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A REMARK ON COMMON FIXED SETS OF COMMUTING MAPPINGS

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This remark is a supplement to the paper [1]. The terminology and the notation used there is preserved. If for a map $f : X \to X$ holds $f(Y) \subset Y$ we use the notation $f \| Y$ for the induced mapping $Y \to Y$. There was proved in [1] that an N-map of an N-space into itself, which is homotopical with a constant one, has a simple fixed set.* (Let us define, in a connected N-space a metric $\rho(a, b)$ as the least integer n such that $a = a_0$, $b = a_n$, $a_i R a_{i+1}$ $(i = 0, \dots, n-1)$ if a and b are different. Then the simple sets are just the sets of diameter ≤ 1). Here we prove that commuting mappings such that at least one of them is homotopical with a constant one, have a common simple set.

<u>Theorem</u>: Let $g_0, g_1, \dots, g_n$ be N-maps of an N-space X into itself. Let $g_i g_j = g_j g_i$ for every $i, j = 0, \dots, n$. Then all the g_i have a common fixed set. If one of them, e. g. g_0, is homotopical with a constant mapping, they have a common simple fixed set.

<u>Proof</u>: Let r be an integer such that $g_0(g_0^r(X)) = g_0^r(X)$ (see [1], 6.1). Put $M_0 = g_0^r(X)$. Then $g_0(M_0) = M_0$ (and hence, see [1] 2.4 , $g \| M_0$ is an isomorphism) and $g_i(M_0) \subset M_0$ for every i. Really, let $y \in M_0$, i.e. $y = g_0^r(x)$, where $x \in X$. Hence $g_i(y) = g_i g_0^r(x) = g_0^r(g_i(x)) \in M_0$.

* By fixed set of a mapping φ we mean such a set A, that $\varphi(A) = A$.

Let M_k be a set such that $g_i(M_k) = M_k$ (and hence $g_i \| M_k$ isomorphism) for $i \leq k$, $g_i(M_k) \subset M_k$ for every i. Put $M_{k+1}^{\ell} = g_{k+1}(M_k)$, where ℓ is an integer such that $g_{k+1}^{\ell+1}(M_k) = g_{k+1}^{\ell}(M_k)$.

Let $y \in M_{k+1}$, i.e. $y = g_{k+1}^{\ell}(x)$, $x \in M_k$. We have $g_i(y) = g_i g_{k+1}^{\ell}(x) = g_{k+1}^{\ell} g_i(x) \in M_{k+1}$, as $g_i(M_k) \subset M_k$. Hence, we have $g_i(M_{k+1}) \subset M_{k+1}$ for every i. For $i \leq k$, $g_i \| M_k$ are isomorphisms, so that, assuming finiteness, $g_i \| M_{k+1}$ are isomorphisms. For $i = k+1$, $g_{k+1} \| M_{k+1}$ is an isomorphism by construction.

By induction, we get a set M_n such that $g_i \| M_n$ are isomorphisms for every i.

If g_0 is homotopical with a constant map, M_0 is h.t. (see [1], the proof of 6.5). By this fact and by 6.2 and 3.5 in [1] we get M_1 h.t., M_2 h.t., ... and finally M_n h.t. by [1] 6.3 $g_i(K(M_n)) = K(M_n)$ and $K(M_n)$ is a simple set.

Corollary. 1. Let $\varphi, \psi : X \to X$ be N-maps, let $\varphi \circ \psi = \psi \circ \varphi$. If ψ is homotopical with a constant mapping, then φ has a fixed simple set. Particularly, a sufficient condition for φ to have a fixed simple set is φ^n being homotopical with a constant mapping for some n.

Remark: Under assumption of corrolary 1, φ may not be homotopical with a constant, e.g. the identity mapping commutes with any other one. It is not difficult to construct a mapping φ such that, for some n, φ^n is homotopical with a constant one, while φ is not.

Corollary 2. Let S be a commutative semigroup of N-maps of an N-space X into itself. Then either S does not contain any mapping homotopical with a constant, or all the elements of S have a common single fixed set. Particularly, for X h.t., the second alternative allways hold.

R e f e r e n c e

[1] A. PULTR, An analogon of the fixed-point theorem, CMUC 4,3 (1963), pp. 121 - 131.