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ON POINTWISE CONVERGENCE OF SEQUENCES OF CONTINUOUS FUNCTIONS

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If P is a topological space, we denote by $C(P)$ the set of all continuous real-valued functions on P. For f, $f_n \in C(P)$ $(n = 1, 2, \ldots)$ we define $f_n \underset{n}{\rightarrow} f$ if and only if $f_n(x) \underset{n}{\longrightarrow} f(x)$ for every $x \in P$ (if f, $f_n \in C(P)$ and $f_n \underset{n}{\rightarrow} f$ does not hold, $f_n \underset{n}{\nrightarrow} f$ is written). This convergence defines a topology u on $C(P)$ in the well-known manner (for $A \subset C(P)$ uA consists of all $f \in C(P)$ such that $f_n \rightarrow f$ for some $f_n \in A$). Let us point out that, following E. Čech [1], a topology u on a set Q is defined as a mapping u which to every $M \subset Q$ assigns a set $uM \subset Q$ and satisfies the following axioms: $u\emptyset = \emptyset$, $u(x) = (x)$, $u(M_1 \cup M_2) = uM_1 \cup uM_2$.

The condition $u(uM) = uM$, called axiom F by E. Čech, is not required in general; if it is satisfied, then u is called an F-topology and (Q, u) an F-space. For any topology u on Q two further topologies are defined: $\tilde{u}$, the F-reduction of u, which possesses an open base consisting of all $Q - uA$, $A \subset Q$; u^*, the F-modification of u, which is the finest of all F-topologies coarser than u. Clearly $\tilde{u} = u$ or $u = u^*$ if and only if u is an F-topology.

It is known that for some not at all exceptional spaces P, the space $(C(P), u)$ does not satisfy the axiom F; therefore it will be interesting to consider the spaces $(C(P), \tilde{u})$ and $(C(P), u^*)$. In this note it is shown that for the most important spaces P, $(C(P), \tilde{u})$ is a discrete space (part II). In part I, the regularity of the space $(C(P), u^*)$ is studied.

The usual notation and terminology is used (with the above differences ·). The same symbol will be used for a sequence or double sequence and for the set of its members. The power of a set A will be denoted card A .

I would like to take this opportunity to thank Prof. M. Katětov for his useful hints, namely concerning theorem 2 in part II.

I.

We denote by N the set of all positive integers. If X is a set, X^N denotes the collection of all sequences $\{x_m\}_{m=1}^{\infty}$, $x_m \in X$.

Definition : Let $\alpha = \{\alpha_m\} \in N^N$, $\beta = \{\beta_m\} \in N^N$. We write $\alpha \succ \beta$ if there exists an m_0 such that $\alpha_m > \beta_m$ for all $m \geqq m_0$. If $\alpha \succ \beta$ does not hold, we write $\alpha \not\succ \beta$.

Proposition 1 : There exists a set $\mathcal{A} \subset N^N$ such that :

(1) If α , $\beta \in \mathcal{A}$, $\alpha \neq \beta$, then either $\alpha \succ \beta$ or $\beta \succ \alpha$.

(2) If $\gamma \in N^N$, then there exists an $\alpha \in \mathcal{A}$ such that $\gamma \not\succ \alpha$.

(3) If $\alpha^i \in \mathcal{A}$ $(i = 1,2,\dots)$, then there exists an $\alpha \in \mathcal{A}$ such that $\alpha \succ \alpha^i$ $(i = 1,2,\dots)$.

Proof : The existence of a set $\mathcal{A}$ with properties (1),(2) follows from Zorn's lemma; (3) follows from (1) and (2) by means of the diagonal method.

Theorem 1 : If P is a normal space which contains a discrete [1] family of power $\geqq 2^{\aleph_0}$ consisting of open sets, then there exists a closed set $F \subset (C(P), u)$ and a double sequence $A = \{f_{m,n}\} \subset C(P)$ such that :

1) $0 \notin F$ (0 denotes the function identically zero on P) ;

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1) That is, locally finite and such that the closures of any two distinct sets of the family are disjoint.

2) $f_{m,n} \xrightarrow[n]{} 0$ for all $m \in N$;

3) If $\mathcal{U}$ is a neighborhood of 0 in $(C(P), u)$, then $F \cap u(A \cap \mathcal{U}) \neq \emptyset$.

Proof : Let $\{G_g \mid g \in \mathcal{G}\}$ be a discrete family of open subsets of P , $g \in G_g$, card $\mathcal{G} \geq 2^{\aleph_0}$ Let $\mathcal{A} \subset N^N$ possess properties (1), (2), (3) from proposition 1. Let φ be a one-to-one mapping of $\mathcal{A}$ into $\mathcal{G}$. If $\beta \in \mathcal{A}$, we denote $\mathcal{A}_\beta = \{\alpha \in \mathcal{A} \mid \alpha \succ \beta\}$, $Z_\beta = \varphi(\mathcal{A}_\beta)$, $F_\beta = \{f \in C(P) \mid f|Z_\beta \geq 1\}$, $F = \bigcup_{\beta \in \mathcal{A}} F_\beta$. Property (3) of $\mathcal{A}$ implies that F is a closed set. For $g \in \mathcal{G}$ let $f^g \in C(P)$ be such that $f^g(g) = 1$, $f^g | P - G_g = 0$. We put, for $m, n \in N$ $\mathcal{A}_{m,n} = \{\alpha = \{\alpha_k\} \in \mathcal{A} \mid \alpha_m \geq n\}$, $G_{m,n} = \varphi(\mathcal{A}_{m,n})$, $f_{m,n} = \sum_{g \in G_{m,n}} f^g$, $A = \{f_{m,n}\}$.

It is easy to prove that $f_{m,n} \xrightarrow[n]{} 0$ for all $m \in N$.

If $\mathcal{U}$ is a neighborhood of 0 , then there exists a $\varkappa = \{\varkappa_m\} \in N^N$ such that $f_{m,n} \in \mathcal{U}$ for $n \geq \varkappa_m$. By property (2) of $\mathcal{A}$ there exists a $\beta \in \mathcal{A}$, $\beta = \{\beta_m\}$, such that $\varkappa \not\succ \beta$. Let $\{j_n\} \in N^N$ be an increasing sequence such that $\beta_{j_m} \geq \varkappa_{j_m}$; put $h_m = f_{j_m, \beta_{j_m}}$, $f = \sum_{g \in Z_\beta} f^g$. It may be proved that $h_m \xrightarrow[m]{} f$, $f \in F$.

Corrollary : If P is a normal space and there exists a discrete family of power $\geq 2^{\aleph_0}$ consisting of open subsets of P , then the space $(C(P), u^*)$ is not regular

Remark : If P contains a dense countable subset, then for every $A = \{f_{m,n}\} \subset C(P)$ such that $f_{m,n} \xrightarrow[n]{} 0$ for all $m \in N$, there exists a $\{\varkappa_m\} \in N^N$ such that the set $B = \{f_{m,n} \in A \mid n \geq \varkappa_m\}$ has, in

$(C(P), u)$ precisely one cluster point, the point 0.

Problem: Let P contain a countable dense subset. Under what conditions is $(C(P), u^*)$ regular? (Clearly it is such if P itself is countable).

II.

In this part we shall examine the F-reduction of the topology u on $C(P)$. We notice that a space $(C(P), \tilde{u})$ is discreete if and only if for every $f \in C(P)$ there exists a $H \subset C(P)$ such that

1) for every $g \in C(P)$, $g \neq f$, there exists a $\{g_n\} \in H^N$ such that $g_n \xrightarrow{n} g$,

2) if $\{f_n\} \in H^N$, then $f_n \not\xrightarrow{n} f$.

Definitions:

We shall call $\sigma(P)$ every double sequence $\{f_{m,n}\}$ of functions from $C(P)$ such that

1) $f_{m,n} \xrightarrow{n} 0$ for all $m \in N$,

2) if $\{k_m\} \in N^N$ is increasing, $\{n_m\} \in N^N$, then $f_{k_m, n_m} \not\xrightarrow{m} 0$.

We shall call $j(P)$ every double sequence $\{h_{m,n}\}$ of functions from $C(P)$ such that:

1) $h_{m,n} \xrightarrow{n} 1$ for all $m \in N$,

2) if $\{k_m\} \in N^N$ is increasing, $\{n_m\} \in N^N$, $\{g_m\} \in C(P)^N$, then $g_{k_m} \cdot h_{k_m, n_m} \not\xrightarrow{m} 1$.

Theorem 1. For every space P containing a dense countable subset the following properties are equivalent:

1) $(C(P), u)$ is not an F-space;

2) there exists an $\sigma(P)$;

3) there exists a $j(P)$;

4) $(C(P), \tilde{u})$ is discrete.

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2) $\sigma(P)$ is a $\mathcal{G}$-system, as defined in [3].

Proof : No $(C(P), u)$ is discrete and therefore

4) $\Rightarrow$ 1) is trivial.

1) $\Leftrightarrow$ 2) is proved in [3].

2) $\Rightarrow$ 3) : Let $\{f_{m,n}\} \subseteq C(P)$ be an $\sigma(P)$.
Let $\alpha_m \in C(E_1)$ be defined as follows :

$\alpha_m(y) = 0$ if $y \leq 1 - \frac{1}{m}$

$\alpha_m(y) = (m+1)(my - m + 1)$ if $1 - \frac{1}{m} \leq$
$\leq y \leq 1 - \frac{1}{m+1}$

$\alpha_m(y) = 1$ if $1 - \frac{1}{m+1} \leq y$.

For every $x \in P$ we put $h_{m,n}(x) =$
$= \alpha_m(1 - |f_{m,n}(x)|)$.

It is easy to prove that $\{h_{m,n}\}$ is a $j(P)$.

3) $\Rightarrow$ 4) : Let $\{a_1, a_2, \dots\}$ be dense in P.
For $k, l \in N$ denote by $M_{k,l}$ the set of $f \in C(P)$ such that $|f(a_k) - 1| \geq \frac{1}{l}$;

clearly $\bigcup_{k,l} M_{k,l} = C(P) - (1)$. Let φ be a one-to-one mapping of $N \times N$ onto N.
Let $\{h_{s,n}\}$ be a $j(P)$; since $h_{s,n} \xrightarrow[n]{} 1$, we may suppose (replacing, if necessary, $\{h_{s,n}\}$ by $\{h_{s,p_s+n}\}$ with p_s sufficiently large) that for any k, l, $n \in N$ and $s = \varphi(k,l)$, $|h_{s,n}(a_k) - 1| \leq \frac{1}{3l}$. Then, evidently,

$$(*) \quad |f(a_k) \cdot h_{s,n}(a_k) - 1| \geq \frac{1}{3l} \quad \text{if } s = \varphi(k,l),\ f \in M_{k,l}$$

Denote by H_1 the set of all $f \cdot h_{s,n}$ where $k \in N$, $l \in N$, $f \in M_{k,l}$, $s = \varphi(k,l)$. Clearly $f \cdot h_{\varphi(k,l),n} \xrightarrow[n]{} f$ if $f \in M_{k,l}$; hence $u H_1 \supset C(P) - (1)$. Suppose $g_i \in H_1$, $g_i \xrightarrow[i]{} 1$. Then, for some $\{k_i\}$, $\{l_i\}$, $\{n_i\}$ from N^N, we have $g_i = f_i \cdot h_{s_i, n_i}$, $f_i \in M_{k_i, l_i}$, $s_i = \varphi(k_i, l_i)$. Since $\{h_{s,n}\}$

is a $j(P)$ and $f_i \cdot h_{s_i, n_i} \xrightarrow[i]{} 1$, almost all s_i are equal to some $s = \varphi(k,\ell)$. Therefore, for large i, $f_i \in M_{k,\ell}$ and by $(*)$, $|f_i(a_k) \cdot h_{s_i, n_i}(a_k) - 1| \geq \frac{1}{3\ell}$; this contradicts $g_i \xrightarrow[i]{} 1$. Thus $1 \notin u H_1$. Putting, for $g \in C(P)$, $H_g = \{f - 1 + g \mid f \in H_1\}$ we obtain $u H_g = C(P) - (g)$.

There follow several criteria which show, in some cases, when $(C(P), u)$ is an F-space and when it is not so (and therefore, if P contains a countable dense subset, $(C(P), \tilde{u})$ is discrete). Proofs are omitted.

Proposition 1. (contained in [4]). If P is a countable space, then $(C(P), u)$ is an F-space (metrizable even).

Proposition 2. If for every $f \in C(P)$ there exists a countable $A \subset P$ such that f is constant on $P - A$, then $(C(P), u)$ is an F-space.

Proposition 3. If P is a dense-in-itself non-meager [3] normal space containing a countable metrizable dense subspace, then $(C(P), u$ is not an - F -space.

Proposition 4. If P is the β-compactification of some infinite discrete space, then $(C(P), u)$ is not an F-space.

Proposition 5. Let A be a closed G_δ subset of a normal space P. $(C(A), u)$ is not an F-space, then nor is $(C(P), u)$. If $(C(A), \tilde{u})$ is discrete and $P - A$ contains a countable dense (in $P - A$) subset, then $(C(P), \tilde{u})$ is discrete also.

If in theorem 1 we omit the condition of separability of P, the theorem ceases to hold. In this case only properties (1), (2), (3) are equivalent. Property (4) is not equivalent to them,

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3) That is, P is not a union of countably many nowhere dense sets.

as following example shows.

Let A be some space such that $(C(A), u)$ is not an F space. Let M be a set, $\operatorname{card} M > (\operatorname{card} C(A))^{\aleph_0}$. Let $B = M \cup (\xi)$ be a compact Hausdorff space with a unique non-isolated point ξ. We put $P = A \cup B$, $A \cap B = \emptyset$, A, B are closed in P. Clearly, if A is a compact Hausdorff space, P is such also. $(C(P), u)$ is not an F-space (the existence of an $\sigma(P)$ is evident), but $(C(P), \tilde{u})$ is mot discrete - this may be proved using proposition 2.

Nevertheless $(C(P), \tilde{u})$ is also discrete for some important non-separable spaces, e.g. for α-separable metric spaces with $\alpha = \alpha^{\aleph_0}$, and also for their β-compactifications, as shown in the following theorem.

In this theorem and its proof, $\overline{A}$ denotes the closure of a set $A \subset P$ in P.

Theorem 2. If P is a normal space containing a discrete normally imbedded [4] subset of power $\alpha = \alpha^{\aleph_0}$ and a dense subset of power $\leq 2^{\alpha}$, then $(C(P), \tilde{u})$ is discrete.

Lemma : Let Z, Ξ be sets, card $Z = \alpha = \alpha^{\aleph_0}$, card $\Xi = 2^{\alpha}$. Then there exists a collection $\{ Z_{\xi,n} \mid \xi \in \Xi, n \in N\}$ of subsets of Z such that

(1) if $\xi \in \Xi$, $n_1, n_2 \in N$, $n_1 \neq n_2$, then $Z_{\xi,n_1} \cap Z_{\xi,n_2} = \emptyset$;

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4) A set Q is said to be normally imbedded in space P if $Q \subset P$ and every bounded continuous function on Q can be extended continuously to P. By discrete subset we means, in this theorem, simply a subset which, as a subspace, contains isolated points only.

(2) if $\xi_i \in \Xi$ are distinct, $n_i \in N$, then $\bigcap_{i=1}^{\infty} Z_{\xi_i, n_i}$ is uncountable.

This lemma follows easily from the following proposition (cf. [1], p. 489). If R is the topological product of 2^{α} intervals $\langle 0,1 \rangle$, then R contains a dense subset of power $\leq \alpha^{\aleph_0}$ whose intersection with every non-void G_δ subset of R is uncountable. (Let φ be a one-to-one mapping of A into Z. If $R = \{\{r_\eta\} \mid r_\eta \in \langle 0,1 \rangle, \eta \in \Xi\}$, $A_{\xi,n} = \{\{r_\eta\} \in A \mid r_\xi \in (\frac{1}{n+1}, \frac{1}{n})\}$, we put $Z_{\xi,n} = \varphi(A_{\xi,n})$).

Proof of theorem 2 : Let $X \subset P$ be dense, $Z \subset P$ discrete normally imbedded, card $Z = \alpha = \alpha^{\aleph_0}$, card $X \leq 2^{\alpha}$. Put $\Xi = X \times N$ and let $\{Z_{\xi,n} \mid \xi \in \Xi, n \in N\}$ be a collection of subsets of Z with properties (1), (2) from the lemma, and such that if $\xi = [x, m] \in \Xi$, then $x \notin Z_{\xi,n}$. Since Z is normally imbedded, the collection $\{\overline{Z}_{\xi,n}\}$ is disjoint, be (1), for any fixed $\xi \in \Xi$. This implies, by normality of P, the existence of open sets $G_{\xi,m} \supset \overline{Z}_{\xi,n}$ such that, for any fixed $\xi = [x, m]$ $\{G_{\xi,n}\}$ is disjoint, $x \in G_{\xi,n}$. Now choose, for any $\xi \in \Xi$, $n \in N$, a function $h_{\xi,n} \in C(P)$ equal to 0 on $Z_{\xi,n}$ and to I on $P - G_{\xi,n}$. Clearly (a) for any $\xi \in \Xi$ $h_{\xi,n} \xrightarrow{n} 1$, (b) for any $\xi \in [x,m] \in \Xi$ $h_{\xi,n}(x) = 1$, (c) if $\xi_k \in \Xi$ are distinct, $n_k \in N$, $f_k \in C(P)$, then $f_k \cdot h_{\xi_k, n_k} \not\xrightarrow{k} 1$ (since $h_{\xi_k, n_k}(y) = 0$ where $y \in \bigcap_{k=1}^{\infty} Z_{\xi_k, n_k}$).

Let H_1 denote the set of all $f \cdot h_{\xi,n}$ with $\xi = [x, m]$, $f \in C(P)$, $|f(x) - 1| \geq \frac{1}{m}$. By (a), $f \cdot h_{\xi,n} \xrightarrow{n} f$; hence $u H_1 \supset C(P) - (1)$. Suppose that $g_k \in H_1$, $g_k \xrightarrow{k} 1$. Then $g_k = f_k \cdot h_{\xi_k, n_k}$, $\xi_k = [x_k, m_k]$, $f_k \in C(P)$,

$|f_k(x_k) - 1| \geqq \frac{1}{m_k}$. Since $g_k \xrightarrow[k]{} 1$, we obtain, by (c), that almost all ξ_k are equal to some $[x, m] \in \Xi$. Hence, for large k , $|f_k(x) - 1| \geqq \frac{1}{m}$ and by (b) $g_{\xi_k, n_k}(x) = 1$; therefore $|g_k(x) - 1| \geqq \frac{1}{m}$ which is contradiction. We have proved that $u H_1 = C(P) - (1)$. Now, if $g \in C(P)$, put $H_g = \{g + f - 1 \mid f \in H_1\}$; then $u H_g = C(P) - (g)$.

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References :

[1] E. Čech : Topologické prostory, Praha 1959.

[2] M. Neubauer : Sur l'espace des fonctions continues. Fund.Math. 31 (1938), 269-278.

[3] J. Novák, L. Mišík : O $\mathcal{L}$ -priestoroch spojitých funkcí. Mat.fyz.sborník SAVU 1 (1951), 1-13.

[4] B. Pospíšil : Sur les fonctions continues. Fund.Math. 31 (1938), 262-268.