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ON INTEGRATION IN COMPACT METRIC SPACES

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Theorem. Let μ be a finite Borel measure on a compact metric space X. Then there exist $x_k \in X$, $k = 1,2\ldots$, such that, for any continuous function f,

$$\int_X f d\mu = \mu(X) \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} f(x_k)$$

Proof. Clearly we may suppose that $\mu(X) = 1$ and $\mu(G) > 0$ for any open non-void $G \subset X$. It is easy to prove that there exist finite closed coverings

$\mathcal{A}_m = \{A_1^m, \ldots, A_r^m\}$, $m = 1, 2, \ldots$ such that

(1) $\mathcal{A}_{m+1}$ refines $\mathcal{A}_m$,

(2) $d(\mathcal{A}_m) \to 0$ where $d(\mathcal{A}_m)$ denotes the maximum of diameters of A_j^m,

(3) $\mu(A_i^m \cap A_j^m) = 0$ for $i \neq j$,

(4) $\mu(A_j^m) > 0$ for any m, j;

then, evidently,

(5) for any $m, m', m > m'$, there exists, for every $i = 1, \ldots, r_m$, exactly one $j = j(i)$ with $A_i^m \subset A_j^{m'}$. Moreover, we may suppose that

(6) $i_1 < i_2$ implies $j(i_1) < j(i_2)$.

Denote by B_j^m, $m = 1, 2, \ldots$, $j = 1, \ldots, r_m$, the closed interval with endpoints $\sum_{i=1}^{j-1} \mu(A_i^m)$, $\sum_{i=1}^{j} \mu(A_i^m)$. For any set $M \subset \langle 0, 1 \rangle$, denote by χ_M the characteristic function of M; that is, χ_M is defined on $\langle 0, 1 \rangle$, equal to 1 on M, and to 0 on its complement. It is well known that there exists a sequence $\{\xi_k\}$, $0 \leqq \xi_k \leqq 1$, such that, for any $0 \leqq \alpha \leqq \beta \leqq 1$,

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \chi_{\langle \alpha, \beta \rangle} (\xi_k) = \beta - \alpha$$

It is easy to see that ξ_k may be chosen to be distinct from any endpoint of the intervals B_j^m.

For a given k, there exists, for any m, exactly one $j = j(m)$ such that $\xi_k \in B_j^m$; since $A_{j(m+1)}^{m+1} \subset \subset A_{j(m)}^m$ (this follows from the above property (6)), the intersection C_k of all $A_{j(m)}^m$ is non-void. Now, choose a point x_k from every C_k. We are going to show that x_k possess the required properties.

For any $Y \subset X$, put

$$\mu^*(Y) = \limsup_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \chi_Y (x_k),$$

$$\mu_*(Y) = \liminf_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \chi_Y (x_k),$$

where χ_Y is the characteristic function of Y (i.e. $\chi_Y(x) = 1$ if $x \in Y$, $\chi_Y(x) = 0$ if $x \in X - Y$).

Denote by $\mathfrak{X}$ the collection of those $Z \subset X$ which can be represented, for some m, as union of certain A_j^m. Clearly, $Z \in \mathfrak{X}$ implies $\mu_*(Z) \geqq \mu(Z)$. On the other hand, for any closed $F \subset X$ and any $\varepsilon > 0$, there exists $Z \in \mathfrak{X}$ such that $F \cap Z = \emptyset$, $\mu(Z) > 1 - \mu(F) - \varepsilon$. Hence

$$\mu^*(F) \leqq 1 - \mu_*(Z) \leqq 1 - \mu(Z) < \mu(F) + \varepsilon$$

Therefore $\mu^*(F) \leqq \mu(F)$ for any closed F, and

$$\mu^*(Z) = \mu_*(Z) = \mu(Z) \quad \text{for } Z \in \mathcal{Z} .$$

Moreover,

$$\mu^*(F) = 0 \quad \text{whenever } F \text{ is closed, } \mu(F) = 0 .$$

It follows that

$$\int_X g \, d\mu = \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} g(x_k)$$

for any function g which is, for some m, constant in the interior of each A_j^m. This concludes the proof since every continuous function can be uniformly approximated by functions of this kind.