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A NOTE ON PERFECTLY NORMAL SPACES

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A topological space is said to be <u>perfectly normal</u> if it is a normal space in which every closed set is a G_δ. It is well known [1, p. 110] that

(*) any subspace of a perfectly normal space is perfectly normal.

The difficulty in the proof of (*) lies in the verification that every subspace of a perfectly normal space is normal; it is obvious that a subspace will inherit the property that every closed set is a G_δ. In this note we present a characterization of perfectly normal spaces that will yield (*) as a simple corollary and may have other advantages as well.

The essential idea is to replace the notion of closed G_δ with that of zero-set (see [2]). The zero-set of a continuous real-valued function f on a topological space X is the set $\{x \in X : f(x) = 0\}$. Any zero-set in X is a closed G_δ: it is obviously closed, and it is the intersection of the open sets $\{x : |f(x)| < 1/n\}$. In a normal space, conversely, every closed G_δ is a zero-set. To prove this, let K be a closed set that is the intersection of the open sets U_n, $n = 1, 2, \dots$. By Urysohn's lemma, there exist continuous functions f_n satisfying $0 \leq f_n(x) \leq 1$ on the whole space, $f_n(x) = 0$ for all $x \in K$, and $f_n(x) = 1$ for all $x \notin U_n$. Then the continuous function $\sum^{\infty} 2^{-n} f_n(x)$ has K as its zero-set. In a non-normal space, a closed G_δ need not be a zero-set. [2, pp. 50, 97].

THEOREM. <u>A space X is perfectly normal if and only if every closed set in X is a zero-set.</u>

<u>Proof.</u> That every closed set in a perfectly normal space is a zero-set was proved above. Conversely, suppose that every closed set in X is a zero-set. We saw above that every closed set is then a G_δ. Let K and L be any two

disjoint closed sets in X and let them be the zero-sets of the functions f and g, respectively. Then the continuous function $f^2/(f^2+g^2)$ vanishes on K and is equal to 1 identically on L. Consequently, X is a normal space.

COROLLARY. Any subspace of a perfectly normal space is perfectly normal.

Proof. The intersection of a zero-set (of a space) with a subspace is a zero-set of the subspace.

REMARK. We did not assume above that we were dealing with T_1-spaces. If one wishes to include the T_1 separation axiom in the definition of "perfectly normal", then of course, one must assume in the Theorem that X is a T_1-space.

REFERENCES

[1] E. ČECH: Topologické prostory, Praha, 1959.

[2] L. GILLMAN and M. JERISON: Rings of Continuous Functions, Princeton, 1960.