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ON THE SOLUTION OF HOMOGENEOUS FUNCTIONAL EQUATIONS IN HILBERT SPACE

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This paper contains the proofs of theorems (theorem 1 and 4) which were published previously without proofs in Commentationes Mathematicae Universitatis Carolinae 1,3(1960) [1].

Let the equation

$$Ay - \mu y = \theta \tag{1}$$

be given, where A throughout this paper will denote a linear operator bounded in complex Hilbert space H, μ is a real parameter. Suppose that A is a positive operator $(Ay, y) > \theta$ for every $y \in H$, $y \neq 0$ and $(Ay, y) = 0 \Longleftrightarrow y \equiv \theta$). This assumption will be later omitted. We solve the equation (1) by iterative process

$$y_{n+1} = \frac{1}{\mu_{n+1}} A y_n , \tag{2}$$

where the parameters $\{\mu_n\}$, $(n = 1, 2, \dots)$ are to be determined from the condition that the functional $\|Ay - \tau y\|^2$ for the given element $y = y_{n-1} \in H$ shall catch the minimal value on the set $\mathcal{R}$ of all real numbers. Let us denote that value of τ (dependent on n) by μ_n. We get

$$\mu_{n+1} = \frac{(Ay_n, y_n)}{\|Ay_n\|^2} . \tag{3}$$

Then

$$y_{n+1} = \frac{\|y_n\|^2}{(Ay_n, y_n)} A y_n , \quad y_0 \neq 0 , \quad y_0 \in H, \ (n = 0, 1, 2 \dots). \tag{4}$$

<u>Lemma 1.</u> Let A be a positive operator in H. Then the sequence $\{\mu_n\}$ defined by (2), is monotone, increasing

and convergent.

Proof: The sequence $\{\mu_n\}$ is bounded because

$$\mu_{n+1} \leqq \frac{\|Ay_n\| \, \|y_n\|}{\|y_n\|^2} \leqq \|A\| .$$

From (2) and (3)

$$\|Ay_n\| \leqq \frac{\|y_n\| \, \|Ay_n\|}{\|y_n\|^2} \|y_{n+1}\| .$$

Hence $\|y_n\| \leqq \|y_{n+1}\|$ for every n. Since

$$(Ay_{n-1}, y_{n-1}| \leqq \frac{(Ay_{n-1}, y_{n-1})}{\|y_{n-1}\|^2} \|y_n\|^2 \leqq (Ay_{n-1}, y_n)$$

we have from Schwarz´s inequality

$$(Ay_{n-1}, y_{n-1})^2 \leqq (Ay_{n-1}, y_n)^2 \leqq (Ay_{n-1}, y_{n-1})(Ay_n, y_n)$$

Thus

(5) $\quad (Ay_{n-1}, y_{n-1}) \leqq (Ay_n, y_n)$ for every n.

From the equality

$$(Ay_n, y_n) = \frac{(Ay_n, y_n)}{\|y_n\|^2} (y_{n+1}, y_n)$$

follows that $\|y_n\|^2 = (y_{n+1}, y_n)$. In view of (2) and of the precedent equality we get

$$\|y_n\|^2 = \frac{1}{\mu_n} (Ay_{n-1}, y_n), \quad \|y_n\|^2 = \frac{1}{\mu_{n+1}} (Ay_n, y_n).$$

We have now

$$\mu_n (Ay_n, y_n) = \mu_{n+1} (Ay_{n-1}, y_n)$$

and from (5)

$$\mu_n^2 (Ay_n, y_n)^2 = \mu_{n+1}^2 (Ay_{n-1}, y_n)^2 \leqq \mu_{n+1}^2 (Ay_{n-1}, y_{n-1})(Ay_n, y_n)$$
$$\leqq \mu_{n+1}^2 (Ay_n, y_n)^2 .$$

Hence $\mu_n \leqq \mu_{n+1}$ for every n. It follows from the fact that $\mu_n > 0$ for every n and A is a positive operator. Since $\{\mu_n\}$ is increasing and bounded, there exists $\lim_{n\to\infty} \mu_n = \mu$ and $\mu_1 \leqq \mu \leqq \|A\|$.

Lemma 2. Let A be a positive operator in H. Then the sequence $\{\|y_n\|\}$ defined by (2) is monotone, increasing and bounded.

Proof: Let us denote $g_n = \dfrac{y_n}{\|y_n\|}$. According to

(2) $g_{n+1} = \varkappa_{n+1} A y_n$, where $\varkappa_{n+1} = \|y_n\|^2 / \|y_{n+1}\| (Ay_n, y_n)$.

Hence

$$(6) \qquad \frac{\|y_{n+1}\|}{\|y_n\|} = \frac{\|y_n\|}{(y_n, g_{n+1})} = \frac{1}{(g_n, g_{n+1})} .$$

It is sufficient to show that $\prod_{n=1}^{\infty} \dfrac{1}{(g_{n-1}, g_n)}$ converges.

Because $(g_n, g_{n+1}) \leqq 1$, the product converges, when the series $\sum_{n=1}^{\infty} [1 - (g_{n-1}, g_n)]$ is convergent.

From (6) and (2) we obtain

$$(7) \qquad A g_n = \frac{Ay_n}{\|y_n\|} = \mu_{n+1} \frac{\|y_{n+1}\| \, g_{n+1}}{\|y_n\|} =$$

$$= \mu_{n+1} \frac{1}{(g_n, g_{n+1})} g_{n+1} .$$

From (7) we get

$$(8) \qquad (g_n, g_{n+1}) = \frac{(g_n, g_{n+1})}{\mu_{n+1}} (g_{n-1}, A g_n)$$

$$(g_n, g_{n-1}) = \frac{(g_n, g_{n+1})}{\mu_{n+1}} (A g_{n-1}, y_n) = \frac{\mu_n}{\mu_{n+1}} \frac{(g_n, g_{n+1})}{(g_{n-1}, g_n)} .$$

Further

$$0 \leqq (g_n - g_{n-1}, A(g_n - g_{n-1})) = (g_n - g_{n-1}, \mu_{n+1} \frac{1}{(g_n, g_{n+1})} g_{n+1}) -$$

$$- (g_n - g_{n-1}, \mu_n \frac{1}{(g_{n-1}, g_n)} g_n) =$$

$$= \mu_{n+1} - \frac{\mu_{n+1}}{(g_n, g_{n+1})} (g_{n-1}, g_{n+1}) - \frac{\mu_n}{(g_{n-1}, g_n)} + \mu_n .$$

It follows from (8) that

$$\mu_{n+1} + \mu_n - 2\mu_n \frac{1}{(g_{n-1}, g_n)} \geqq 0$$

and hence

$$1 - (g_{n-1}, g_n) \leqq 1 - \frac{2\mu_n}{\mu_n + \mu_{n+1}} = \frac{\mu_{n+1} - \mu_n}{\mu_n + \mu_{n+1}} \leqq \frac{\mu_{n+1} - \mu_n}{2\mu_1} .$$

$$(n = 1, 2, \ldots)$$

Therefore
$$\sum_{n=1}^{\infty} [1-(g_{n-1}, g_n)] \leq \sum_{n=1}^{\infty} \frac{\mu_{n+1}-\mu_n}{2\mu_1} .$$

The sequence $\{\mu_n\}$ converges, and hence the series $\sum_{n=1}^{\infty} [1-(g_{n-1}, g_n)]$ is convergent. This concludes the proof.

Theorem 1. Let A be a non-negative $((Ay, y) \geq 0$ for every $y \in H)$ completely continuous operator in a complex Hilbert space H. Let N be a null set of A and let $y_0 \in H \ominus N$ be not orthogonal to the eigenspace $H_{\tilde{\mu}_1}$ corresponding to the first eigenvalue $\tilde{\mu}_1$ of (1). Then the sequence $\{\mu_n\}$ defined by (3), (2) is monotone, increasing and it converges to $\tilde{\mu}_1$. The sequence $\{y_n\}$ defined by (2), (3), is convergent in $H \ominus N$ to one of the eigenfunctions corresponding to $\tilde{\mu}_1$.

Proof: The inequality $\|Ay\|^2 \leq \|A\| (Ay, y)$ holds for every $y \in H$. Hence A is a positive operator on $H \ominus N$. *The sequence $\{y_n\}$ is contained in $H \ominus N$.* According to our assumption $y_0 \in H \ominus N$. Suppose that $y_n \in H \ominus N$. Then $\mu_{n+1} > 0$ and from (2) $Ay_{n+1} = \frac{1}{\mu_{n+1}} A^2 y_n$. The null set N of A coincides with the null set of A^2. Hence $y_{n+1} \in H \ominus N$.

Now we use lemma 1 and 2. There exists a positive number C so that $\|y_n\| \leq C$. The sequence is bounded, because $\frac{\|y_n\|}{\mu_n} \leq \frac{c}{\mu_1}$. Hence it contains the subsequence $\left\{ \frac{y_{n_k}}{\mu_{n_k}} \right\}$ such that $\left\{ \frac{1}{\mu_{n_k}} A y_{n_k} \right\}$ converges. We set $\lim \frac{1}{\mu_{n_k}} A y_{n_k} = \tilde{y}$. Because $\frac{1}{\mu_{n+1}} Ay_n - y_{n+1} = \theta$ for every n $(n = 0, 1, 2, \dots)$, then $\frac{1}{\mu_{n+1}} Ay_n - y_{n+1} \to 0$

Therefore $y_{n_k} \to \tilde{y}$ and according to lemma 1 $A\tilde{y} = \mu \tilde{y}$ $(\tilde{y} \neq 0)$. We shall prove (see 2, Chapt.XV) that $\mu = \tilde{\mu}_1$.

Let P_k $(k = 1, 2, \dots)$ be projectors from H on eigenspace $H_{\tilde{\mu}_k}$ corresponding to different eigenvalues $\tilde{\mu}_k$. We set

$$\tilde{g}_k = \frac{P_k g_0}{\|P_k g_0\|} \; ; (P_k g_0 \neq 0) \quad , \text{ where } \quad g_0 = \frac{y_0}{\|y_0\|} .$$

Then $\tilde{g}_k \in H_{\tilde{\mu}_k}$, $g_0 = \sum_k P_k g_0 = \sum_k \|P_k g_0\| g_k = \sum_k a_{0k} \tilde{g}_k$,

where $\sum a_{0k}^2 = 1$, $a_{0k} = \|P_k g_0\|$, $a_{01} > 0$.

According to (2)

$g_1 = \sum a_{1k} \tilde{g}_k$, where $a_{1k} = \dfrac{\|y_0\| \tilde{\mu}_k}{\|y_1\| \mu_1} a_{0k}$ and

$g_1 = \dfrac{y_1}{\|y_1\|}$. Generally

(9) $g_n = \sum_k a_{nk} \tilde{g}_k$, where

$$a_{nk} = \frac{\tilde{\mu}_k \|y_{n-1}\|}{\mu_n \|y_n\|} a_{n-1,k} \; ; \; g_n = \frac{y_n}{\|y_n\|} .$$

Suppose now that $\mu = \tilde{\mu}_p$ $(p > 1)$. Since $y_{nk} \to \tilde{y}$, then $g_{n_k} \to \tilde{g}$, where $\tilde{g} = \dfrac{\tilde{y}}{\|\tilde{y}\|}$, $\tilde{g} = \sum_k a_k \tilde{g}_k$

and $a_k = \lim\limits_{j \to \infty} a_{n_j k}$ $(k = 1, 2, \dots)$.

Because $\tilde{g} \in H_{\tilde{\mu}_p}$, $\tilde{g}_k \in H_{\tilde{\mu}_k}$, then $(\tilde{g}, \tilde{g}_k) = 0$ for $k \neq p$. Hence $\tilde{g} = a_p \tilde{g}_p$ and $|a_p| = 1$. From $a_{n_k} \geq 0$ follows that $a_p = 1$ and $\tilde{g} = \tilde{g}_p$. From (9) we get

(10)
$$\frac{a_{np}}{a_{n_1}} = \frac{\tilde{\mu}_p}{\tilde{\mu}_1} \; \frac{a_{n-1,p}}{a_{n-1,1}} < \frac{a_{n-1,p}}{a_{n-1,1}} .$$

Further $\lim\limits_{j \to \infty} a_{n_j p} = a_p = 1$, $\lim\limits_{j \to \infty} a_{n_j 1} = a_1 = \theta$,

So that
$$\lim_{j \to \infty} \frac{a_{n_j p}}{a_{n_j 1}} = \infty .$$

This is a contradiction with (10) which shows that $\mu = \tilde{\mu}_1$.

Let us denote $r = \tilde{\mu}_1 - \tilde{\mu}_2$, then

$$\| g_n - \tilde{g}_1 \| = 2(1 - a_{n1}) \leqq 2(1 - a_{n1}^2) = 2 \sum_{k \geq 2} a_{nk}^2 \leqq$$

$$\leqq \frac{2}{r} \sum_k (\tilde{\mu}_1 - \tilde{\mu}_k) a_{nk}^2 \leqq \frac{2}{r} (\tilde{\mu}_1 - \mu_{n+1}) \longrightarrow 0 .$$

Hence $g_n \longrightarrow \tilde{g}_1$, where $\tilde{g}_1 \neq 0$. By lemma 2 the sequence $\{\| y_n \|\}$ converges and $\lim_{n\to\infty} \| y_n \| = \sup_n \| y_n \| = h > \theta$. We have $y_n = g_n \| y_n \| \longrightarrow \tilde{g}_1 h$. Hence the sequence $\{y_n\}$ converges to eigenfunction $\tilde{y}$ corresponding to $\tilde{\mu}_1$. The theorem 1 has been now established.

Let the equation

$$Ay - \lambda By = \theta$$

be given, where A, B (not necessarily bounded) are linear operators in H .

Theorem 2 . Let B be a linear operator such that B^{-1} exists and let $T = B^{-1} A$ be a non-negative completely continuous operator in H . Let N be a null set of T and let $y_0 \in H \ominus N$ be not orthogonal to the eigenspace $H\tilde{\mu}_1$ corresponding to the first eigenvalue $\tilde{\mu}_1$ of T . Then the sequence $\{\mu_n\}$ defined by the equalities

$$y_{n+1} = \frac{1}{\mu_{n+1}} T y_n , \quad \mu_{n+1} = \frac{(Ty_n, y_n)}{\| y_n \|^2}$$

is monotone, increasing and it converges to $\tilde{\mu}_1$. The sequence $\{y_n\}$ converges in $H \ominus N$ to one of the eigenfunctions corresponding to $\tilde{\mu}_1$.

Let H be a real Hilbert space. We say that an operator A is symmetrizable by a positive operator B , if the equality $(BAx,y)=(X,BAy)$ holds for every $x,y \in H$. We define on H a new inner product:

$$[x,y]=(Bx,y). \tag{11}$$

The product (11) defines on the set of all $x,y \in H$ a new Hilbert space $\mathcal{H}$ which is not generally complete.

Adding to $\mathcal{H}$ the limit points, we get a complete Hilbert space. We denote it by $\mathcal{H}_0$.

The norm in $\mathcal{H}_0$ is defined by the equality

$$\|y\|_{\mathcal{H}_0}=(By,y)^{\frac{1}{2}}.$$

<u>Lemma 3</u>. ([3], [4]) Let A be a bounded operator in H. Then A is bounded in $\mathcal{H}$ and $\|A\|_{\mathcal{H}} \leqq \|A\|$.

The operator A is bounded and symmetric in $\mathcal{H}$. It can be extended to the self-adjoint operator $\tilde{A}$ in $\mathcal{H}_0$.

<u>Lemma 4</u>. ([3], [4]) The spectrum of the operator $\tilde{A}$ in $\mathcal{H}_0$ is a subset of the spectrum of A in H.

<u>Lemma 5</u>. ([3], [4]) Let A be a completely continuous operator in H. Then $\tilde{A}$ is completely continuous in $\mathcal{H}_0$. The sets of eigenvalues of A in H and $\tilde{A}$ in $\mathcal{H}_0$ are identical. The eigenspaces of A in H and $\tilde{A}$ in $\mathcal{H}_0$ corresponding to the eigenvalue μ_1 are equal.

Hence in view of lemma 5 we may investigate instead the eigenvalues and eigenfunctions of the symmetrizable completely continuous operator A in H the eigenvalues and eigenfunctions of the self-adjoint completely continuous operator $\tilde{A}$ in $\mathcal{H}_0$.

<u>Theorem 3</u>. Let A be a completely continuous operator which is symmetrizable by a positive operator B in a real Hilbert space H. Let BA be a positive operator in

H and let $y_0 \in H$ be not orthogonal to the eigenspace $H_{\tilde{\mu}_1}$ corresponding to the first eigenvalue $\tilde{\mu}_1$ of A. Then the sequence $\{\mu_n\}$ defined by

$$\mu_{n+1} = \frac{(BAy_n, y_n)}{(By_n, y_n)} \qquad y_{n+1} = \frac{1}{\mu_{n+1}} A y_n$$

is monotone, increasing and convergent to $\tilde{\mu}_1$. The sequence $\{y_n\}$ converges in $\mathcal{H}_0$ to one of the eigenfunctions corresponding to $\tilde{\mu}_1$.

I.A. Birger [5] gave another method for calculation of the characteristic values and characteristic functions, but without any conditions and without proof of the convergence. We give the sufficient conditions for convergence of his method.

Let the equation

$$y - \lambda A y = 0 \tag{12}$$

be given, where λ is a parameter, A a linear bounded operator in H. To solve it, I.A. Birger used the iterative formula

$$y_n = \lambda_n A y_{n-1}, \quad \lambda_n = \frac{(Ay_{n-1}, y_{n-1})}{\|Ay_{n-1}\|^2}, \tag{13}$$

where λ_n are Schwarz's parameters. Let N be a null set of A. We prove the following theorem.

Theorem 4. Let A be a non-negative completely continuous operator in complex Hilbert space H. If an element $y_0 \in H \ominus N$ is not orthogonal to the space $H_{\tilde{\lambda}_1}$ generated by characteristic functions corresponding to the first characteristic number $\tilde{\lambda}_1$ of (12), then the sequence $\{\lambda_n\}$ is monotone, increasing and convergent to $\tilde{\lambda}_1$.

The sequence $\{y_n\}$ is convergent in $H \ominus N$ to one of the characteristic functions corresponding to $\tilde{\lambda}_1$.

Proof: Because

$$\|y_n\| = \frac{(Ay_{n-1}, y_{n-1})}{\|Ay_{n-1}\|^2} \|Ay_{n-1}\| \leqq \|y_{n-1}\| ,$$

we have

(14) $$\|y_n\| \leqq \|y_{n-1}\| \leqq \dots \leqq \|y_0\| .$$

The sequence $\{\|y_n\|\}$ is decreasing and bounded. Therefore it is convergent. Let us denote $\lim_{n\to\infty} \|y_n\| = \varkappa$. According to (13)

(15) $$\lambda_n (Ay_{n-1}, y_n) = \|y_n\|^2, \quad \lambda_{n+1}(Ay_n, y_{n+1}) = \|y_{n+1}\|^2 .$$

Hence

(16) $$\lambda_{n+1}(Ay_n, y_{n+1}) \leqq \lambda_n (Ay_{n-1}, y_n) ,$$

and from (14) we get

$$\lambda_{n+1}^2 \|Ay_n\|^2 \leqq \lambda_n^2 \|Ay_{n-1}\|^2 ,$$

so that

(17) $$\lambda_{n+1} (Ay_n, y_n) \leqq \lambda_n (Ay_{n-1}, y_{n-1}) \leqq \dots \leqq \lambda_1 (Ay_0, y_0)$$

in view of (13). The sequence $\{\lambda_n (Ay_{n-1}, y_{n-1})\}$ is decreasing and bounded. Hence it converges. From (13) follows that

(18) $$(Ay_{n-1}, y_n) = (Ay_{n-1}, y_{n-1}) \text{ for every } n = 1, 2, \dots .$$

According to (17) and (18)

$$\lambda_{n+1} (Ay_{n-1}, y_n)^2 \leqq \lambda_{n+1} (Ay_{n-1}, y_{n-1})(Ay_n, y_n) \leqq$$
$$\leqq \lambda_n (Ay_{n-1}, y_{n-1})^2 = \lambda_n (Ay_{n-1}, y_n)^2 .$$

Hence $\lambda_{n+1} \leqq \lambda_n$ for every n $(n = 1, 2, \dots)$. In view of (18) and from the fact that $\lambda_n > \theta$ and that A is a positive operator in $H \ominus N$, the sequence $\{\lambda_n\}$ is decreasing and bounded. There exists $\lim_{n\to\infty} \lambda_n = \lambda$ and $\lambda \geqq 0$. Further according to (13)

(19) $\quad \| \lambda_n A y_{n-1} - y_{n-1} \|^2 = \| y_{n-1} \|^2 - \lambda_n (A y_{n-1}, y_{n-1})$.

From $\| y_n \| \to \kappa$ and in view of (17), (15) we have

(20) $\quad \| \lambda_n A y_{n-1} - y_{n-1} \|^2 \to 0$, when $n \to \infty$.

The sequence $\{ \lambda_n y_n \}$ is bounded:

$$\| \lambda_n y_n \| \leqq \lambda_1 \| y_0 \| = Const.$$

It contains the subsequence $\{ \lambda_{n_k} y_{n_k} \}$ such that $\{ \lambda_{n_k} A y_{n_k} \}$ converges. Let us denote $\lim_{k \to \infty} \lambda_{n_k} A y_{n_k} = \tilde{y}$. From (20) $y_{n_k} \to \tilde{y}$. Because $A y_{n_k} \to A \tilde{y}$ and $\lambda_n \to \lambda$, we get that $\tilde{y} - \lambda A \tilde{y} = 0$. We shall prove that $\lambda > 0$ and $\tilde{y} \not\equiv 0$.

From (18) follows that

(21) $\quad 0 < (A y_0, y_0) \leqq \dots \leqq (A y_{n-1}, y_{n-1}) \leqq (A y_n, y_n) \leqq \dots$.

The sequence $\{ (A y_n, y_n) \}$ is increasing and bounded:

$$(A y_n, y_n) \leqq \| A \| \, \| y_n \|^2 \leqq \| A \| \, \| y_0 \|^2 \text{ .}$$

There exists $\lim_{n \to \infty} (A y_n, y_n) = p$ and $p > \theta$.

According to (13) and (18)

(22) $\quad (A y_{n-1}, y_n) = \| A y_{n-1} \| \cdot \| y_n \|$

for every n $(n = 1, 2, \dots)$. From (22), (18) and (21)

$$\| A y_{n-1} \| \, \| y_n \| \leqq (A y_n, y_n) \leqq \| A y_n \| \cdot \| y_n \| \text{ ,}$$

so that

$$0 < \| A y_0 \| \leqq \| A y_1 \| \leqq \dots \leqq \| A y_n \| \leqq \dots \text{ ,}$$

$$\| A y_n \| \leqq \| A \| \, \| y_n \| = \| A \| \, \| y_0 \| \text{ .}$$

Hence the sequence $\{ \| A y_n \| \}$ is increasing and bounded. There exists $\lim_{n \to \infty} \| A y_n \| = q$ and $q > 0$. Since

$\lambda_n \to \frac{p}{q^2}$ and $\frac{p}{q^2} = \lambda$, then $\lambda > 0$.

From the fact that $\lambda = \inf_n \lambda_n$ and from (21), (15) and (18) we get

$$\|y_n\|^2 = \lambda_n (Ay_{n-1}, y_{n-1}) \geqq \lambda (Ay_0, y_0) > 0 .$$

Since $y_{n_k} \to \tilde{y}$, we have that $\|y_{n_k}\| \to \|\tilde{y}\|$ and $\|\tilde{y}\|^2 \geqq \lambda(Ay_0, y_0) > \theta$. Hence $\tilde{y} \neq 0$. Further the proof can be performed similarly as the proof of theorem 1 .

H.F. Bückner [6] investigated the iterative process (13) for linear and non-linear problems. I. Marek [7], [8] generalized the methods (3), (13) for bounded operators which have a dominant eigenvalue.

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R e f e r e n c e s

[1] J. KOLOMÝ, On convergence of the iterative methods, Comm.Math.Univ.Carol. 1,3(1960), 18-24

[2] L.V. KANTOROVITCH - G.P. AKILOV, Funkcional´nyj analiz v normirovannych prostranstvach, Moskva 1959

[3] M.G. KREIN, O linějnych vpolne nepreryvnych operatorach v funkcional´nych prostranstvach s dvumja normami, Sb.Tr.in-ta AN SSSR, No 9,1948

[4] P. LAX , Symmetrizable linear transformations, Comm. on Pure and Appl.Math., Vol.XII,No,4,1954, 633-647

[5] I.A. BIRGER, Nekotoryje matematičeskije metody rešenija inženernych zadač, Oborongiz, Moskva 1956

[6] H.F. BUECKNER, An iterative method for solving non-linear integral equations, PICC Symposium, Rome 1960, 614-643

[7] I. MAREK, On iterations of bounded linear operators and Kellogg´s iterations in not self-adjoint eigenvalue problems, Comm.Math.Univ.Carol. 1,4(1960), 12-21; Czech.Math.Journ. 12, 1962, 4, 536-554

[8] I. MAREK, Further applications of iterations of linear bounded operators in not self-adjoint problems. Comm.Math.Univ.Carol.,2,3 (1961), 13-23.