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ON THE HOMOLOGY THEORY OF DISCRETE SPACES

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In present paper the homology theory from category of couples of discrete spaces is constructed. It is shown that there exist sufficiently different homology theories from the category of couples of finite discrete spaces, satisfying all seven Eilenberg-Steenrod axioms. On the other hand, we get the uniqueness theorem adding another axiom.

§ 1.

1.1. <u>Definition</u>: A topological space X is said to be a discrete space if $\overline{\bigcup_\iota A_\iota} = \bigcup_\iota \overline{A_\iota}$ for arbitrary system $\{A_\iota\}$ of subsets of X.

1.2. The following statement is obvious:

<u>Theorem</u>: Let X be a discrete space (discrete T_0-space, respectively). Then a function $\mathcal{Cl} : X \to \exp X$, defined by formula $\mathcal{Cl}(x) = \overline{(x)}$, has the following properties:

I) $x \in \mathcal{Cl}(x)$

II) $x \in \mathcal{Cl}(y),\ u \in \mathcal{Cl}(x) \Rightarrow u \in \mathcal{Cl}(y)$

(III) $x \in \mathcal{Cl}(y),\ y \in \mathcal{Cl}(x) \Rightarrow x = y$, respectively).

On the contrary, for every function $F : X \to \exp X$, satisfying the conditions I, II there exists one and only one discrete topology over the set X such that $F(x) = \overline{(x)}$ for every x. If F satisfies the condition III, this topology is T_0.

1.3. Let us denote

$$St(x)=\{y \mid x \in Cl(y)\}, \quad St(A)=\bigcup_{x\in A} St(x).$$

Obviously $St(A)$ is always an open set and it is the least open set containing A.

1.4. Theorem: Let X, Y be discrete spaces, $f: X \to Y$ be a mapping. Then f is a continuous mapping iff $f(Cl(x)) \subset Cl(f(x))$ for every $x \in X$.

Proof: Obviously f is a continuous mapping iff $f(St(x)) \subset St(f(x))$ for every x. Let $y \in f(Cl(x))$. Then there exists a $u \in Cl(x)$, such that $y = f(u)$. Because $x \in St(u)$, we have $f(x) \in St(f(u))$ and hence $y = f(u) \in Cl(f(x))$. The rest of the proof is obvious.

§ 2.

2.1. Definition: A sequence $(x_0, \dots, x_n)$ of elements of X such that $x_i \in Cl(x_{i+1})$, $i = 0, \dots, n-1$, is said to be an elementary n-chain over the discrete space X. Let G be an abelian group; n-chain over X is every formal combination of finite number of elementary n-chains with coefficients from G. The set of all n-chains over X with obviously defined structure of an abelian group is said to be the group of n-chains of X (with coefficients from G and will be denoted $C_n(X;G)$ (or simply $C_n(X)$), if there is no danger of misunderstanding). We define yet $C_n(\emptyset) = 0$.

2.2. Remark: Let us call simple n-chains such n-chains $\sum g_\alpha \alpha$, that at most one g_α is non-zero element of G. It is easy to prove that for every additive function from the set of all simple n-chains into some abelian group there exists just one homomorphism which is its extension over $C_n(X)$. Therefore it is sufficient to define a homomorphism from $C_n(X)$ in its simple n-chains only.

2.3. Definition: Let X, Y be discrete spaces, $f: X \to Y$ be a continuous mapping. Because of 1.4 we can define the homomorphism $f_n : C_n(X) \to C_n(Y)$ by formula

$$f_n(g \cdot (x_0, \dots, x_n)) = g \cdot (f(x_0), \dots, f(x_n)) .$$

Homomorphism $d_n : C_n(X) \to C_{n-1}(X)$ is defined by formula

$$d_n(g \cdot (x_0, \dots, x_n)) = \sum_{i=0}^{n} (-1)^i g \cdot (x_0, \dots, \hat{x}_i, \dots, x_n) .$$

2.4. Definition: Let (X, A) be a couple of discrete spaces (in the ordinary sense; i.e. A is a subspace of X). Then we define

$$C_n(X, A; G) = C_n(X; G) / C_n(A; G)$$

Obviously $d_n(C_n(A)) \subset C_{n-1}(A)$ and therefore we can define $\bar{d}_n : C_n(X, A) \to C_n(X, A)$ by formula $\bar{d}_n([a]) = [d_n(a)]$. For a mapping $f: (X, A) \to (Y, B)$ we have $f_n(C_n(A)) \subset C_n(B)$ and therefore we can define $\bar{f}_n : C_n(X, A) \to C_n(Y, B)$ in an analogous way.

2.5. It is easy to prove

Lemma: $\bar{d}_{n-1} \circ \bar{d}_n = 0, \quad (\bar{g} \circ \bar{f})_n = \bar{g}_n \circ \bar{f}_n \quad ,$

$$\bar{f}_{n-1} \circ \bar{d}_n = \bar{d}_n \circ \bar{f}_n .$$

2.6. Corollary: The system $\{C_n(X, A), \bar{d}_n\}$ (if we define $C_k(X, A) = 0$, $\bar{d}_k = 0$ for $k < 0$) is a chain complex in the sense of [E-S, chapter V]. If $f: (X, A) \to (Y, B)$ is a continuous mapping, the system $\{\bar{f}_n\}$ is a mapping of

$\{C_n(X,A)\}$ into $\{C_n(Y,B)\}$. We define homology groups of couples of discrete spaces and induced homomorphisms of their continuous mappings as homology groups and induced mappings of corresponding chain complexes and their mappings, respectively. We use the denotations $H_n(X,A)$, f_{*n}, and $\partial(n,X,A)$ for homology groups, induced homomorphisms and boundary homomorphisms, respectively. For groups of cycles and boundaries we use the denotations $Z_n(X,A)$, $B_n(X,A)$, respectively. In this way we get a homology theory from the category of couples of discrete spaces which satisfies obviously first four Eilenberg-Steenrod axioms and, as it is easy to see, also the axiom of dimension. In the following paragraph we are going to prove that remaining axioms of homotopy and excission are satisfied, too (with a slight change in the defini - tion of homotopy).

2.7. Remark: If we define $C_{-1}(X,A;G) = G$ instead of $C_{-1}(X,A) = 0$, $d_{-1}(g(x_0)) = g$, f_{-1} identical homomorphism of G, we get (2.4 holds obviously, too) the "reduced homology theory". All things said about homology theory defined in 2.6, except of dimension axiom in the ordinary form, are true for reduced homology theory, too (the same for homotopy and excission axiom).

§ 3.

3.1. Definition: Let $\alpha \in C_n(X)$, $\alpha = \sum a_i(x_{i0}, \ldots, x_{in})$. The set $\{x_{ij}\}$ is said to be the carrier of α.

3.2. Definition: Let $\alpha \in C_n(X)$, let R be its carrier. Let φ, ψ be mappings from R into Y such that $\varphi(x) \in Cl(\psi(x))$ for every $x \in R$. Then we define the homomorphisms $D_n: C_n(R) \to C_{n+1}(Y)$ as follows:

$$D_n(a(x_0,\dots,x_n)) = \sum_{k=0}^{n} (-1)^k (\varphi(x_0),\dots,\varphi(x_k), \psi(x_k),\dots,\psi(x_n)).$$

3.3. Lemma: $d_{n+1} D_n(\alpha) = \psi_n(\alpha) - \varphi_n(\alpha) - D_{n-1} d_n(\alpha)$.

Proof: Is a matter of counting.

3.4. Definition: Mappings $f, g: X \to Y$ are said to be homotopical if there exists a continuous mapping $h: X \times I \to Y$ such that

a) $h(x,0) = f(x)$, $h(x,1) = g(x)$ for every $x \in X$.

b) The set $h((x) \times I)$ is finite for every $x \in X$.

In the analogous way the relation of homotopy between two mappings of couples of discrete spaces is defined.

3.5. Remark: If Y is a star-finite space (i.e. $St(x)$ is finite for every $x \in X$), the condition b) is satisfied automatically. In general, it is possible to prove that image of any compact space in a star-finite space is finite.

3.6. Lemma: Let T be a topological space, X be a discrete space. Let $M \subset T$ $a \in \overline{M}$, let $f: T \to X$ be a continuous mapping. Then there exists an element $y \in M$ such that $f(a) \in Cl(f(y))$.

Proof: Because of continuity of f there exists a neighborhood V of the point a such that $f(V) \subset St(a)$. $a \in \overline{M}$ and therefore $V \cap M \neq \emptyset$. Let us take some $y \in V \cap M$. We have $f(y) \in St(a)$ and hence $f(a) \in Cl(f(y))$.

3.7. Lemma: Let X, Y be discrete spaces, $f, g: X \to Y$ be homotopical mappings. Let R be a finite subspace of X. Then there exists a finite sequence of mappings

$$h_0, h_1, \dots, h_m : R \to Y$$

such that $h_0 = f|R$, $h_m = g|R$ and for $i = 1, \dots, m$ the following alternative holds

either $h_i(x) \in Cl(h_{i-1}(x))$ for every $x \in R$,

or $h_{i-1}(x) \in Cl(h_i(x))$ for every $x \in R$.

Proof: Let us take a homotopy mapping $h : X \times I \to Y$, $h(x, 0) = f(x)$, $h(x, 1) = g(x)$. Obviously $h(R \times I) = \bigcup_{x \in R} h((x) \times I)$ and hence finite. Let us define mappings $h_\lambda : R \to Y$ by formula $h_\lambda(x) = h(x, \lambda)$. Because of finiteness of $h(R \times I)$ we have only finite number of different mappings h_λ. Let $I_1, \dots, I_n$ be equivalence classes in I with respect to the equivalence relation defined as follows:

λ equivalent λ' iff $h_\lambda = h_{\lambda'}$.

Let I_{k_0} be the class containing zero and let us define $h_0 = h_\lambda, \lambda \in I_{k_0}$. Let us denote $a_0 = \sup I_{k_0}$. Then either $a_0 \in I_{k_0}$ or $a_0 \notin I_{k_0}$. In the first case either $a_0 = 1$ or $a_0 < 1$. If $a_0 = 1$ it is $f|R = g|R$. If $a_0 < 1$ there exists a class I_{k_1} such that a_0 is its condensation point and that there exists a point in I_k, lying behind a_0. Let us denote $h_1 = h_\lambda$, $\lambda \in I_{k_1}$. According to 3.6 we have $h_0(x) \in Cl(h_1(x))$ for every $x \in R$.

Let $a_0 \notin I_{k_0}$; in that case let us denote I_{k_1} the class containing a_0 and $h_1 = h_\lambda$, $\lambda \in I_{k_1}$. Because of $a_0 \in \overline{I}_{k_0}$ we get immediately that $h_1(x) \in Cl(h_0(x))$ for every $x \in R$.

Now let us assume h_{l-1} to be found, $h_{l-1} = h_\lambda$, $\lambda \in I_{k_{l-1}}$, and denote $a_{l-1} = \sup I_{k_{l-1}}$.

If $a_{l-1} \in I_{k_{l-1}}$, we have the two possibilities: either $a_{l-1} = 1$ and there is no need of further proof, or $a_{l-1} < 1$.

In the second case there exists a class I_{k_l} such that $a_{l-1} \in \overline{I}_{k_l}$, and I_{k_l} contains a point which lies behind a_{m-1} . We define $h_l = h_\lambda$, $\lambda \in I_{k_l}$ and get $h_{l-1}(x) \in \mathcal{U}(h_l(x))$ for every $x \in R$.
If $a_{l-1} \notin I_{k_{l-1}}$ let us denote I_{k_l} the class containing a_{l-1} . Let us define $h_l = h_\lambda$, $\lambda \in I_{k_l}$ and we get $h_l(x) \in \mathcal{U}(h_{l-1}(x))$ for every $x \in R$. It is easy to see that our h_i never repeat (because of choice of h_i such that I_{k_i} contains always an element lying behind all elements of $I_{k_{i-1}}$). Hence we must, after a finite number of steps, get $h_m = g \mid R$.

3.8. __Theorem__: Let $f, g : X \to Y$ be homotopical mappings. Then $f_{*n} = g_{*n}$ for every n .

__Proof__: Let us take an $[\alpha] \in H_n(X), \alpha \in [\alpha], \alpha \in Z_n(X)$. Let R be the carrier of α, h_i mappings from the lemma 3.7 . Because of $\alpha \in Z_n(R)$ we can use the lemma 3.3 and we get

either $d_n D_n(\alpha) = (h_i)_n(\alpha) - (h_{i+1})_n(\alpha)$

or $d_n D_n(\alpha) = (h_{i+1})_n(\alpha) - (h_i)_n(\alpha)$.

Anyway, the cycles $(h_i)_n(\alpha)$ and $(h_{i+1})_n(\alpha)$ are homological for every i and we get immediately $f_{*n}([\alpha]) = g_{*n}([\alpha])$.

3.9. __Theorem__: Let $f, g : (X, A) \to (Y, B)$ be homotopical mappings of couples of discrete spaces. Then $f_{*n} = g_{*n}$ for every n .

__Proof__: Let us denote f', g' corresponding mappings $X \to Y$. f' and g' are also homotopical. Further let us denote $i : X \subset (X, A)$, $i' : Y \subset (Y, B)$. We have (see 3.8) $f'_n(\beta) - g'_n(\beta) \in B_n(Y)$ for every $\beta \in Z_n(X)$. i_n are epimorphisms and hence

$$f_{*n}([\alpha]) - g_{*n}([\alpha]) = [f_n(\alpha) - g_n(\alpha)] =$$
$$= [f_n\, i_n(\beta) - g_n\, i_n(\beta)] = [i'_n(f'_n(\beta) - g'_n(\beta))] =$$
$$= [i'_n\, d_{n+1}(\gamma)] = [d_{n+1}\, i_{n+1}(\gamma)] = 0 \ .$$

3.10. <u>Lemma</u>: Let $U \subset A \subset X$, $f: (X-U, A-U) \subset (X, A)$. Then $f_n: C_n(X-U, A-U) \longrightarrow C_n(X, A)$ are monomorphisms. If $Cl\, St(U) \subset A$, f_n are isomorphisms.

<u>Proof</u>: Let $[\alpha] \in C_n(X-U, A-U)$, $\alpha = \sum a_i(x_{i0}, \dots, x_{in}) \in C_n(X-U)$. We have $f'_n(\alpha) = \sum a_i(x_{i0}, \dots, x_{in})$, where $f': X-U \subset X$. Let $f_n([\alpha]) = 0$. Hence $f'_n(\alpha) \in C_n(A)$ and therefore $x_{ij} \in A$ for every i, j. Because of $\alpha \in C_n(X-U)$, no x_{ij} is an element of U. Hence $x_{ij} \in A-U$ and we get $\alpha \in C_n(A-U)$ and therefore $[\alpha] = 0$. Now, let $Cl\, St(U) \subset A$. We are going to prove that f_n are epimorphisms. At first we show that every elementary chain containing some element of U contains elements of A only. Let in $(x_0, \dots, x_n)$ $x_k \in U$.

If $i \leqq k$, $x_i \in Cl(x_k) \subset Cl(U) \subset Cl\, St(U) \subset A$.

If $i \geqq k$, $x_i \in St(x_k) \subset St(U) \subset Cl\, St(U) \subset A$.

Now let us take an $\alpha = \sum a_i(x_{i0}, \dots, x_{in})$ and decompose it in $\alpha' + \alpha''$, $\alpha' = \sum a'_i(x'_{i0}, \dots, x'_{in})$, $\alpha'' = \sum a''_i(x''_{i0}, \dots, x''_{in})$, where $(x'_{i0}, \dots, x'_{in})$ are all elementary chains of α which do not contain any element of U, $(x''_{i0}, \dots, x''_{in})$ are remaining ones. Hence $\alpha'' \in C_n(A)$ and therefore $[\alpha'] = [\alpha]$. Let us denote β the chain α' as an element of $C_n(X-U)$. We have $f'_n(\beta) = \alpha'$ and therefore $f_n([\beta]) = [f'_n(\beta)] = [\alpha'] = [\alpha]$.

3.11. <u>Theorem:</u> Let

$$Cl\, St(U) \subset A \subset X, \quad f: (X-U, A-U) \subset (X, A).$$

Then f_{*n} are isomorphisms.

Proof: f_{*n} are monomorphisms:

Let $f_{*n}([\alpha]) = 0$, $\alpha \in [\alpha] \in H_n(X-U, A-U)$, $\alpha \in Z_n(X-U, A-U)$, i.e. $f_n(\alpha) \in B_n(X, A)$. Hence there exists an element $\beta \in C_{n+1}(X, A)$ such that $f_n(\alpha) = \bar{d}_{n+1}(\beta)$. Hence

$$f_n(\alpha) = \bar{d}_{n+1} f_{n+1} f_{n+1}^{-1}(\beta) = f_n \bar{d}_{n+1}(f_{n+1}^{-1}(\beta))$$

and therefore $\alpha = \bar{d}_{n+1}(f_{n+1}^{-1}(\beta))$, $[\alpha] = 0$.

f_{*n} are epimorphisms:

Let $[\gamma] \in H_n(X, A)$, $\gamma \in Z_n(X, A)$, $\gamma \in [\gamma]$. Let us denote $\alpha = f_n^{-1}(\gamma)$. It is $\alpha \in C_n(X-U, A-U)$, $f_n(\alpha) = \gamma$. Because of $f_{n-1} \bar{d}_n(\alpha) = \bar{d}_n f_n(\alpha) = \bar{d}_n(\gamma) = 0$, we have $\alpha \in Z_n(X-U, A-U)$ and therefore $f_{*n}([\alpha]) = [f_n(\alpha)] = [\gamma]$.

3.12. Remark: We see that we got the "excission axiom" in a slightly stronger form. We can define excission as an imbedding $f: (X-U, A-U) \subset (X, A)$, where $Cl\, St(U) \subset A$, though in formal translation of Eilenberg-Steenrod excission axiom we get the assumption U open and $St\, Cl(U) \subset A$.

§ 4.

4.1. Definition: Let X be a discrete space. The space $\mathscr{C}(X)$ is the set of all finite subsets $\{x_i\}$ of sets of type $Cl(y)$, $y \in X$, with topology defined by inclusion (i.e. $\alpha \in Cl(\beta) \iff \alpha \subset \beta$).

Let $f: X \to Y$ be a continuous mapping. We define the mapping $\mathscr{C}(f): \mathscr{C}(X) \to \mathscr{C}(Y)$ by formula $\mathscr{C}(f)(\{x_i\}) = \{f(x_i)\}$. (It is correct, because if $x_i \in Cl(y)$, $f(x_i) \in Cl(f(y))$ according to the continuity of f.)

If $A \subset X$ we have obviously $\mathscr{C}(A) \subset \mathscr{C}(X)$ and we can define $\mathscr{C}(X,A) = (\mathscr{C}(X), \mathscr{C}(A))$. Let $f: (X,A) \to (Y,B)$ be a continuous mapping. Then we have $\mathscr{C}(f')(\mathscr{C}(A)) \subset \mathscr{C}(B)$, where f' is the corresponding mapping $X \to Y$ and we can define $\mathscr{C}(f)$ in the natural way.

4.2. Theorem: $\mathscr{C}$ is a covariant h-functor from the category of couples of finite discrete spaces into itself.

Proof: The fact that $\mathscr{C}(f)$ is continuous (for f continuous) is obvious. The same for the facts about $\mathscr{C}$ being a covariant functor preserving pairs of mappings of a type $j: A \subset X$, $i: X \subset (X,A)$ and preserving one-point spaces. It remains to prove the homotopy and excission preserving.
Homotopy: Let $f, g: X \to Y$ be homotopical mappings. Because of 3.7 we can assume without loss of generality

(1) $f(x) \in Cl(g(x))$ for every $x \in X$.

We define $h: \mathscr{C}(X) \times I \to \mathscr{C}(Y)$ by formulae

$$h((\{x_i\}), 0) = \{f(x_i)\}$$

$$h((\{x_i\}), 1) = \{g(x_i)\}$$

$$h((\{x_i\}), t) = \{f(x_i)\} \cup \{g(x_i)\} \quad \text{for } 0 < t < 1 .$$

($\{f(x_i)\} \cup \{g(x_i)\}$ is an element of $\mathscr{C}(Y)$, for according to (1), $\{f(x_i)\} \cup \{g(x_i)\} \subset Cl(g(y))$, where y is such a point of X, that $\{x_i\} \subset Cl(y)$). The continuity of h is obvious.
Excission: Let $Cl\, St(U) \subset A \subset X$. We are going to prove that then $Cl\, St(\mathscr{C}(U)) \subset \mathscr{C}(A)$. Let $\{x_i\}_{i=1,\dots,m} \in Cl\, St(\mathscr{C}(U))$. Hence there is $\{x_i\}_{i=1,\dots,m} \in St(\mathscr{C}(U)), m \geq n$. ~~Hence at first there exists~~ some element $y \in X$, $x_i \in Cl(y)$ for every $i = 1, \dots, m$. Because of $\{x_i\}_{i=1,\dots,m} \in St(\mathscr{C}(U))$, there exists an

i_0, $x_{i_0} \in U$. Hence $\{x_i\}_{i=1,\dots,m} \subset Cl(y)$ and $y \in St(x_{i_0}) \subset A$ and therefore $\{x_i\}_{i=1,\dots,m} \in \mathcal{C}(A)$.

4.3. Corollary: We can define new homology theory from the category of couples of finite discrete spaces by formulae

$$H'_n(X, A; G) = H_n(\mathcal{C}(X), \mathcal{C}(A); G), \quad f_{\# n} = (\mathcal{C}(f))_{* n},$$
$$\partial'(n, X, A) = \partial(n, \mathcal{C}(X), \mathcal{C}(A)).$$

4.4. Theorem: There exist non-isomorphical homology theories from the category of couples of finite discrete spaces satisfying all seven Eilenberg-Steenrod axioms and having the same coefficient group.

Proof: The homology theory defined in 2.6 and the one defined in 4.3 satisfy Eilenberg-Steenrod axioms. Let us construct the space X, consisting of four points a, b, c and d with the topology defined as follows:

$$Cl(a) = \{a, c, d\}, \; Cl(b) = \{b, c, d\}, \; Cl(c) = (c), \; Cl(d) = (d).$$

It is only a matter of counting to prove that

$$H_0(X; G) \approx G, \; H'_0(X; G) \approx G, \; H_1(X; G) \approx G, \text{ but } H'_1(X, G) = 0.$$

4.5. Remark: In the following paragraphs we shall prove further important property of the homology theory from 2.6. We shall prove that the theorem of uniqueness for homology theories, satisfying Eilenberg-Steenrod axioms and having this property, does hold for the category of couples of finite discrete spaces.

§ 5.

5.1. Definition: Let X be a discrete space. In the set of all finite sequences of elements of X of a type $(x_0, \dots, x_n)$ with $x_i \in Cl(x_{i+1})$, $x_i \neq x_{i+1}$, we define a topology by inclusion. Let us denote $\mathcal{B}(X)$ the space obtained this way. Let us define a mapping $\varkappa : \mathcal{B}(X) \to X$ (more precisely $\varkappa_X$) by formula $\varkappa((x_0, \dots, x_n)) = x_n$.

5.2. Lemma: $\varkappa$ - is a continuous mapping onto.

Proof: The fact that $\varkappa$ is onto is obvious, because $\mathcal{B}(X)$ contains the sequences consisting of one element, too. Continuity: Let $\xi = (x_0, \dots, x_n) \in \mathcal{B}(X)$. Let $\xi' \in Cl(\xi)$. Hence $\xi' = (x_{k_0}, \dots, x_{k_l}), k_l \leqq n$ and therefore

$$\varkappa(\xi') = x_{k_l} \in Cl(x_n) = Cl(\varkappa(\xi))$$

5.3. Definition: Let $f : X \to Y$ be a continuous mapping. Let us define $\mathcal{B}(f) : \mathcal{B}(X) \longrightarrow \mathcal{B}(Y)$ by formula $\mathcal{B}(f)(x_0, \dots, x_n) = (f(x_0), \dots, f(x_n))'$ ($(y_0, \dots, y_n)'$ means the maximal strictly monotone subsequence of $(y_0, \dots, y_n)$; according to the continuity of f , $(f(x_0), \dots, f(x_n))'$ is really an element of $\mathcal{B}(Y)$). If $A \subset X$, we have $\mathcal{B}(A) \subset \mathcal{B}(X)$ and therefore we can define $\mathcal{B}(X, A) = (\mathcal{B}(X), \mathcal{B}(A))$. The mapping

$\varkappa_{(X,A)} : \mathcal{B}(X, A) \longrightarrow (X, A)$ is defined in the obvious way. If $f : (X, A) \longrightarrow (Y, B)$ is a continuous mapping of couples of discrete spaces, we have $\mathcal{B}(f)(\mathcal{B}(A)) \subset \mathcal{B}(B)$ and therefore we can define $\mathcal{B}(f)$ for such mappings in the natural way.

5.4. Theorem: $\mathcal{B}$ is a covariant functor from the category of couples of discrete spaces into itself. The system $\{\varkappa_{(X,A)}\}$ is a transformation of functor $\mathcal{B}$ into identical functor of our category.

Proof: Continuity of $\mathcal{B}(f)$ for continuous f and functorial properties of $\mathcal{B}$ are obvious. It remains to prove the commutativity of diagrams of a type

$$\begin{array}{ccc} \mathcal{B}(X,A) & \xrightarrow{\mathcal{B}(f)} & \mathcal{B}(Y,B) \\ \downarrow \varkappa_{(X,A)} & & \downarrow \varkappa_{(Y,B)} \\ (X,A) & \xrightarrow{f} & (Y,B) \end{array}$$

Let us take a $\xi = (x_0, \dots, x_n) \in B(X)$. We have

$$f \varkappa_{(X,A)}(\xi) = f(x_n),$$

$\varkappa_{(Y,B)} B(f)(\xi) = \varkappa_{(Y,B)}(f(x_0), \dots, f(x_n))' = f(x_n)$ q.e.d.

5.5. It is very easy to prove

Lemma: $\varkappa_m : C_m(\mathcal{B}(X,A)) \to C_n(X,A)$ is an epimorphism for every n .

5.6. Lemma: Let us define a homomorphism

$$s_n : C_n(\mathcal{B}(X)) \longrightarrow C_n(\mathcal{B}(X))$$

by formula

$s_n(g(\xi_0, \dots, \xi_n)) = g((\varkappa(\xi_0))', (\varkappa(\xi_0), \varkappa(\xi_1))', \dots, (\varkappa(\xi_0), \dots, \varkappa(\xi_n))')$

Then $\operatorname{Ker} s_n = \operatorname{Ker} \varkappa_n$.

Proof: We must at first prove that the definition of s_n is correct, i.e. to prove the fact that $(\varkappa(\xi_0), \dots, \varkappa(\xi_k))' \in \mathcal{B}(X)$. Let us take a non-negative integer $i < n$. $(\xi_0, \dots, \xi_n)$ is an elementary chain, hence $\xi_i \in \mathcal{Cl}(\xi_{i+1})$, i.e. $\xi_i \subset \xi_{i+1}$ and therefore $\varkappa(\xi_i) \in \mathcal{Cl}(\varkappa(\xi_{i+1}))$.

It holds:

(1) $\varkappa_n(g(\xi_0, \dots, \xi_n)) = \varkappa_n(g(\eta_0, \dots, \eta_n))$

iff (2) $\quad s_m(g(\xi_0,\dots,\xi_m)) = s_m(g(\eta_0,\dots,\eta_m))$

(because both (1) and (2) are equivalent with the assertion $\varkappa(\xi_i) = \varkappa(\eta_i)$ for $i = 0,\dots,n$).

We have to prove the equivalence:

$$(1') \quad \varkappa_m(\sum_{i=1}^{m} g_i(\xi_{i0},\dots,\xi_{in})) = 0 ,$$

$$\text{iff } (2') \quad s_m(\sum_{i=1}^{m} g_i(\xi_{i0},\dots,\xi_{in})) = 0 .$$

In the set $\{1,\dots,m\}$ let us define the equivalence relation by formula

(3) $i \sim j$ iff there exists a $g \neq 0$ such that $\varkappa_m(g(\xi_{i0},\dots,\xi_{in})) = \varkappa_m(g(\xi_{j0},\dots,\xi_{jn}))$.

Let us denote $I_0,\dots,I_k$ the equivalence classes. The same equivalence classes will be obtained if we substitute $\varkappa_m$ by s_m in (3) (see (1) and (2)).

Now, both (1′) and (2′) are equivalent with the assertion

$$\sum_{i\in I_j} a_i = 0 \qquad \text{for every } j .$$

5.7. __Lemma__: Let $(\xi_0,\dots,\xi_n)$ be an elementary chain. Let us define $\xi_k' = (\varkappa(\xi_0),\dots,\varkappa(\xi_k))'$. Then $\xi_k' \in Cl(\xi_k)$ for every k and $\varkappa(\xi_l') = \varkappa(\xi_k)$.

__Proof__: I. For $k = 0$, $(\varkappa(\xi_0))$ is obviously a subset of ξ_0 .

II. Let $\xi_{k-1}' \subset \xi_{k-1}$. Hence $\xi_{k-1}' \subset \xi_k$ and therefore $\xi_k' = (\xi_{k-1}', \varkappa(\xi_k))' \subset \xi_k$ (because of $\varkappa(\xi_k) \in \xi_k$).

5.8. __Lemma:__ Let us preserve the denotation of preceding lemma. According to this lemma we can define a homomorphism $D_n : C_n(\mathcal{B}(X)) \longrightarrow C_{n+1}(\mathcal{B}(X))$ by formula

$$D_n(g(\xi_0,\dots,\xi_n)) = \sum_{i=0}^{n} (-1)^i g(\xi_0',\dots,\xi_i',\xi_i,\dots,\xi_n) .$$

Then we have

$$d_1 D_0(\alpha) = \alpha - s_0(\alpha),$$

$$d_{n+1} D_n(\alpha) = \alpha - s_n(\alpha) - D_{n-1} d_n(\alpha) \quad (n \geqq 1),$$

and
$$\varkappa_n D_{n-1} d_n(\alpha) = -d_{n+1} \varkappa_{n+1} D_n(\alpha) \quad (n \geqq 1).$$

Proof: Is a matter of counting.

5.9. Theorem: $\varkappa_{*n} : H_n(\mathcal{B}(X,A)) \to H_n(X,A)$ are isomorphisms.

Proof: I. $\varkappa_{*n}$ is a monomorphism:

In this proof let us denote $\varkappa = \varkappa_{(X,A)}$, $\varkappa' = \varkappa_X$, $\varkappa'' = \varkappa_A$.

Let $\varkappa_{*n}([a]) = 0$, i.e. $\varkappa_n(a) \in B_n(X,A)$.

Hence there exists a $b \in C_{n+1}(X,A)$, $b = [\beta]$ such that $\bar{d}_{n+1}(b) = \varkappa_n(a)$. $a \in Z_n(\mathcal{B}(X,A))$; let $a = [\alpha]$;

hence we have $[d_{n+1}(\beta)] = [\varkappa'_n(\alpha)]$ and hence

$d_{n+1}(\beta) - \varkappa'_n(\alpha) = \gamma \in C_n(A)$.

$\varkappa_n$ is an epimorphism and hence there exists a $b' \in C_n(\mathcal{B}(X,A))$, $b' = [\beta']$ such that $b = \varkappa_{n+1}(b')$ and hence $[\beta] = [\varkappa'_{n+1}(\beta')]$ and therefore

(1) $\quad d_{n+1} \varkappa'_{n+1}(\beta') - \varkappa'_n(\alpha) - d_{n+1}(\gamma') - \gamma = 0.$

$\varkappa''_n$ is an epimorphism and therefore there exists a $\sigma \in C_n(\mathcal{B}(A))$ such that $d_{n+1}(\gamma') - \gamma' = \varkappa''_n(\sigma) = \varkappa'_n(\sigma)$. We get the formula (1) in the form

$$\varkappa'_n(d_{n+1}(\beta') - \alpha - \sigma) = 0$$

and therefore $\lambda = d_{n+1}(\beta') - \alpha - \sigma \in \operatorname{Ker} \varkappa'_n = \operatorname{Ker} s_n$. Hence we have

(A) if $n = 0$:

$d_1 D_0(\lambda) = \lambda$ and hence

$\bar{d}_1([D_0(\lambda)]) = [\lambda] = \bar{d}_1([\beta]) - [\alpha] \qquad ([\sigma] = 0).$

Hence $\quad a = \bar{d}_1(b - [D_0(\lambda)])$.

B) If $n > 0$:

$$d_{n+1} D_n(\lambda) = \lambda - D_{n-1} d_n(\lambda) \quad \text{and hence}$$

$$\bar{d}_{n+1}([D_n(\lambda)]) = [\lambda] - [D_{n-1} d_n(\lambda)] =$$

$$= [d_{n+1}(\beta)] - [\alpha] - [D_{n-1}(d_{n+1}(\beta) - \alpha - \delta))] =$$

$$= \bar{d}_{n+1}(b) - a - [D_{n-1}(-d_n(\alpha) - d_n(\delta))] .$$

Because of $a \in Z_n(\mathcal{B}(X,A))$, we have $d_n(\alpha) \in C_{n-1}(\mathcal{B}(A))$; obviously $d_n(\delta) \in C_{n-1}(\mathcal{B}(A))$, too. Because $D_i(C_i(\mathcal{B}(A)) \subset C_{i+1}(\mathcal{B}(A))$, we have $a = \bar{d}(b - [D_n(\lambda)])$ and hence $[a] = 0$.

II) $\varkappa_{*n}$ is an epimorphism:

Let $[b] \in H_n(X,A)$, $b \in Z_n(X,A)$, $b \in [b]$.

$\varkappa_n$ is an epimorphism and hence there is an $a \in C_n(\mathcal{B}(X,A))$ such that $\varkappa_n(a) = b$. Let $n = 0$. Then $C_0(\mathcal{B}(X,A)) = Z_0(\mathcal{B}(X,A))$ and hence

$$[a] \in H_0(\mathcal{B}(X,A)), \quad \varkappa_{*0}([a]) = [\varkappa_0(a)] = [b] .$$

Let $n > 0$, $a = [\alpha]$, $b = [\beta]$. We have $\bar{d}_n(b) = 0$, i.e. $d_n(\beta) \in C_{n-1}(A)$. We have $\varkappa_{n-1}(\bar{d}_n(a)) = \bar{d}_n \varkappa_n(a) = 0$ and hence $\varkappa'_{n-1} d_n(\alpha) \in C_{n-1}(A)$. Because $\varkappa''_{n-1}$ is an epimorphism, there exists a $\gamma \in C_{n-1}(\mathcal{B}(A))$ such that

$$d_n(\alpha) - \gamma \in \operatorname{Ker} \varkappa'_{n-1} = \operatorname{Ker} s_{n-1} .$$

Hence we have for $n = 1$:

$d_1 D_0(d_1(\alpha) - \gamma) = d_1(\alpha) - \gamma$ and therefore $\bar{d}_1([D_0(d_1(\alpha) - \gamma)]) = d_1(a)$. Let us denote $a' = a - [D_0(d_1(\alpha) - \gamma)]$. It is $\bar{d}_1(a') = 0$, i.e. $a' \in Z_1(\mathcal{B}(X,A))$, and $\varkappa_1(a') = \varkappa_1(a) - [\varkappa_1 D_0 d_1(\alpha)] =$
$= b - [d_2 \varkappa_2 D_1(\alpha)] = b - \bar{d}_2([\varkappa_2 D_1(\alpha)])$ and hence $\varkappa_{*1}([a']) = [b]$.

If $n > 1$, we have $d_n(D_{n-1}(d_n(\alpha) - \gamma)) =$

$= d_n(\alpha) - \gamma - D_{n-2}(d_{n-1}(d_n(\alpha) - \gamma)) = d_n(\alpha) - \gamma + D_{n-2}(d_{n-1}(\gamma))$.

On the other hand we have $\gamma \in C_{n-1}(\mathcal{B}(A))$, hence $d_{n-1}(\gamma) \in C_{n-2}(\mathcal{B}(A))$ and hence $D_{n-2}\, d_{n-1}(\gamma) \in C_{n-1}(\mathcal{B}(A))$. We get $\bar{d}_n([D_{n-1}(d_n(\alpha) - \gamma)]) = \bar{d}_n(a)$ and hence (because of $D_{n-1}(\gamma) \in C_n(\mathcal{B}(A))$) $\bar{d}_n(a - [D_{n-1}(d_n(\alpha))]) = 0$.

Now let us denote $a' = a - [D_{n-1}(d_n(\alpha))]$ and we get easily $\varkappa_{*n}([a']) = [b]$.

5.10. __Remark:__ Because of 5.4 we can formulate the preceding result in the following way: For every n, the system $\{(\varkappa_{(X,A)})_{*n}\}$ is a natural isomorphism between the functors $H_n \circ \mathcal{B}$ and H_n.

5.11. __Theorem:__ For every couple (X, A) of discrete spaces the commutativity holds in the diagram

$$\begin{array}{ccc} H_n(\mathcal{B}(X,A)) & \xrightarrow{(\varkappa_{(X,A)})_{*n}} & H_n(X,A) \\ \downarrow{\scriptstyle \partial(\mathcal{B}(X,A),n)} & & \downarrow{\scriptstyle \partial(X,A,n)} \\ H_{n-1}(\mathcal{B}(A)) & \xrightarrow{(\varkappa_A)_{*n-1}} & H_{n-1}(A) \end{array}$$

__Proof__: It is an easy consequence of commutativity of the diagrams

$$\begin{array}{ccc} C_n(\mathcal{B}(A)) & \xrightarrow{(\mathcal{B}j)_n} & C_n(\mathcal{B}(X)) \\ \downarrow{\scriptstyle (\varkappa_A)_n} & & \downarrow{\scriptstyle (\varkappa_X)_n} \\ C_n(A) & \xrightarrow{j_n} & C_n(X) \end{array} \quad \text{and} \quad \begin{array}{ccc} C_n(\mathcal{B}(X)) & \xrightarrow{(\varkappa_X)_n} & C_n(X) \\ \downarrow{\scriptstyle d_n} & & \downarrow{\scriptstyle d_n} \\ C_{n-1}(\mathcal{B}(X)) & \xrightarrow{(\varkappa_X)_{n-1}} & C_{n-1}(X) \end{array}$$

(where $j : A \subset X$), and the definition of ∂ (see, for example [E-S]).

5.12. __Remark:__ Let us agree to call, for further purposes, "the property (B)" the property of homology theory $\{H_{n,*}, \partial\}$, formulated in 5.10 and 5.11.

§ 6.

6.1. Definition: Let X be a triangulable space, T some triangulation of X. Let us denote $\mathcal{D}^T(X)$ the set of all simplexes of this triangulation with the topology defined by formula:

$$s' \in \mathrm{Cl}(s) \quad \text{iff } s' \text{ is a face of } s.$$

Let A be a subspace of X triangulated by T. Then $\mathcal{D}^T(A) \subset \mathcal{D}^T(X)$ and hence we can define $\mathcal{D}^T(X,A) = (\mathcal{D}^T(X), \mathcal{D}^T(A))$. Let $f: X \to Y$ be a simplicial mapping with respect to the triangulations T, U of X, Y respectively. We define $\mathcal{D}^{T,U}(f): \mathcal{D}^T(X) \to \mathcal{D}^U(Y)$ by formula $\mathcal{D}^{T,U}(f)(s) = f(s)$ (the image of the set). Obviously, $\mathcal{D}^{T,U}(f)$ is a continuous mapping. $\mathcal{D}^{T,U}(f)$ for a mapping f of couples of triangulated spaces is defined in the obvious way.

6.2. Theorem: Let $\{H_n, *, \partial\}$ be some homology theory from the category of couples of finite discrete spaces, satisfying all Eilenberg-Steenrod axioms and having the property Then there exists a homology theory $\{H'_n, *', \partial'\}$ from the category of triangulable couples which agree with $\{H_n, *, \partial\}$, i.e. for every triangulable couple (X,A) and for its every triangulation T there can be defined isomorphisms

$$i(X,A;T;n): H'_n(X,A) \approx H_n(\mathcal{D}^T(X,A))$$

such that the system $\{i(X,A;T,n)\}$ has the following properties:

(1) For every (X,A), T the commutativity holds in the rectangle

$$\begin{array}{ccc} H'_n(X,A) & \xrightarrow{i(X,A;T)} & H_n(\mathcal{D}^T(X,A)) \\ \downarrow \partial' & & \downarrow \partial \\ H'_{n-1}(A) & \xrightarrow{i(A;T)} & H_{n-1}(\mathcal{D}^T(A) \end{array}$$

2) For every $f:(X,A)\to(Y,B)$ simplicial with respect to triangulations T, U the commutativity holds in the rectangle

$$\begin{array}{ccc} H'_n(X,A) & \xrightarrow{i(X,A;T)} & H_n(\mathcal{D}^T(X,A)) \\ \Big\downarrow f_{*n} & & \Big\downarrow (\mathcal{D}^{T,U}(f))_{*n} \\ H'_n(Y,B) & \xrightarrow{i(Y,B;U)} & H_n(\mathcal{D}^U(Y,B)) \end{array}$$

Proof: Let us construct the Čech homology theory from the category of triangulable couples in the way that we use our homology theory for nerves (of finite coverings) taken as discrete spaces. It is easy to show that the nerve of covering of (X,A) by stars of edges in the triangulation T is homeomorphic with $\mathcal{D}^T(X,A)$. Further, it is easy to show that we have $\mathcal{D}^U(X,A)$ homeomorphic with $\mathcal{B}\mathcal{D}^T(X,A)$ if U is the barycentrical subdivision of T, and that the corresponding projection, induced by projection of nerves, is homotopical to the mapping $\varkappa$ from § 5. Finally, it is obvious that the system of coverings by stars of edges in barycentrical subdivisions of a given triangulation is confinal in the directed set of all coverings. For wanted isomorphism i we can take the projection from the limit-group in the construction to the group of covering by stars of edges. The commutativity relations are not difficult to prove.

6.3. Theorem: For every two homology theories $\{H_n, *, \partial\}$, $\{\overline{H}_n, *, \overline{\partial}\}$ from the category of couples of finite discrete spaces, satisfying all Eilenberg-Steenrod axioms and having the property (B), and for every homomorphism $h_0 : H_0(P)\to\overline{H}_0(P)$ (where P is a one-point space) there exists a system of homomorphisms

$$h(n, X, A) : H_n(X, A) \longrightarrow \overline{H}_n(X, A)$$

such that:

(1) $h(0, P) = h_0$

(2) For every continuous mapping $f : (X, A) \to (Y, B)$ the commutativity holds in the rectangle

$$\begin{array}{ccc} H_n(X,A) & \xrightarrow{h(n,X,A)} & \overline{H}_n(X,A) \\ \downarrow f_{*n} & & \downarrow f_{\#n} \\ H_n(Y,B) & \xrightarrow{h(n,Y,B)} & \overline{H}_n(Y,B) \end{array}$$

(3) For every couple (X, A) the commutativity holds in the rectangle

$$\begin{array}{ccc} H_n(X,A) & \xrightarrow{h(n,X,A)} & \overline{H}_n(X,A) \\ \downarrow \partial & & \downarrow \partial \\ H_{n-1}(A) & \xrightarrow{h(n-1,A)} & \overline{H}_{n-1}(A) \end{array}$$

If h_0 is an isomorphism, so is every $h(n, X, A)$.

Proof: For every couple (X, A) of finite discrete spaces let us take some triangulated couple $\mathcal{K}(X, A)$ with triangulation T such that $\mathcal{D}^T(\mathcal{K}(X, A))$ is homoemorphical with $\mathcal{B}(X, A)$. (Such a $\mathcal{K}(X, A)$ obviously exists, moreover, for every continuous mapping f the mapping $\mathcal{B}(f)$ can be represented as a simplicial mapping of corresponding triangulated pairs.)

Now, let us construct for $\{H_n, *, \partial\}$, $\{\overline{H}_n, \#, \overline{\partial}\}$ the homology theories $\{H'_n, *', \partial'\}, \{\overline{H}'_n, \#', \overline{\partial}'\}$ (see 6.2) from the category of triangulable couples agreeing with the given ones. According to the theorem 10.1 from [E-S, chapter III] there exists a system of homomorphisms $\{h'(n, X, A)\}$

for $h_o' = \varkappa_{*o}^{-1} (\bar{\iota}(P_o; 0))^{-1} h_o (i(P_o; 0)) \varkappa_*$ (i , $\bar{\iota}$ are the isomorphisms from 6.2), having corresponding properties.

If we take now

$$h(n, X, A) = (\varkappa_{(X,A)})_{*n} \bar{\iota}(\mathcal{K}(X,A), n)(h'(\mathcal{K}(X,A), n))(i(\mathcal{K}(X,A), n))^{-1} (\varkappa_{(X,A)})_{*n}^{-1}$$

all assertions can be easily verified.

Bibliography

[E-S] EILENBERG - STEENROD, Foundations of Algebraic Topology, Princeton Univ. Press, 1962.