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ON ITERATIONS OF BOUNDED LINEAR OPERATORS AND KELLOG'S ITERATIONS IN NOT SELF - ADJOINT EIGENVALUE PROBLEMS

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1. Kellog's method is one of the method of determining the eigenvectors of linear operators in Hilbert space. Since Kellog's method is very simple, it is useful for practical calculations. When solving homogenous problems, in contrary to the solving of the absolute majority of unhomogenous functional equations, one usually cannot go over to the respective self - adjoint equations, which is equivalent to the original equation. The problem of finding an eigenvector of a generally not self - adjoint operator cannot be reduced to the problem of finding the eigenvector of a self - adjoint operator. Thus any extension of the class of operators, for which Kellog's iteration is applicable, is important. One can divide the conditions, which satisfy operators, for which Kellog's iterations converge, into two categories. The conditions which refer to the operator itself, belong to the first category; those which refer to the space in which the operator is investigated, belong to the second one. One can expect that operators in special spaces will satisfy more general conditions and vice versa. In finite - dimensional spaces, for instance, Kellog's iteration series converges to the generalized eigenvector, corresponding to the dominant eigenvalue, for arbitrary linear bounded operators ([1], [5]). The convergence of Kellog's iteration series has been investigated mainly in Hilbert spaces ([9], [15], [16]). The class of operators, for which Kellog's method is correct, was extended from compact linear symmetric operators, to compact symmetrizable operators and from there either to compact operators ([14]), or to bounded symmetrizable operators ([15], [16], [12], [13]).

The convergence of Kellog's iteration series ensures only the existence of a dominant eigenvalue of a linear bounded operator. Thus the class of operators, for which

Kellog's method is applicable, is extended in both mentioned directions: Kellog's iterations converge for bounded operators in any Banach space.

One can even use Kellog's method for some unbounded operators, by passing over to inverse operators. Something similar is valid for the construction of the solution of the equation

$$Lx = \lambda Bx \qquad (1)$$

where L is generally an unbounded operator. It is not necessary to construct the inverse operator L^{-1}, since it is sufficient to know the solutions of the equation $Lu = f$ for special right sides $f \in \mathfrak{X}$. This is the case of modified Kellog's iterations. Of course, convergent iteration series for finite type spectral operators exist, the convergence of which follows from the convergence of the respective Kellog's iteration series. The methods, which I.A.Birger states without proof of convergence in [2], and the convergence of which is proved by J.Kolomý in [6], belong to these. The Birger and Kolomý methods can be adapted for solving equations of the typ (1), whereby one obtains modified Birger / Kolomý iterations.

2. Let $\mathfrak{X}$ be a Banach space. We use the symbol $\mathfrak{X}_1$ for the Banach space of linear bounded operators, mapping space $\mathfrak{X}$ into itself. We will use small latin letters for elements of the space and will use the symbol σ for its null - vectors. We distinguish norms in the spaces $\mathfrak{X}, \mathfrak{X}_1$ by and index of the space in the symbol of the norm: $x \in \mathfrak{X}$, $\|x\|_{\mathfrak{X}}$; $T \in \mathfrak{X}_1$, $\|T\|_{\mathfrak{X}_1}$. If no misunderstanding can occur, we will drop the indexes. We design the spectrum of the operator T by the symbol $\sigma(T)$, the resolvent set of the operator T by $\varrho(T)$.

Further, let T be a definite linear bounded operator. The symbol $E_T(\{\mu_0\})$ denotes the projector correspon - ding to the value $\mu_0 \in \sigma(T)$ of the operator T, which is defined as follows:

$$E_T(\{\mu_0\}) = \frac{1}{2\pi i}\int_{C\sigma_0} R(\lambda, T)\, d\lambda$$

where $\bar{\sigma}_0$ is a circle for which $\bar{\sigma}_0 \cap \sigma^*(T) = \{\mu_0\}$, $C\bar{\sigma}_0 \subset \rho(T)$.

Let $R(\lambda, T) = (\lambda \mathcal{I} - T)^{-1}$ be the resolvent of the operator T and μ_0 the dominant value of spectrum, i.e. the value for which

$$|\lambda| < |\mu_0| \tag{2}$$

for any point $\lambda \in \sigma^*(T)$, $\lambda \neq \mu_0$.

The convergence of Kellog's iteration series is a direct result of the validity of the following fundamental lemma.

LEMMA. Let T be a linear bounded operator and let μ_0 be the dominant eigenvalue of operator T, which is a simple pole of the resolvent $R(\lambda, T)$. Then the following inequality is correct:

$$\|\mu_0^{-m} T^m - E_T(\{\mu_0\})\|_{\mathfrak{X}_1} \leqq K_1 g(m) \tag{3}$$

where K_1 is independent on m,

$$g(m) = \left(\frac{\mu}{\mu_0}\right)^m$$

and μ is the radius of a circle, which contains the whole spectrum $\sigma(T)$ except the value μ_0, so that (taking into account (2))

$$\lim_{m \to \infty} g(m) = 0. \tag{4}$$

Taking into account the equation (4), we can write the inequality (3) in another form:

$$\lim_{m \to \infty} \mu_0^{-m} T^m = E_T(\{\mu_0\}).$$

3. Kellog's iterations can be constructed according to the rule:

$$x^{(m)} = T^m x^{(0)}, \quad x_{(m)} = \frac{x^{(m)}}{\|x^{(m)}\|}, \tag{5}$$

$$\mu_{(m)} = \frac{\|x^{(m+1)}\|}{\|x^{(m)}\|} \tag{6}$$

An arbitrary vector $x^{(0)} \in \mathfrak{X}$ can be taken as the initial element, for which

$$E_T(\{\mu_0\}) x^{(0)} \neq \sigma \tag{7}$$

The convergence of the series (5), (6) is treated in the following statements:

THEOREM 1. Let T be a bounded linear operator, $E_T(\{\mu_o\})$ the projector corresponding to the dominant eigenvalue μ_o of the operator T, which is a simple pole of the resolvent $R(\lambda, T)$. Let $x^{(o)} \in \mathfrak{X}$ be such a vector that (7) holds and let

$$x_o = \frac{E_T(\{\mu_o\})x^{(o)}}{\| E_T(\{\mu_o\})x^{(o)} \|} \tag{8}$$

If these conditions are fulfilled, then (5) and (6) converge and the following relations are valid:

$$x_o = \lim_{m \to \infty} x_{(m)},$$

$$|\mu_o| = \lim_{m \to \infty} \mu_{(m)}.$$

To estimate the remainder we obtain the following inequalities:

$$\| x_{(m)} - x_o \| \leqq K_2\, g(m) \| x^{(o)} \|,$$

$$|\mu_{(m)} - |\mu_o|| \leqq \| T \|_{\mathfrak{X}_1} \| x_{(m)} - x_o \|_{\mathfrak{X}}.$$

One can apply this fundamental theorem to various special types of operators and thus obtain the well-known assertions about the convergence of Kellog´s iterations. For instance in the case of a symmetric bounded operator in Hilbert space we obtain the convergence of Kellog´s iteration series directly from theorem 1.

4. The modified Kellog´s method, as was stated in the introduction, is used for calculating the eigenvalues and eigensolutions of equations of the type (1). Terms of the iteration series are constructed according to the following rule:

$$y^{(k)} = Bx^{(k)}, \quad Lx^{(k+1)} = y^{(k)}, \tag{9}$$

$$x_{(k)} \doteq \frac{x^{(k)}}{\| x^{(k)} \|}, \quad \lambda_{(k)} = \frac{\| x^{(k)} \|}{\| x^{(k+1)} \|} \tag{10}$$

Another scheme of the modified Kellog's method is:

$$L u^{(k)} = y^{(k-1)}, \quad y^{(k)} = B u^{(k)}, \tag{11}$$

$$u_{(k)} = \frac{u^{(k)}}{\| y^{(k)} \|}, \quad \hat{\lambda}_k = \frac{\| y^{(k-1)} \|}{\| y^{(k)} \|} . \tag{12}$$

In the first case, the iterations correspond to the operator $P = L^{-1} B$; in the second one to the operator $Q = B L^{-1}$.

We will assume that the operators L and B in equation (1) have the following properties:

The property (B) . Operator B is a bounded linear operator, mapping the space $\mathfrak{X}$ into itself.

The property (L) . The generally unbounded linear operator L maps its range $\mathfrak{D}(L)$ into $\mathfrak{X}$. Besides that there exists a bounded inverse operator L^{-1} .

We will call attention to the fact, that the properties (B) and (L) are used in the proofs of some of the assertions, by writing corresponding brackets in the headings of such assertions.

THEOREM 2. (B), (L) . Let $P = L^{-1} B$ be a bounded linear operator in space $\mathfrak{X}$, $E_P(\{\mu_0\})$ the projector corresponding to the dominant eigenvalue μ_0 of the operator P , which is a simple pole of the resolvent $R(\lambda, P)$. Let $x^{(0)} \in \mathfrak{X}$ be such a vector, that

$$E_P(\{\mu_0\}) x^{(0)} \neq \sigma .$$

The equations

$$x_0 = \lim_{k\to\infty} x_{(k)} , \quad |\mu_0^{-1}| = \lim_{k\to\infty} \lambda_{(k)}$$

then hold for the series (9), (10) , where x_0 is the eigenvector of the equation (1) , to which the eigenvalue $\lambda_0 = \mu_0^{-1}$ corresponds.

Owing to the property (B) , $\mathcal{Y} = B\mathfrak{X}$ is a closed set and hence a subspace of the space $\mathfrak{X}$.

THEOREM 3. (B), (L) . Let $Q = B L^{-1}$ be a bounded linear operator in space $\mathcal{Y} = B\mathfrak{X}$, $E_Q'(\{\mu_0\})$ the projector corresponding to the dominant eigenvalue μ_0 of the operator Q , which is a simple pole of the resolvent $R(\lambda, Q)$.

Let $y^{(0)} \in \mathcal{Y}$ be such a vector that

$$E_Q(\{\mu_0\})x^{(0)} \neq \sigma.$$

In this case the equations

$$x_0 = \lim_{k\to\infty} u_{(k)}, \ |\mu_0^{-1}| = \lim_{k\to\infty} \hat{\lambda}_{(k)}$$

hold for the series (11), (12), where x_0 is the eigenvector of equation (1), to which the eigenvalue $\lambda_0 = \mu_0^{-1}$ corresponds.

5. One often uses the theorems 1., 2., 3. for the case of positive operators (u_0-bounded operators).

Let $\mathcal{K}$ be a cone in the space $\mathfrak{X}$ (for instance [8] or [10]). The following lemma ensures the fulfilment of the respective conditions for theorems 1. - 3.

DEFINITION. Operator T, mapping $\mathfrak{X}$ into itself, is called a positive operator, if there exists a cone $\mathcal{K}$ such, that $T\mathcal{K} \subset \mathcal{K}$. Positive operator T is called an u_0-bounded operator, if there exists an element $u_0 \in \mathcal{K}$ such, that for arbitrary vector $x \in \mathcal{K}$, $x \neq \sigma$ positive constants α, β and a natural p can be found, for which

$$\alpha u_0 \leqq T^p x \leqq \beta u_0$$

and $\mathfrak{X}$ is partially ordered by cone $\mathcal{K}$ so, that $x \leqq y$ just if $y - x \in \mathcal{K}$.

The u_0-bounded operator T is called strongly u_0-bounded, if it has an eigenvector $x_0 \in \mathcal{K}$, correspon - ding to the eigenvalue μ_0, which is the principal ei - genvalue, i.e. for any other point μ of the spectrum $\sigma(T)$ the relation

$$|\mu| < |\mu_0|$$

is valid.

LEMMA. Let $\mathcal{K}$ be a normal cone in $\mathfrak{X}$ ([8], [10]). Let T be a strongly u_0-bounded linear operator, $E_T(\{\mu_0\})$ the projector corresponding to the dominant eigenvalue μ_0. Let $x \in \mathcal{K}$ be an arbitrary (but not null) vector.

Then

$$E_T(\{\mu_0\})x \neq \sigma.$$

Thus all the conditions of the theorem 1. are fulfilled for strongly u_0-bounded operators.

THEOREM 4. Assumptions: 1) $\mathcal{K}$ is normal cone in $\mathcal{X}$, $u_0 \in \mathcal{K}$

2) T is strongly u_0-bounded linear operator

3) $x^{(0)} \in \mathcal{K}$, $x^{(0)} \neq \sigma$.

Statement: The formulae

$$\lim_{m \to \infty} x_{(m)} = x_0, \quad \lim_{m \to \infty} \mu_{(m)} = \mu_0$$

hold for the series (5) and (6), where x_0 is the eigen-vector of the operator T, to which the positive eigen-value μ_0 corresponds. The vector x_0 is the only eigenvector of the operator T, belonging to $\mathcal{K}$, except for positive multiples.

The following theorem is an analogy of the theorem 4.:

THEOREM 5. (B),(L) Assumptions: a) The assumptions 1) and 3) of the theorem 4. are fulfilled.

b) One of the following conditions is fulfilled:

(I) The operator $P = L^{-1}B$ is a strongly u_0-bounded linear operator in $\mathcal{X}$.

(II) The operator $Q = BL^{-1}$ is a strongly v_0-bounded linear operator in $\mathcal{Y}$, i.e. $v_0 \in \mathcal{K}' = B\mathcal{K}$ and the normality of the cone $\mathcal{K}$ implies the normality of the cone $\mathcal{K}'$.

Statement: There exists one and only one solution x_0 of the equation (1) in $\mathcal{K}$. This solution corresponds to the positive eigenvalue λ_0 of the equation (1) and we have:

$$\lambda_0 = \lim_{k \to \infty} \lambda_{(k)}, \quad \lambda_0 = \lim_{k \to \infty} \hat{\lambda}_{(k)},$$

$$x_0 = \lim_{k \to \infty} u_{(k)}, \quad x_0 = \lim_{k \to \infty} x_{(k)}.$$

6. To construct eigenvectors in Hilbert spaces one uses Birger-Kolomý series instead of Kellog´s iteration series, because they converge more rapidly. One constructs them according to the following rules:

(BZ) $$z_{k+1} = \lambda_{k+1} T z_k,$$

$$\lambda_{k+1} = \frac{(Tz_k, z_k)}{(Tz_k, Tz_k)},$$

$$(KZ)\quad y_{k+1} = \frac{1}{\mu_{k+1}} T y_k ,\qquad \mu_{k+1} = \frac{(Ty_k , y_k)}{(y_k , y_k)} .$$

Here $\mathcal{H}$ is a given Hilbert space, y_k, z_k its elements, (x, y) is the scalar product of the vectors $x \in \mathcal{H}$, $y \in \mathcal{H}$.

Let $\mathcal{K}$ be such a cone in $\mathcal{H}$, that $(x, y) \geqq 0$ if $x \in \mathcal{K}$, $y \in \mathcal{K}$. Then $\mathcal{K}$ is a normal cone in $\mathcal{H}$. In the following text we will presume, that $\mathcal{K}$ is such a cone.

When constructing the eigensolution of equations of type (1) for operators, fulfilling the conditions (B),(L), we obtain modified Birger-Kolomý iterations in this form:

$$(B1)\quad w^{(k)} = Bw_{(k)} ,\ Lw_{k+1} = w^{(k)},\ w_{(k+1)} = \tilde{\lambda}_{k+1} w_{k+1} ,\qquad \tilde{\lambda}_{k+1} = \frac{(w_{k+1} , w_{(k)})}{(w_{k+1} , w_{k+1})},\quad w_{(0)} = x^{(0)} ,$$

$$(B2)\quad Lz_{(k)} = z_k ,\ z^{(k+1)} = Bz_{(k)} ,\ z_{k+1} = \lambda'_{k+1} z_k ,\qquad \lambda'_{k+1} = \frac{(z^{(k+1)} , z_k)}{(z^{(k+1)} , z^{(k+1)})},\quad z_0 = Bx^{(0)} ,$$

$$(K1)\quad v^{(k)} = Bv_{(k)} ,\ Lv_{k+1} = v^{(k)},\ v_{(k+1)} = \frac{1}{\tilde{\mu}_{k+1}} v_{k+1} ,\qquad \tilde{\mu}_{k+1} = \frac{(v_{k+1} , v_{(k)})}{(v_{(k)} , v_{(k)})},\quad v_{(0)} = x^{(0)} ,$$

$$(K2)\quad Ly_{(k)} = y_k ,\ y^{(k+1)} = By_{(k)} ,\ y_{k+1} = \frac{1}{\mu'_{k+1}} y^{(k+1)},\qquad \mu'_{k+1} = \frac{(y^{(k+1)} , y_k)}{(y_k , y_k)},\quad y_0 = Bx^{(0)} .$$

The following theorems are valid for the Birger-Kolomý series.

THEOREM 6. Let T be a bounded linear operator in Hilbert space $\mathcal{H}$ and $E_T(\{\mu_0\})$ the projector corresponding to the dominant eigenvalue μ_0 of the operator T, which is a simple pole of the resolvent $R(\lambda, T)$. Let $x^{(0)}$ be such an element in $\mathcal{H}$, for which $E_T(\{\mu_0\}) x^{(0)} \neq \sigma$.

Then the series (BZ) and (KZ) converge and $\lim_{k\to\infty} z_k = x_0$, $\lim_{k\to\infty} \lambda_k = \mu_0^{-1}$, $\lim_{k\to\infty} y_k = \tilde{x}_0$, $\lim_{k\to\infty} \mu_k = \mu_0$,

where x_0 is the eigenvector of the operator T, corresponding

corresponding to the value μ_0 and $\tilde{x} = \gamma x_0$ where $\gamma \neq 0$.

A similar theorem holds for modified iterations. We bring only a special case of this theorem, which is often met in applications.

THEOREM 7. (B), (L) Let one of the following conditi - ons be fulfilled: (I) The operator $P = L^{-1}B$ is a strongly u_0-bounded operator in the space $\mathcal{H}$.

(II) The operator $Q = BL^{-1}$ is a strongly v_0-bounded operator with respect to $\mathcal{K}' = B\mathcal{K}$.

Then such a $\lambda_0 > 0$ exists, that the equation (1) has one and only one solution x_0 in $\mathcal{K}$, $(\|x_0\| = 1)$.

In case (I) $\lim\limits_{k\to\infty} w_{(k)} = \tilde{x}_0$, $\lim\limits_{k\to\infty} \tilde{\lambda}_k = \lambda_0$,

$$\lim_{k\to\infty} v_{(k)} = x_0^{\wedge}, \quad \lim_{k\to\infty} \tilde{\mu}_k = \lambda_0^{-1},$$

if we choose an arbitrary vector $x^{(0)} \in \mathcal{K}$, $x^{(0)} \neq \sigma$ as the initial element of the iterations.

In the case (II)

$$\lim_{k\to\infty} z_k = x_0', \quad \lim_{k\to\infty} \lambda_k' = \lambda_0,$$

$$\lim_{k\to\infty} y_k = x_0'', \quad \lim_{k\to\infty} \mu_k' = \lambda_0^{-1},$$

if we choose an arbitrary vector $y^{(0)} \in \mathcal{K}'$, $y^{(0)} \neq \sigma$ as the initial element of the iterations.
Further, there exists such not null constants $\tilde{\gamma}, \gamma^{\wedge}, \gamma', \gamma''$, that $\tilde{x}_0 = \tilde{\gamma} x_0$, $x_0^{\wedge} = \gamma^{\wedge} x_0$, $x_0' = \gamma' x_0$, $x_0'' = \gamma'' x_0$.

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