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ON FINITELY GENERATED COMMUTATIVE SEMIGROUPS

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In this paper we show that in every finitely generated commutative semigroup the maximality condition for congruence-relations is satisfied. This may seem to be important because of the fact that for those commutative semigroups which satisfy this maximality condition the noetherian congruence-theory is true. This theory has been discovered by the author a short time before (see [1] or [2]).

We shall use the following notation: S_1 is the additive semigroup of all non-negative integers, $S_n (n > 1)$ is defined recurrently by $S_n = S_1 \oplus S_{n-1}$, $\oplus$ being the symbol of the direct sum. S_n is in fact the free commutative semigroup with n generators and with the unit element. Congruence-relations on S_n will be denoted by $\mathscr{C}$, $\mathscr{D}$ and especially a chain

(1) $$\mathscr{C}_0 \subset \mathscr{C}_1 \subset \mathscr{C}_2 \subset \dots \subset \mathscr{C}_k \subset \dots$$

of congruence-relations on S_n will be considered. $\underline{1}$ is the identity-relation. The notion of an ideal will be used in the usual way but the empty set will be regarded as an ideal, too.

If $n > 1$ we often write A instead of S_1 and B instead of S_{n-1} so that $S_n = A \oplus B$ and we use $a, a_1, a_2, \dots, d, x_1, x_2$ for to denote elements of A whereas b, b_1, b_2, y_1, y_2 belong always to B . Elements of S_n

will be mostly written as $a \oplus b$, etc.

Having $\mathcal{C}$ on S_n and $a \in A$ we define a congruence-relation $\mathcal{C}^a$ on B by

$$b_1 \; \mathcal{C}^a \; b_2 \Longleftrightarrow a \oplus b_1 \; \mathcal{C} \; a \oplus b_2$$

It is clear that $\mathcal{C} \subset \mathcal{D}$ implies $\mathcal{C}^a \subset \mathcal{D}^a$ and that $\mathcal{C}^a \subset \mathcal{C}^{a+t}$ for every $t \in A$.

Having $\mathcal{C}$ on S_n and $a_1, a_2 \in A$ we define an ideal $J(\mathcal{C}, a_1, a_2)$ in B as the ideal consisting of all $b_1 \in B$ such that there exists at least one $b_2 \in B$ with $a_1 \oplus b_1 \; \mathcal{C} \; a_2 \oplus b_2$. It is clear that $\mathcal{C} \subset \mathcal{D}$ implies $J(\mathcal{C}, a_1, a_2) \subset J(\mathcal{D}, a_1, a_2)$ and that $J(\mathcal{C}, a_1, a_2) \subset$ $\subset J(\mathcal{C}, a_1 + t, a_2 + t)$ holds for every $t \in A$.

Having $d \in A$ we define $K(d)$ as the ideal in S_n consisting of all $a \oplus b$ with $a \geq d$. In connection with this $K(d, \mathcal{C})$ is defined as the ideal consisting of all $X \in S_n$ such that there exists at least one $Y \in K(d)$ with $X \; \mathcal{C} \; Y$. It is clear that $\mathcal{C} \subset \mathcal{D}$ implies $K(d, \mathcal{C}) \subset K(d, \mathcal{D})$.

Having $\mathcal{C}$ on S_n an ideal B' in B will be called an <u>$\mathcal{C}$-ideal</u> if and only if we can find numbers $a_1 \prec a_1'$, $a_2 \prec a_2'$, ..., $a_r \prec a_r'$ in A such that

$$B' = \bigcup_{i=1}^{r} J(\mathcal{C}, a_i, a_i')$$

It will be shown that in S_n always the maximality condition for ideals is satisfied so that a maximal $\mathcal{C}$-ideal in B can be found. Of course, this maximal $\mathcal{C}$-ideal in B is the greatest $\mathcal{C}$-ideal in B, so it is uniquely determined and it will be denoted by $M(\mathcal{C})$. It is clear that $\mathcal{C} \subset \mathcal{D}$ implies $M(\mathcal{C}) \subset M(\mathcal{D})$.

Finally, having an ideal K in S_n $(n > 1)$ and $a \in A$ we define K^a as the ideal in B consisting of all $b \in B$

such that $a \oplus b \in K$. It is clear that $K_1 \subset K_2$ implies $K_1^a \subset K_2^a$ and that $K^a \subset K^{a+t}$ for every $t \in A$.

In the proof of the maximality condition for ideals a chain

(2) $K_0 \subset K_1 \subset K_2 \subset \dots \subset K_k \subset \dots$

of ideals in S_n will be considered.

Before coming to our main theorem we have to prove some propositions. Proofs, when they are simple, are omitted.

Proposition 1. For every $\mathcal{C}$ on S_1, $\mathcal{C} \neq \underline{1}$, the factor semigroup $S_1/\mathcal{C}$ is finite.

Proposition 2. In S_1 the maximality condition for congruence-relations and the maximality condition for ideals are satisfied.

Proposition 3. In S_n $(n = 1, 2, 3, \dots)$ the maximality condition for ideals is satisfied.

Proof: We can suppose that $n > 1$ and that our proposition is proved in B . Consider the chain (2). Let $K_{k^*}^{a^*}$ be maximal in $\{K_k^a\}_{a,k \in A}$ and for every $a < a^*$ let $K_{k(a)}^a$ be maximal in $\{K_k^a\}_{k \in A}$. Putting $\hat{k} = \max\{k^*, k(a) \text{ for } a < a^*\}$ we have $K_k^a = K_{\hat{k}}^a$ for every $k \geq \hat{k}$ and for every $a \in A$. Hence $k \geq \hat{k}$ implies $K_k = K_{\hat{k}}$.

Proposition 4. Consider a chain (1) on S_n $(n > 1)$ and suppose that the maximality condition for congruence-relations holds in B . Then it is possible to find $\ell \in A$ such that $\mathcal{C}_k^a = \mathcal{C}_\ell^a$ holds for every $k \geq \ell$ and for every $a \in A$.

Proof: Let $\mathcal{C}_{k^*}^{a^*}$ be maximal in $\{\mathcal{C}_k^a\}_{a,\, k \in A}$ and for every $a < a^*$ let $\mathcal{C}_{k(a)}^a$ be maximal in $\{\mathcal{C}_k^a\}_{k \in A}$. Now we put $\ell = \max\{k^*, k(a) \text{ for } a < a^*\}$.

Proposition 5. Let be $\mathscr{C} \subset \mathscr{D}$ on S_n $(n > 1)$, a_1, $a_2 \in A$. Suppose that $J(\mathscr{C}, a_1, a_2) = J(\mathscr{D}, a_1, a_2)$ and that $\mathscr{C}^{a_2} = \mathscr{D}^{a_2}$. Then $a_1 \oplus b_1 \mathscr{D} a_2 \oplus b_2$ implies $a_1 \oplus b_1 \mathscr{C} a_2 \oplus b_2$.

Proof: Let $a_1 \oplus b_1 \mathscr{D} a_2 \oplus b_2$. We conclude step by step: $b_1 \in J(\mathscr{D}, a_1, a_2)$; $b_1 \in J(\mathscr{C}, a_1, a_2)$; $a_1 \oplus b_1 \mathscr{C} a_2 \oplus b_2'$ for some b_2'; $a_1 \oplus b_1 \mathscr{D} a_2 \oplus$ $\oplus b_2'$; $a_2 \oplus b_2 \mathscr{D} a_2 \oplus b_2'$; $b_2 \mathscr{D}^{a_2} b_2'$; $b_2 \mathscr{C}^{a_2} b_2'$; $a_2 \oplus b_2 \mathscr{C} a_2 \oplus b_2'$; $a_1 \oplus b_1 \mathscr{C} a_2 \oplus$ $\oplus b_2$.

Proposition 6. Consider the chain (1) on S_n $(n > 1)$ and let $0 < d \in A$. Then it is possible to find $k\langle d\rangle \in A$ such that $J(\mathscr{C}_k, a, a+d) = J(\mathscr{C}_{k\langle d\rangle}, a, a+d)$ holds for every $k \geq k\langle d\rangle$ and for every $a \in A$.

Proof: Let $J(\mathscr{C}_{k^*}, a^*, a^*+d)$ be maximal in $\{J(\mathscr{C}_k, a, a+d)\}_{a,\, k\in A}$ and for every $a < a^*$ let $J(\mathscr{C}_{k(a)}, a, a+d)$ be maximal in $\{J(\mathscr{C}_k, a, a+d)\}_{k\in A}$. Now we put $k\langle d\rangle = \max\{k^*, k(a)$ for $a < a^*\}$.

Proposition 7. Consider the chain (1) on S_n $(n > 1)$ and let $0 < d \in A$. Then it is possible to find $k[d] \in A$ such that $J(\mathscr{C}_k, a_1, a_2) = J(\mathscr{C}_{k[d]}, a_1, a_2)$ holds for every $k \geq k[d]$ and for all $a_1, a_2 \in A$ such that $0 < a_2 - a_1 \leq d$.

Proof: We put $k[d] = \max\{k\langle t\rangle\}_{0 < t \leq d}$.

Theorem. In S_n $(n = 1, 2, 3, \ldots)$ the maximality condition for congruence-relations is satisfied.

Proof: For the case $n = 1$ see proposition 2. We can suppose that $n > 1$ and that our theorem is proved in B. Consider the chain (1) on S_n. We can clearly find $m \in A$ such that $M(\mathscr{C}_k) = M(\mathscr{C}_m)$ for all $k \geq m$. Let

$$M(\mathcal{C}_m) = \bigcup_{i=1}^{r} J(\mathcal{C}_m, a_i, a_i')$$

for some $a_1 < a_1'$, $a_2 < a_2'$, ..., $a_r < a_r'$ in A . Of course, it is $M(\mathcal{C}_k) = \bigcup_{i=1}^{r} J(\mathcal{C}_k, a_i, a_i')$ for all $k \geqslant m$. Put $d = \max \{a_i'\}_{1 \leqslant i \leqslant r}$ and find $k[d] \in A$ as in proposition 7 . Then find $\ell \in A$ as in proposition 4 . Finally, find $m' \in A$ such that $K(d, \mathcal{C}_k) = K(d, \mathcal{C}_{m'})$ for all $k \geqslant m'$.

Now let us observe that m can be chosen in such a way that $m \geqslant k[d]$, $m \geqslant \ell$, $m \geqslant m'$ and that $J(\mathcal{C}_k, a_i, a_i') = J(\mathcal{C}_m, a_i, a_i')$ holds for every $k \geqslant m$ and for every $i = 1, 2, \ldots, r$.

Now we shall prove that $\mathcal{C}_k = \mathcal{C}_m$ is true for every $k \geqslant m$. Let us fix any $k > m$ and assume that $x_1 \oplus y_1 \mathcal{C}_k x_2 \oplus y_2$ and $x_1 \oplus y_1$ (non $\mathcal{C}_m$) $x_2 \oplus y_2$ hold for some $x_1, x_2 \in A$ and for some $y_1, y_2 \in B$. We may obviously suppose that $x_1 \leqslant x_2$. Consider now three cases:

I. $d \leqslant x_1 \leqslant x_2$. In this case we may suppose that $x_2 - x_1$ is minimal in regard to all possible cases which preserve all conditions mentioned up to this point. Now $x_1 = x_2 = x$ is not possible for it would be $y_1 \mathcal{C}_k^x y_2$ and y_1(non $\mathcal{C}_m^x$)y_2 contrary to $k, m \geqslant \ell$. In the case $0 < x_2 - x_1 \leqslant d$ we get a contradiction when using proposition 5 after having observed that k, $m \geqslant k[d]$ and $k, m \geqslant \ell$ so that $J(\mathcal{C}_k, x_1, x_2) = J(\mathcal{C}_m, x_1, x_2)$ and $\mathcal{C}_k^{x_2} = \mathcal{C}_m^{x_2}$ are true.

Hence we have $d < x_2 - x_1$. Starting with $x_1 \oplus y_1 \mathcal{C}_k x_2 \oplus y_2$ we conclude step by step: $y_1 \in J(\mathcal{C}_k, x_1, x_2)$; $y_1 \in M(\mathcal{C}_k)$; $y_1 \in M(\mathcal{C}_m)$; $y_1 \in J(\mathcal{C}_m, a_i, a_i')$ for some $i = 1, 2, \ldots, r$; $a_i \oplus y_1 \mathcal{C}_m a_i' \oplus y_2'$ for some $i = 1, 2, \ldots, r$ and for some $y_2' \in B$.

As $a_i < d < x_1$ we can write $x_1 = a_i + t$, $t \in A$.

Putting $a_i' + t = x_2'$ we have

(3) $$x_1 \oplus y_1 \ \mathscr{C}_m \ x_2' \oplus y_2'$$

It follows that $x_1 \oplus y_1 \ \mathscr{C}_k \ x_2' \oplus y_2'$ and $x_2' \oplus y_2' \ \mathscr{C}_k \ x_2 \oplus y_2$. Now $0 < x_2' - x_1 = a_i' - a_i \leq d < x_2 - x_1$, $x_1 < x_2' < x_2$ and so, using the minimality of $x_2 - x_1$ we have $x_2' \oplus y_2' \ \mathscr{C}_m \ x_2 \oplus y_2$. Observing (3) we get $x_1 \oplus y_1 \ \mathscr{C}_m \ x_2 \oplus$ $\oplus \ y_2$ - a contradiction.

II. $x_1 \leq x_2 < d$. In this case we have $x_1 = x_2$ or $0 < x_2 -$ $- x_1 < d$ and we use the same way as that contained in I to get a contradiction.

III. $x_1 < d \leq x_2$. We write $x_1 \oplus y_1 = X_1$, $x_2 \oplus y_2 =$ $= X_2$ so that $X_1 \mathscr{C}_k X_2$ is assumed. We conclude step by step: $X_2 \in K(d)$; $X_1 \in K(d, \mathscr{C}_k)$; $X_1 \in K(d, \mathscr{C}_m)$; $X_1 \mathscr{C}_m Y$ for some $Y \in K(d)$; $X_1 \mathscr{C}_k Y$; $X_2 \mathscr{C}_k Y$.

Now, using the results of I , we have $X_2 \mathscr{C}_m Y$, hence $X_1 \mathscr{C}_m X_2$.

<u>Remark:</u> In our theorem the maximality condition has been proved in the case of any finitely generated free commutative semigroup with unit element. But it is quite clear that our theorem can be generalized to any finitely generated commutative semigroup.

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