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On the logarithmic potential

ON THE LOGARITHMIC POTENTIAL

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I n t r o d u c t o r y   r e m a r k. Suppose we are
given a simple oriented curve C of finite length and a
continuous function F on it. It is well known that the
corresponding potential of the double distribution can be
defined by the formula

$$W_C (z; F) = \mathcal{I}m \int_C \frac{F(\xi)}{\xi - z} \, d\xi, \; z \in E_2 - C .$$

Investigation of the behaviour of $W_C (z; F)$ (as well as
of the corresponding Cauchy's integral) as z approaches C
is of importance for a number of applications. Accordingly,
solution of the following problems I – III seems to be of
interest.

P r o b l e m   I. Fix a point $\xi \in C$ and suppose
that

$$C \cap \{z; \; z = \xi \pm \rho \, exp \, i \, \vartheta, \; 0 < \rho < R\} = \emptyset$$

for every

$$\vartheta \in (\vartheta_0 - \delta, \vartheta_0 + \delta) = \{\vartheta; \; \vartheta_0 - \delta < \vartheta < \vartheta_0 + \delta\} \; (\delta > 0).$$

Put $\mathcal{U} = \{z; \; z = \xi + \rho \, exp \, i \, \vartheta_0, \; 0 < \rho < R\}$.

Find a necessary and sufficient condition to secure the e-
xistence of

$$\lim_{\substack{z \to \xi \\ z \in \mathcal{U}}} W_C (z; F)$$

for every continuous function F on C .

P r o b l e m   II. Suppose that C is a simple clo-
sed curve and write G for its bounded complementary

domain. What (necessary and sufficient) restrictions are to be imposed on C that $W_C(z; F)$ be uniformly continuous on G for every continuous distribution F on C? Or, which is the same: Under what conditions $W_C(z; F)$ (so far considered for $z \in G$ only) can be extended to a continuous function on $\overline{G} = G \cup C$ whenever F is continuous on C?

The problem II having been settled, one can consider the operator

$$TF(\xi) = \sigma F(\xi) - \lim_{\substack{z \to \xi \\ z \in G}} W_C(z; F)$$

on the Banach space B of all continuous functions F on C equipped with the norm $\|F\| = \max_{\xi \in C} |F(\xi)|$; here $\sigma = \pi \; (-\pi)$ provided C is positively (negatively) oriented respectively. In connection with the classical Fredholm´s method for solution of the Dirichlet problem it is useful to have an expression (or, at least, some estimates) for the quantity

$$(1) \qquad \omega = \inf_L \|T - L\|,$$

L ranging over the system of all completely continuous linear operators acting on B . (ω^{-1} is so-called Fredholm´s radius of T .) Thus we arrive at the following

P r o b l e m III. Find an expression for ω making clear its dependence of the shape of C .

It is the purpose of the present paper to show that methods of Real Variables make it possible to solve the problems quoted above. The problem I will be treated in a slightly more general fashion for path-curves with any num-

ber of self-intersections. We shall introduce certain geo-
metric indicatrices and variations (which can also be des-
cribed purely analytically) and announce some theorems sol-
ving I - III in terms of them.

Definition 1. Let φ be a continuous complex-
valued function on $\langle a, b \rangle = \{t; t \in E_1, a \le t \le b\}$
and let f be a real-valued function on $\langle a, b \rangle$. Given
a point $z \in E_2 - \varphi(\langle a, b \rangle)$ we fix a continuous
real-valued function ϑ_z on $\langle a, b \rangle$ with

(2) $\qquad |\varphi(t) - z| \exp i \vartheta_z(t) = \varphi(t) - z, \quad a \le t \le b,$

and define

(3) $\qquad w_\varphi(z; f) = \int_a^b f(t) \, d \vartheta_z(t)$

provided the Stieltjes integral on the right-hand side e -
xists.

Remark . The definition (3) is independent of the
choice of ϑ_z fulfilling (2).

Definition 2. Let φ have the same mean-
ing as in the definition 1. Fix $\xi \in E_2$ and define on
$\langle 0, 2\pi \rangle$ the function $\mu^\varphi(\alpha; \xi)$ of the variab-
le α as follows: For $\alpha \in \langle 0, 2\pi \rangle$ put

$\mu^\varphi(\alpha; \xi) = n$ (where $n \ge 0$ is an integer)
if and only if the path-curve φ meets the half-line

$\{z; z = \xi + \rho \exp i \alpha, \rho > 0\} = P_\xi^\alpha$ exactly n -times.
Further put $\mu^\varphi(\alpha; \xi) = +\infty$ if φ meets P_ξ^α
infinitely many times. Thus

$\mu^\varphi(\alpha; \xi) (0 \le \mu^\varphi(\alpha; \xi) \le +\infty)$ is equal to the num-
ber of points in $\varphi^{-1}(P_\xi^\alpha)$ Similarly, define for

- 5 -

any $\kappa > 0$ the function $\mu_\kappa^\varphi(\alpha; \xi)$ of the varia-
ble α on $\langle 0, 2\pi \rangle$ as follows: For every
$\alpha \in \langle 0, 2\pi \rangle$, $\mu_\kappa^\varphi(\alpha; \xi)$ $(0 \leqq \mu_\kappa^\varphi(\alpha; \xi) \leqq + \infty)$
is equal to the number of points in

$$\varphi^{-1}(\rho_\xi^\alpha \cap \{z; |z - \xi| < \kappa\}).$$

R e m a r k . It can be proved that (for fixed φ and
ξ) the functions $\mu^\varphi(\alpha; \xi)$, $\mu_\kappa^\varphi(\alpha; \xi)$ $(\kappa > 0)$
are measurable. Therefore we may introduce the following

N o t a t i o n .

$$\nu^\varphi(\xi) = \int_0^{2\pi} \mu^\varphi(\alpha; \xi) \, d\alpha,$$

$$\nu_\kappa^\varphi(\xi) = \int_0^{2\pi} \mu_\kappa^\varphi(\alpha; \xi) \, d\alpha,$$

the integrals on the right-hand side being taken in the sen-
se of Lebesgue.

D e f i n i t i o n 3. The meaning of φ, ξ is the
same as in the definition 2. For any $\rho > 0$ denote by
$\nu^\varphi(\rho; \xi)$ $(0 \leqq \nu^\varphi(\rho; \xi) \leqq + \infty)$ the number
of points in $\{t; t \in \langle a, b \rangle, |\varphi(t) - \xi| = \rho\}$. The
function $\nu^\varphi(\rho; \xi)$ (φ, ξ are fixed) is measu-
rable on $(0, + \infty)$ and, consequently, the Lebesgue in-
tegrals

$$\int_0^\infty \nu^\varphi(\rho; \xi) \, d\rho = \mu^\varphi(\xi),$$

$$\int_0^\kappa \nu^\varphi(\rho; \xi) \, d\rho = \mu_\kappa^\varphi(\xi) \quad (\kappa > 0)$$

are available.

Now we are able to announce the following

T h e o r e m 1. Let φ be a continuous complex-va-
lued function on $\langle a, b \rangle$, $\xi \in \varphi (\langle a, b \rangle)$. Put

$$\mathcal{U} = \{ z; \ z = \xi + \rho \exp i \vartheta_0, \ \ 0 < \rho < R \}$$. Suppose

that the set $\varphi^{-1}(\xi)$ is finite and that there exists a
$\delta > 0$ such that $\varphi(\langle a,b \rangle) \cap \{ z ; z = \xi \pm \rho \exp i \vartheta, 0 < \rho < R \} = \emptyset$
whenever $|\vartheta - \vartheta_0| < \delta$. If

$$\lim_{\substack{z \to \xi \\ z \in \mathcal{U}}} w_{\varphi}^{\cdot}(z; f)$$

exists for every continuous (real-valued) function f on
$\langle a, b \rangle$, then

(4) $v^{\varphi}(\xi) < + \infty ,$

(5) $\sup_{\varkappa > 0} \varkappa^{-1} u_{\varkappa}^{\varphi}(\xi) < + \infty .$

The converse of this theorem is also true. More precise-
ly, we have the following

T h e o r e m 2. Let φ be a continuous complex-valu-
ed function on $\langle a, b \rangle$, $\xi \in \varphi (\langle a, b \rangle)$ and suppo-
se that the set $\varphi^{-1}(\xi) = \{ t_1 < \ldots < t_n \}$ is finite.
If (4) holds, then there exist the limits

$$\lim_{t \to t_k +} \frac{\varphi(t) - \varphi(t_k)}{|\varphi(t) - \varphi(t_k)|} = \tau^{\varphi}(t_k +) \quad \text{whenever } t_k < b,$$

$$\lim_{t \to t_k -} \frac{\varphi(t) - \varphi(t_k)}{|\varphi(t) - \varphi(t_k)|} = \tau^{\varphi}(t_k -) \quad \text{whenever } t_k > a.$$

For the sake of simplicity, let us agree to write
$\tau^{\varphi}(t_1 -) = \tau^{\varphi}(a +)$ in the case $t_1 = a$, $\tau^{\varphi}(t_n +) = \tau^{\varphi}(b -)$
in the case $t_n = b$. If, moreover, the condition (5)
takes place, then, for every continuous function f on
$\langle a, b \rangle$, $w_{\varphi}(\xi + \varkappa \xi; f)$ tends to a limit as

$\kappa \to 0+$ uniformly with respect to ξ on any compact

set
$$K \subset \{ \xi ; \xi \in E_2 , |\xi| = 1 \} - \bigcup_{k=1}^{n} \{ \tau^{\varphi}(t_k -), \tau^{\varphi}(t_k +) \} .$$

R e m a r k . If $\varphi^{-1}(\xi) \subset (a, \ell)$, in the above

theorem, $\lim\limits_{\kappa \to 0+} w_{\varphi}(\xi + \kappa \xi ; f)$ is constant on every

component of $\{ \xi ; |\xi| = 1 \} - \bigcup_{k=1}^{n} \{ \tau^{\varphi}(t_k -), \tau^{\varphi}(t_k +) \}$. The same

is true provided $\varphi(a) = \varphi(\ell)$

R e m a r k . If φ is a rectifiable path-curve on

$\langle a, \ell \rangle$, then the set of all $\xi \in \varphi(\langle a, \ell \rangle)$ with

$\sup\limits_{\kappa > 0} \kappa^{-1} u_{\kappa}^{\varphi}(\xi) = +\infty$ is of (Hausdorff) linear mea-

sure zero. On the other hand, example can be given of a sim-

ple rectifiable path-curve φ on $\langle a, \ell \rangle$ such that

the set $\{ \xi ; \xi \in \varphi(\langle a, \ell \rangle), v^{\varphi}(\xi) = +\infty \}$ is of positive

linear measure. This, of course, does not mean, that there

exists a continuous function F on $C = \varphi(\langle a, \ell \rangle)$

such that non-tangential limits of $W_C(z; F)$ do not

exist on a set of positive linear measure. (In fact, the con-

trary is known to be true.)

By theorems 1, 2 the problem I is solved. Let us now

proceed to the problem II.

N o t a t i o n . From now on we shall assume that φ

is a complex-valued function on $\langle a, \ell \rangle$ such that

$\varphi(a) = \varphi(\ell)$ and $\varphi(t_1) \neq \varphi(t_2)$ whenever

$$0 < |t_1 - t_2| < \ell - a, \ t_1 , t_2 \in \langle a, \ell \rangle .$$

The symbol C will be used to denote the set $\varphi(\langle a, \ell \rangle)$

as well as the oriented curve determined by φ . For any con-

tinuous real-valued function F on C and any $z \in E_2 - C$

we put $W_C(z; F) = w_\varphi(z; f)$, where
$$f(t) = F(\varphi(t)), \quad a \le t \le b.$$

This is, clearly, in accordance with the notation used in the above remarks. Further we shall write G for the bounded complementary domain of C and B for the Banach space of all continuous real-valued functions F on C with the usual norm $\|F\| = \max_{\xi \in C} |F(\xi)|$.

 T h e o r e m 3. Suppose that $W_C(z; F)$ is uniformly continuous on G whenever F is continuous on C . Then

(6) $\quad \sup_{\xi \in C} v^\varphi(\xi) < +\infty$,

and (since $v_r^\varphi(\xi) \leqq v^\varphi(\xi)$) $\sup_{\xi \in C} v_r^\varphi(\xi) < +\infty$ for every $r > 0$.

 Conversely, the following theorem holds.

 T h e o r e m 4 . Suppose that ,
$$\sup_{\xi \in C} v_r^\varphi(\xi) < +\infty$$
for a certain $r > 0$. Then (6) holds and, for every continuous function F on C , $W_C(z; F)$ $(z \in G)$ can be extended to a continuous function on $G \cup C$. The operator T on B defined by
$$TF(\xi) = \sigma F(\xi) - \lim_{\substack{z \to \xi \\ z \in G}} W_C(z; F), \quad \xi \in C$$
($\sigma = \pm \pi$ according to whether C is positively or negatively oriented) is bounded and its norm is equal to $\sup_{\xi \in C} v^\varphi(\xi)$.

 As to the problem III, the following theorem can be proved.

- 9 -

T h e o r e m 5 . Suppose that (6) holds and let ω have the same meaning as in (1). Then

$$\omega = \lim_{z \to 0+} \sup_{\xi \in C} v_z^{\varphi}(\xi).$$

Proofs of the above theorems together with further results in this direction will appear later.