equation $x = F(x) + y$, where F is an everywhere defined Lipschitzian function, which is either "supra-unitary" or "infra-unitary" in a real Hilbert space H. This equation is solved by contractive averanging.

Correction to my paper [28]. In the theorems 2,3 shall be: "for every $x \in E$ ... " instead of "for every $x \in H$ ...". The assumptions of the theorem 9 shall be completed by: $\|F(x_0)\| \leq \mu(1 - \psi(\rho_0)\frac{1}{\mu})\rho_0$ and the assumptions of the theorem 4 by: H real, $F(x)$ is weakly continuous mapping such that Gateaux's derivative $F'(x)$ is continuous on E and commutative with a symmetric operator P, $(I - PF)E \subset E$. The theorem 4 does not generalize the result of M.M. Vajnberg [3], § 10.

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