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REMARK ON TOPOLOGICAL SPACES WITH GIVEN SEMIGROUPS

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J. de Groot proved in [1] that an arbitrary group G can be represented as a group of all autohomeomorphisms A(T) of a topological space T. To represent semigroups in the similar way we need to replace A(T) by some semigroup of transformations of T. Therefore we denote by E(T) the semigroup of all local homeomorphisms into, that is: for $f : T \to T$, $f \in E(T)$ if and only if there exists a neighborhood $O(x)$, for every $x \in T$, such that $f | O(x) : O(x) \to f(O(x))$ is a homeomorphism. It is easy to see that every $f \in E(T)$ is continuous, and E(T) forms a semigroup under composition.

The aim of this remark is to present, using a previous result of the authors, a simple proof of the following theorem:

<u>Theorem</u>. Let S^1 be a semigroup with a unity element, cardinality of S^1 being less than the first unaccessible cardinal. Then there exists a T_0 - topological space T such that E(T) is isomorphic with S^1.

If the cardinal of S^1 is less or equal τ_i, $\tau_0 = \aleph_0$, $\tau_{i+1} = 2^{\tau_i}$ for some natural i, then the proof of the theorem can be made without the use of the axiom of choice.

Proof. If the cardinality of S^1 is less than the first unaccessible cardinal, then there exists a relation R on a set X such that the semigroup C(R, X) of all compatible transformations of X - under the composition - is isomorphic with S^1. Moreover, if $x \in X$ then there exists $y \in X$ such that either xRy or yRx . If the cardinality of S^1 is less or equal $\aleph_i$ for some natural i , the proof can be made without the use of the axiom of choice. See [2],[3].

Let $T = X \cup Y_1 \cup Y_2$, where Y_1 is the set of all triples $(x, y, 1)$, $x, y \in X$, xRy, and Y_2 is the set of all triples $(x, y, 2)$, $x, y \in X$, xRy .

A set $O \subset T$ is open in T if and only if

(i) $(x, y, 1) \in O$ implies $x \in O$,

(ii) $(x, y, 2) \in O$ implies $x, y, (x, y, 1) \in O$.

Evidently, T is a T_0 - topological space.

We are going to prove that the mapping $\{f \to f|X\}$, $f \in E(T)$, is an isomorphism of $E(T)$ onto $C(R, X)$.

Let $f \in E(T)$. Take any $(x, y, 2) \in T$. There exists an open set O containing $(x, y, 2)$ such that $f|O : O \to f(O)$ is a homeomorphism. By the definition, the set O' , $O' = \{x\} \cup \{y\} \cup \{(x, y, 1)\} \cup \{(x, y, 2)\}$ is open and contained in O . Therefore $f|O' : O' \to f(O')$ is a homeomorphism. It follows that $f((x, y, 2)) = (x', y', 2)$, $x', y' \in X$, $f((x, y, 1)) = (x', y', 1)$, $f(x) = x'$, $f(y) = y'$. As $(x, y, 2)$ was arbitrary, we get $f(X) \subset X$, xRy implies $f(x)Rf(y)$.

Let $g : X \to X$, $g \in C(R, X)$. We define $f : T \to T$ as follows:

$$f(x) = g(x) \quad \text{for} \quad x \in X ,$$

$$f((x, y, 1)) = (g(x), g(y), 1) \quad \text{for all} \quad (x, y, 1) \in T,$$

$$f((x, y, 2)) = (g(x), g(y), 2) \quad \text{for all} \quad (x, y, 2) \in T.$$

As $g \in C(R, X)$, f is well defined, and $f \in E(T)$. If $f' \in E(T)$, $f'|X = g$, then $f' = f$.

The proof is finished. The question, whether the T_0 -space can be replaced in the theorem by a Hausdorff one, seems to be open.

R e f e r e n c e s :

[1] J. de GROOT, Groups represented by homeomorphism groups I, Math.Annalen 138(1959),80-102.

[2] Z. HEDRLÍN, A. PULTR, Relations (Graphs) with given finitely generated semigroups, to appear in Monatshefte für Matematik.

[3] A. PULTR, Z. HEDRLÍN, Relations (Graphs) with given infinite semigroups, submitted to Monatshefte für Mathematik.