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ON A NUMBER OF COMMUTING TRANSFORMATIONS

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The aim of this remark is to prove the following:

<u>Theorem</u>: Let f be a transformation of an n-point set X. Then there exist at least n different transformations of X commuting with f, that is, there exist transformations $g_1, g_2, \ldots, g_s$, $s \geq n$, $g_i \neq g_j$ for $i \neq j$, such that

$$g_i[f(x)] = f[g_i(x)] \quad \text{for all } x \in X \text{ and all } i = 1, 2, \ldots, s .$$

We use the following notation:

i denotes the identity transformation of X and we write $i = \overset{o}{f}$, $\overset{i}{f} = \overset{i-1}{f}(f)$, and $F = \bigcup_{i=o}^{n^n} \{\overset{i}{f}\}$. We write $F(x) = \bigcup_{i=o}^{n^n} \{\overset{i}{f}(x)\}$.

$y \in X$ is said to be maximal according to f, or simply maximal, if $f(x) \neq y$ for every $x \in X$. If Y is a set, $|Y|$ denotes the cardinal of Y. We assume that $|X| = n$.

<u>Lemma 1</u>. Let g commute with f, $g(x_1) = x_2$. Then $|F(x_1)| \geq |F(x_2)|$.

Proof. We have $\overset{i}{f}(x_2) = \overset{i}{f}[g(x_1)] = g[\overset{i}{f}(x_1)]$. Hence, if $\overset{i}{f}(x_2) \neq \overset{j}{f}(x_2)$, then also $\overset{i}{f}(x_1) \neq \overset{j}{f}(x_1)$.

<u>Lemma 2</u>. Let $F(x_1) \cap F(x_2) \neq \emptyset$, $F(x_2) \cap F(x_3) \neq \emptyset$. Then $F(x_1) \cap F(x_3) \neq \emptyset$.

Proof. We have $\overset{i}{f}(x_1) = \overset{j}{f}(x_2)$, $\overset{k}{f}(x_2) = \overset{l}{f}(x_3)$, for some i, j, k, l . Hence, $\overset{i+k}{f}(x_1) = \overset{j+l}{f}(x_3)$, and the lemma is proved.

$x_1, x_2 \in X$ are said to belong to the same component according to f , or simply to the same component, if $F(x_1) \cap \cap F(x_2) \neq \emptyset$. By lemma 2 , two components are either equal or disjoint.

Lemma 3. Let there exists only one component and only one maximal element according to f . Then $F(y) = X$, where y is the maximal element.

Proof. Let x_0 non $\in F(y)$. As $x_0 \neq y$, x_0 is not maximal, there exists x_1 such that $f(x_1) = x_0$. Evidently, x_1 non $\in F(y)$. Continuing this process we get a sequence x_m . As $|X| = n$, there exists only finite number of different x_i . Therefore there must exist x_j and natural k such that $\overset{k}{f}(x_j) = x_j$, x_j non $\in F(y)$. Hence, $F(x_j) \cap \cap F(y) = \emptyset$, and x_j and y belong to different components.

Lemma 4. Let there exist only one component, and no maximal element according to f . Then $F(y) = X$ for all $y \in X$. The proof is evident.

Now, we are going to prove the theorem by induction. If $n = 1$, then the theorem is evidently true. Let $n > 1$. We assume that the theorem is proved for all $m \leq n - 1$. We divide the proof in three sections according to the properties of f .

(a) There exist more than one component.

(b) There exists only one component containing at most one maximal element.

(c) The exists only one component containing more than one maximal element.

(a) We denote the components $Y_1, Y_2, \dots, Y_k$. We may assume that $|Y_i| \leqq |Y_{i+1}|$, $i = 1, 2, \dots, k-1$. Evidently, every Y_i is fixed under f, that is $f(Y_i) = \bigcup_{y_i \in Y_i} \{f(y_i)\} \subset Y_i$. If we denote by $f|Y_i$, as usual, the restriction of f onto Y_i, then $f|Y_i$ is a transformation of Y_i. By assumption, there exist at least $|Y_i|$ different transformations of Y_i commuting with $f|Y_i$. We denote this set by F_i. Let G be a system of transformations of X such that

$g \in G$ if and only if $g|Y_i \in F_i$ for each $i = 1, 2, \dots, k$.

We have

$$|G| = \prod_{i=1}^{k} |F_i| \geqq \prod_{i=1}^{k} |Y_i| .$$

If no $|Y_i| = 1$, then the theorem is true, as every transformation in G commutes with f.

Let $|Y_i| = 1$ for $i = 1, 2, \dots, r$, that is $Y_i = \{y_i\}$. For each $i = 1, 2, \dots, r$, we define a constant transformation $h_i(x) = y_i$ for every $x \in X$. Evidently, all h_i, $i = 1, 2, \dots, r$, commute with f. Let us denote $G' = G \cup (\bigcup_{i=1}^{r} \{h_i\})$. We get

$$|G'| \geqq \prod_{i=r+1}^{k} |Y_i| + r .$$

Evidently, $\sum_{i=r+1}^{k} |Y_i| = n - r$, and every $|Y_i| \geqq 2$, $i = r+1, r+2, \dots, k$. Hence, $|G'| \geqq n$, and the case (a) is proved.

(b) If there exists only one component with at most one element y, then, by lemma 3 and 4, the points y, $f(y)$, $\overset{2}{f}(y)$, ..., $\overset{n-1}{f}(y)$ are different. That proves the case (b).

(c) Let $y, y_1, y_2, \ldots, y_t$ be maximal elements, $|F(y)| \leqq |F(y_i)|$ for $i = 1, 2, \ldots, t$. We denote $X' = X \setminus \{y\}$, $f' = f \mid X'$. Evidently, f' is a transformation of X', $|X'| = n - 1$. Hence, there exist different transformations $g_1', g_2', \ldots, g_s'$, $s \geqq n - 1$, such that every g_i' commutes with f'. We may assume that g_i', $i = 1, 2, \ldots$ ..., s, are all transformations of X' commuting with f'. We are going to prove that every g_i' can be commutatively extended to X, that is, for each g_i', $i = 1, 2, \ldots, s$, there exists a transformation g_i of X such that $g_i \mid X' = g_i'$, and g_i commutes with f. By assumption, y is a maximal element and therefore

$$|F[f(y)]| = |F(y)| - 1 = |F'[f(y)]| \quad ,$$

where F' denotes the set of transformations of X' belonging to f'. As $|F(y)| \leqq |F(y_i)|$, by lemma 1, $g_i'[f(y)]$ is not maximal element according to f.

Thus, there exists at least one element x_i' such that

$$f(x_i') = g_i'[f(y)] \quad .$$

If we define $g_i \mid X' = g_i'$, $g_i(y) = x_i'$, then g_i is the required commutative extension of g_i'.

It remains only to prove the existence of g_i' which can be extended in two different ways. To prove it we show that under assumptions of (c), there exists a natural k such that

$$\overset{k}{f}(y) = f(z) \ , \quad z \neq \overset{k-1}{f}(y) \ .$$

Let $y_1 \neq y$, y_1 be maximal. We have $F(y) \cap F(y_1) \neq \emptyset$, y_1 non $\in F(y)$, y non $\in F(y_1)$. Hence, there exists natural

k such that $f^{k}(y) \in F(y_1)$, $f^{k-1}(y)$ non $\in F(y_1)$. There exists an integer m such that $f^{m}(y_1) = f^{k}(y)$. Put $z = f^{m-1}(y_1)$. Evidently, $z \neq f^{k-1}(y)$, as $f^{k-1}(y)$ non $\in F(y_1)$.

Now, $f^{k-1} | X'$ commutes with f' and can be extended in two different ways. The proof is finished.