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A METHOD FOR IMPROVING THE CONVERGENCE OF ITERATION SEQUENCES

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If a sequence of approximations, for a dominant eigenvalue of a suitable operator is known, then it is possible to improve the convergence of the iteration process for the solution of a linear equation in a Banach space

(1) $$Ax = y.$$

The considered "speeding up" method is a generalization of the method suggested by L.A. Lusternik [1] for speeding up the convergence of the iteration process, by means of which systems of linear algebraic equations can be solved.

Let X be a complex Banach space, X' the adjoint space of linear forms, $X_1 = (X \to X)$ the space of linear continuous transformations of the space X into itself. It is well known that if $A \in X_1$, then $A^{-1} \in X_1$, if and only if such a $P \in X_1$ exists, that $P^{-1} \in X_1$ and (I is the identity operator)

(2) $$\| I - PA \| = q < 1 .$$

It is also known, that the iterations

(3) $$x_{n+1} = Tx_n + Pf ,$$

where

(4) $$T = I - PA ,$$

converge to the solution x of the equation (1) and that the estimation

$$\| x_{n+1} - x \|_X = O(R_T^n)$$

holds for the error. Here R_T is the spectral radius of the operator T.

Let operator T have dominant eigenvalue μ_0 ,i.e. let

(5) $$|\lambda| < |\mu_0| \quad \text{for} \quad \lambda \in \sigma(T),\ \lambda \neq \mu_0,$$

where $\sigma(T)$ is the spectrum of operator T .

Let the inequality

(6) $$|\mu_n - \mu_0| \leq c \left|\frac{\mu}{\mu_0}\right|^n$$

hold for the terms of the sequence $\{\mu_n\}$, where μ is the radius of the smallest circle, in which the whole spectrum $\sigma(T)$ except the point μ_0 lies. We construct the iteration process

(7) $$\hat{x}_{n+1} = \frac{1}{1-\mu_n}\left(x_{n+1} - \mu_n x_n\right),$$

where x_n, x_{n+1} are terms of the process (3).

Theorem 1. The sequence $\{\hat{x}_n\}$ defined in (7) converges in the norm of space X to the solution x of equation (1) and the following estimation of the error holds

(8) $$\|\hat{x}_{n+1} - x\| \leq c_1 \frac{1}{|1-\mu_0|} \sup_n \frac{1}{|1-\mu_n|} \mu^n = O(\mu^n).$$

Remark. The position of the point μ_0 with respect to a unity circle with its centre in the origin influences the speed of the convergence of the sequence (7). If μ_0 would lie near to the value 1 the speed of the convergence could be spoiled by the large factors $|1-\mu_0|^{-1}$, $|1-\mu_n|^{-1}$. So as to get rid of such inconvenient influences on the convergence we can use the sequence

(9) $$\tilde{x}_n = \frac{1}{1-\mu_n^p}\left(x_{n+p} - \mu_n^p x_n\right)$$

instead of the sequence (7), where the x_{n+p}, x_n , similarly as in (7) are terms of the sequence (3). It is clear, that for p large enough the mentioned difficulty disappears.

Theorem 2. Under the assumptions of theorem 1 the sequence (9) converges in the norm of the space X to the solution x of the equation (1) and we have the following estimation

$$\| \tilde{x}_n - x \| \leqq c_2 \frac{1}{|1-\mu_0^r|} \sup_n \frac{1}{|1-\mu_m^r|} \mu^n ,$$

where r is a fixed natural number.

One can also use an iteration process to construct the sequence $\{\mu_m\}$. Let x'_n, y'_n, z'_n, x', y' be elements of space X' and let the equations

$$x'(x) = \lim_{n\to\infty} x'_n(x),$$

(10)

$$y'(x) = \lim_{n\to\infty} y'_n(x) = \lim_{n\to\infty} z'_n(x)$$

hold for every $x \in X$.

Let

$$R(\lambda,T) = \sum_{k=0}^{\infty} (\lambda-\mu_0)^k T_k + \sum_{k=1}^{\infty} (\lambda-\mu_0)^{-k} B_k$$

be a Laurent series for the resolvent $R(\lambda,T) = (\lambda I - T)^{-1}$ of the operator T in the neighborhood of the point μ_0. It is well known that

$$B_1 = \frac{1}{2\pi i}\int_{C_0} R(\lambda,T)\,d\lambda, \quad B_{k+1} = (T-\mu_0 I)\,B_k, \; k=1,2,\ldots,$$

where C_0 is the boundary of the circle in which only the one point μ_0 of the spectrum $\sigma(T)$ lies.

We assume that $x^{(0)} \in X$ fulfills the condition

$$B_1 x^{(0)} \neq \sigma$$

and that such an index $s \geqq 1$ exists that

(11) $$B_s x^{(0)} \neq \sigma, \quad B_{s+1} x^{(0)} = \sigma .$$

Further let

(12) $$x'(B_s x^{(0)}) \neq 0, \quad y'(B_s x^{(0)}) \neq 0 .$$

Let

(13) $$x^{(n)} = T x^{(n-1)}, \quad x_{(n)} = \frac{x^{(n)}}{x'_n(x^{(n)})},$$

(14) $$\mu_n = \frac{z'_n(x^{(n+1)})}{y'_n(x^{(n)})} .$$

Theorem 3. If the operator T has the dominant eigenvalue μ_0 and x_0 is the corresponding eigenvector, then the sequence (13) converges in the norm of space X to the vector x_0 and the numerical sequence (14) converges to μ_0.

If μ_0 is a simple pole of the resolvent $R(\lambda, T)$, the estimation (6) is correct.

If the eigenvalue μ_0 is positive then the sequence of linear forms in process (13), (14) can be replaced by sequences of seminorms.

Reference

[1] L.A.LJUSTĚRNIK: Zamečanija k čislennomu rešeniju krajevych zadač uravněnija Laplasa i vyčislenijam sobstvennych značenij metodom setok. Trudy Mat.Inst.im.V.A. Steklova AN SSSR, XX(1947), 49 - 64.