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ASYMPTOTIC BEHAVIOUR OF THE EQUATION $x'' + p(t)\,x' + q(t)\,x = 0$ WITH COMPLEX-VALUED COEFFICIENTS

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1. Introduction. The aim of this paper is to study the asymptotic behaviour of solutions of the differential equation

$$x'' + p(t)\,x' + q(t)\,x = 0. \tag{1}$$

Although the situation is in the case of complex-valued coefficients more complicated than in the real case, some results similar to those concerning the real case can be proved using the transformation of (1) into a Riccati differential equation and applying Ljapunov's method and Ważewski's principle. Some results concerning the real differential equation can be found in a paper by MAŘÍK–RÁB [1].

The equation (1) will be compared with a differential equation with constant coefficients. In the latter case the associated Riccati differential equation $w' + w^2 + pw + q = 0$ has two singular points $w = \frac{1}{2}(-p \pm (p^2 - 4q)^{\frac{1}{2}})$ if $p^2 - 4q \neq 0$. One of them is stable the other unstable focus. The equation $z' = r(t) - z^2/p(t)$ with complex-valued coefficients has been studied in [2] and [3].

2. Notations. Let G and K denote the sets of all real or complex numbers, respectively. If $z = u + iv$, $u, v \in R$, $i = \sqrt{-1}$, we denote $\operatorname{Re} z = u$, $\operatorname{Im} z = v$, $\bar{z} = u - iv$, $|z| = (z\bar{z})^{\frac{1}{2}}$. If $z \neq 0$, then $\arg z$ denotes the angle φ such that

$$\cos\varphi = \frac{\operatorname{Re} z}{|z|}, \qquad \sin\varphi = \frac{\operatorname{Im} z}{|z|}, \qquad 0 \leqq \varphi < 2\pi.$$

If $z_1, z_2 \in K$, the distance $d(z_1, z_2)$ is defined by $|z_1 - z_2|$. Let $a_0, b_0 \in K$, $a_0 \neq b_0$ and let γ be a real parameter, $-1 \leqq \gamma \leqq 1$. Then

$$\gamma = \frac{2\operatorname{Re}\bar{b}_0(w - a_0)}{|w - a_0|^2 + |b_0|^2} \tag{2}$$

represents an eliptic pencil of circles with singular points $A_0 = a_0 + b_0$, $B_0 = a_0 - b_0$.

For $\gamma \in [-1, 0)$, the circles K_γ cover the half-plane $\operatorname{Re}\bar{b}_0(w - a_0) < 0$, for $\gamma \in (0, 1]$ the half-plane $\operatorname{Re}\bar{b}_0(w - a_0) > 0$. The straight-line $\operatorname{Re}\bar{b}_0(w - a_0) = 0$, which is the radical axis of the pencil, corresponds to the value $\gamma = 0$. The circle K_ϱ corresponding to the value $\gamma = \varrho$ has the centre $a_0 + b_0\varrho^{-1}$ and the radius $r = |b_0\varrho^{-1}|(1 - \varrho^2)^{\frac{1}{2}}$. The singular points A_0, B_0 correspond to the values $\gamma = 1$, $\gamma = -1$, respectively. The symbol int K_ϱ denotes the interior of K_ϱ, $\overline{\text{int } K_\varrho}$ means the closure of this set.

For brevity, we shall often omit the independent variable, writting e.g. f instead of $f(t)$ etc.

3. Definitions. We shall say that the equation

$$g(t)\,w' + w^2 + P(t)\,w + Q(t) = 0 \tag{3}$$

is regular if there exists at least one solution $w(t)$ which is defined for $t \to \infty$ and that it is strongly regular if all its solutions are defined for $t \to \infty$.

The equation (1) is said to be nonoscillatory on $J = [t_0, \infty)$ if each solution has on J only a finite number of zeros.

Note that the equation $x'' + x = 0$ is not nonoscillatory although it has a nonoscillatory solution $x = e^{it}$ since e.g. the solution $x = \sin t$ has infinitely many zeros on J.

4. Lemma. *Let p, q, f, g be complex-valued functions such that*

$$p(t), q(t) \in C^\circ(J), f(t), g(t) \in C^1(J), g(t) \neq 0. \tag{4}$$

Put

$$P = pg - g' + 2f, \qquad Q = qg^2 + f^2 + pfg + f'g - fg'.$$

i) *If x is a solution of* (1) *on an interval $J_0 \subset J$ and $x(t) \neq 0$ on J_0, then the function*

$$w = gx'x^{-1} - f$$

is a solution of (3) *on J_0.*

ii) *If w is a solution of* (3) *on $J_0 \subset J$ and $\beta \in J_0$, then the function*

$$x = \exp \int_\beta^t (w + f)\,g^{-1}$$

is a solution of (1) *on J_0.*

iii) *The equation* (1) *has at least one nonoscillatory solution if and only if* (3) *is regular: the equation* (1) *is nonoscillatory if and only if* (3) *is strongly regular.*

The proof will be omitted here. Its real analog is proved in [1, p. 213].

Let the assumptions (4) be fulfilled. Denote

$$\Delta = (g' - pg)^2 - 4g(f' + qg),$$

$$A = -f + \frac{1}{2}(g' - pg + \Delta^{\frac{1}{2}}), \qquad B = -f + \frac{1}{2}(g' - pg - \Delta^{\frac{1}{2}}). \tag{5}$$

Then there is $A + B = -P$, $AB = Q$ so that the equation (3) may be written in the form

(6) $$g(t)\, w' + [w - A(t)]\,[w - B(t)] = 0.$$

If we put $a = \frac{1}{2}(A + B)$, $b = \frac{1}{2}(A - B)$ then it is

(7) $$g(t)\, w' + [w - a(t)]^2 - b^2(t) = 0.$$

5. Theorem. *Let us suppose* (4). *Let $a_0, b_0 \in K$ be such that $a_0 \neq b_0$ and*

(8) $$\operatorname{Re} b_0 g^{-1}(t) > 0, \qquad \int_{t_0}^{\infty} \operatorname{Re} b_0 g^{-1}(t)\, \mathrm{d}t = \infty.$$

Define

(9) $$h(t) = \frac{b^2(t) - b_0^2}{b_0 g(t)}, \qquad k(t) = \frac{a(t) - a_0}{b_0 g(t)},$$

(10) $$\varkappa(t) = (\left| h(t) - b_0 g(t)\, k^2(t) \right| + \left| b_0 k(t) \right|) \operatorname{Re}^{-1} b_0 g^{-1}(t)$$

and suppose

(11) $$\limsup_{t \to \infty} \varkappa(t) = k_0 < 1.$$

Then there exists a $t_1 > t_0$ such that

(12) $$\varkappa(t) < 1 \qquad \text{for } t \geqq t_1$$

and each solution $u(t)$ (1) *whose initial conditions at t_1 satisfy*

(13) $$u(t_1) \neq 0, \qquad \operatorname{Re} \bar{b}_0 g(t_1)\, u'(t_1)\, u^{-1}(t_1) \geqq \operatorname{Re} \bar{b}_0 [a_0 + f(t_1)]$$

has no zeros for $t > t_1$ and it is

(14) $$\limsup_{t \to \infty} \left| a_0 + b_0 \varrho^{-1} + f(t) - g(t)\, u'(t)\, u^{-1}(t) \right| \leqq \left| b_0 \varrho_0^{-1} \right| (1 - \varrho_0^2)^{\frac{1}{2}}$$

where

(15) $$\varrho_0 = (1 - \varkappa_0^2)^{\frac{1}{2}}.$$

Proof. Let $u(t)$ be a solution of (1) satisfying (13). Then the function

(16) $$w = g u' u^{-1} - f$$

is by i) of Lemma 4 a solution of (3). The statement of the theorem will be proved by means of Ljapunov's method applied to the equation (3). Let us investigate for this purpose the pencil (2). Under the substitution

(17) $$z = w - a_0$$

the equation (3) is equivalent to

(18) $$g(t)\,z' + [z + a_0 - a(t)]^2 - b^2(t) = 0$$

and the pencil (2) to

$$\gamma = \frac{2 \operatorname{Re} \bar{b}_0 z}{|z|^2 + |b_0|^2}\,.$$

If $z = z(t)$ is any trajectory of (18), then the point $z(t)$ pertains to the circle K_{γ}, having the equation

(19) $$\gamma(t) = \frac{2 \operatorname{Re} \bar{b}_0 z(t)}{\zeta(t)}, \qquad \text{where } \zeta(t) = |z(t)|^2 + |b_0|^2.$$

Differentiation yields

$$\begin{aligned} \xi^2\gamma' &= 2\xi \operatorname{Re} \bar{b}_0 z' - 2 \operatorname{Re} \bar{b}_0 z \,.\, 2 \operatorname{Re} \bar{z} z' = \\ &= 2 \operatorname{Re} \bar{b}_0 z \bar{z} z' + 2 \operatorname{Re} b_0 \bar{b}_0 z' - 2 \operatorname{Re} \bar{b}_0 z \bar{z} z' - 2 \operatorname{Re} b_0 \bar{z}^2 z' = \\ &= 2 \operatorname{Re} b_0 z'(\bar{b}_0^2 - \bar{z}^2). \end{aligned}$$

Using the fact that $z(t)$ satisfies (18), we get

$$\xi^2\gamma' = 2 \operatorname{Re} b_0 g^{-1}[b^2 - (z + a_0 - a)^2]\,(\bar{b}_0^2 - \bar{z}^2)$$

and with respect to (9)

$$\begin{aligned} \xi^2\gamma' &= 2 \operatorname{Re} b_0 g^{-1}(b_0^2 + b_0 g h - z^2 + 2 b_0 g k z - b_0^2 g^2 k^2)\,(\bar{b}_0^2 - \bar{z}^2) = \\ &= 2 \operatorname{Re} b_0 g^{-1}(b_0^2 - z^2)\,(\bar{b}_0^2 - \bar{z}^2) + 2\Phi + 4\Psi, \end{aligned}$$

where

$$\Phi = \operatorname{Re} b_0^2(h - b_0 g k^2)\,(\bar{b}_0^2 - \bar{z}^2), \qquad \Psi = \operatorname{Re} b_0^2 k z(\bar{b}_0^2 - \bar{z}^2).$$

Since

(20) $$|b_0^2 - z^2|^2 = \xi^2(1 - \gamma^2),$$

it holds

(21) $$\gamma' = 2(1 - \gamma^2) \operatorname{Re} b_0 g^{-1} + 2\Phi\xi^{-2} + 4\Psi\xi^{-2}.$$

Now using the inequality $\xi = |z|^2 + |b_0|^2 > 2\,|b_0 z|$ and (20) we get

(22) $$\begin{aligned} |\Psi\zeta^{-2}| &\leqq \zeta^{-1}\,|b_0^2 k z|\,(1 - \gamma^2)^{\frac{1}{2}} \leqq \frac{1}{2}\,|b_0 z|^{-1}\,|b_0^2 k z|\,(1 - \gamma^2)^{\frac{1}{2}} = \\ &= \frac{1}{2}\,|b_0 k|\,(1 - \gamma^2)^{\frac{1}{2}} \end{aligned}$$

and

(23) $$|\Phi\xi^{-2}| \leqq \xi^{-1}\,|b_0|^2\,|h - b_0 g k^2|\,(1 - \gamma^2)^{\frac{1}{2}} \leqq |h - b_0 g k^2|\,(1 - \gamma^2)^{\frac{1}{2}}$$

since $\xi > |b_0|^2$. Consequently, (21), (22), (23) and (10) yield the following basic inequality

(24) $$|\gamma'(t) - 2[1 - \gamma^2(t)] \operatorname{Re} b_0 g^{-1}(t)| \leqq 2[1 - \gamma^2(t)]^{\frac{1}{2}} \varkappa(t) \operatorname{Re} b_0 g^{-1}(t).$$

From (11) and (12) it follows that there exists $\varkappa_1$, $\varkappa_0 < \varkappa_1 < 1$ such that $\varkappa(t) < \varkappa_1$ for $t > t_1$. Putting $\varrho_1 = (1 - \varkappa_1^2)^{\frac{1}{2}}$, we get from (24)

$$\gamma'(t) \geqq 2[1 - \gamma^2(t)]^{\frac{1}{2}} ([1 - \gamma^2(t)]^{\frac{1}{2}} - [1 - \varrho_1^2]^{\frac{1}{2}}) \operatorname{Re} b_0 g^{-1}(t)$$

so that $\gamma'(t) > 0$ for all t for which $|\gamma(t)| < \varrho_1$. Now, since (13), (16) and (17) imply $\operatorname{Re} \bar{b}_0 z(t_1) \geqq 0$, we conclude that for each $t_2 > t_1$ for which $z(t)$ is defined there exists a ϱ_2, $0 < \varrho_2 < \varrho_1$ such that $z(t) \in \operatorname{int} K_{\varrho_2}$ for $t \geqq t_2$. This is the consequence of two facts:

1° $\gamma(t)$ is increasing if $|\gamma(t)| < \varrho_1$;

2° the straight line $\operatorname{Re} \bar{b}_0 z = 0$ belongs to the pencil (2) for $\gamma = 0$ and $K_{\varrho_1} \subset \operatorname{int} K_{\varrho_2}$.

Since the trajectory $z(t)$ cannot leave the circle K_{ϱ_2}, it is defined on the whole interval $[t_1, \infty)$.

Now, we shall prove

(25) $$\lim_{t \to \infty} d[z(t), \operatorname{int} K_{\varrho_0}] = 0.$$

Obviously it is sufficient to verify that to each ϱ, $0 < \varrho < \varrho_0$, there exists a time $t_3 \geqq t_1$ such that $z(t) \in \operatorname{int} K_\varrho$ for $t \geqq t_3$. To prove this, choose a ϱ_1, $\varrho < \varrho_1 < \varrho_0 = (1 - \varkappa_0^2)^{\frac{1}{2}}$. Then it is $\varkappa_0 < (1 - \varrho_1^2)^{\frac{1}{2}} < 1$ so that in view of (11) there exists a time $t_2 \geqq t_1$ such that $\varkappa(t) < (1 - \varrho_1^2)^{\frac{1}{2}}$ for $t \geqq t_2$. Hence for $t \geqq t_2$

(26) $$\gamma'(t) > 2[1 - \gamma^2(t)]^{\frac{1}{2}} ([1 - \gamma^2(t)]^{\frac{1}{2}} - (1 - \varrho_1^2)^{\frac{1}{2}}) \operatorname{Re} b_0 g^{-1}(t).$$

It follows from this inequality that $\gamma'(t) > 0$ for all $t \geqq t_2$ for which $|\gamma(t)| < \varrho_1$. If $z(t_2) \in \operatorname{int} K_\varrho$, then $z(t) \in K_\varrho$ for all $t > t_2$ obviously. Suppose consequently $z(t_2) \bar{\in} \operatorname{int} K_\varrho$ and admit there is no $t_3 \geqq t_2$ implying $z(t_3) \in \operatorname{int} K_\varrho$. In this case the trajectory $z(t)$ is situated for $t \geqq t_2$ in the domain D,

$$D = \{z \in K : \operatorname{Re} \bar{b}_0 z > 0, z \bar{\in} \operatorname{int} K_\varrho\}.$$

The values $\gamma(t)$ corresponding to $z(t) \in D$ satisfy $0 < \gamma(t) < \varrho < \varrho_1$ and in view of (26) it holds

(27) $$\gamma'(t) > 2(1 - \varrho^2)^{\frac{1}{2}} [(1 - \varrho^2)^{\frac{1}{2}} - (1 - \varrho_1^2)^{\frac{1}{2}}] \operatorname{Re} b_0 g^{-1}(t) > 0.$$

Integrating this inequality from t_2 to t, we get

$$\gamma(t) \geqq \gamma(t_2) + 2(1 - \varrho^2)^{\frac{1}{2}} [(1 - \varrho^2)^{\frac{1}{2}} - (1 - \varrho_1^2)^{\frac{1}{2}}] \int_{t_2}^{t} \operatorname{Re} b_0 g^{-1}(s) \, ds \to \infty$$

for $t \to \infty$

which contradicts the fact that $|\gamma(t)| \leqq 1$. Hence there exists a $t_3 \geqq t_2$ such that $z(t_3) \in \operatorname{int} K_\varrho$. Then it is of course by (27) $z(t) \in \operatorname{int} K_\varrho$ for all $t > t_3$. Since ϱ was arbitrarily near to ϱ_0, (25) is proved. But (25) is equivalent to

$$\limsup_{t\to\infty} |b_0\varrho_0^{-1} - z(t)| \leqq |b_0\varrho_0^{-1}| (1 - \varrho_0^2)^{\frac{1}{2}}$$

for the circle K_{ϱ_0} has in the z-plane the centre $b_0\varrho_0^{-1}$ and the radius $r = |b_0\varrho_0^{-1}| \cdot (1 - \varrho_0^2)^{\frac{1}{2}}$. Thus, with respect to (17) and (16) the inequality (14) is proved and the proof is complete.

6. Theorem. *Let the assumptions of Theorem 5 be satisfied. Then there exist a* $T > t_0$ *and a solution* $v(t)$ *of the equation* (1) *which has no zeros for* $t \geqq T$ *and which satisfies the condition*

$$\limsup_{t\to\infty} |a_0 - b_0\varrho_0^{-1} + f(t) - g(t)\, v'(t)\, v^{-1}(t)| \leqq |b_0\varrho_0^{-1}| (1 - \varrho_0^2)^{\frac{1}{2}}. \tag{28}$$

Proof. Let $\gamma_1 \in G$, $-\varrho_0 < \gamma_1 < 0$. First we shall prove of all that there exist a $T > t_0$ and a trajectory $z(t)$ of (18) defined for all $t \geqq T$ and contained in the interior of K_{γ_1}. To this purpose choose a γ_0, $\gamma_1 < \gamma_0 < 0$ and a $T_0 \geqq t_0$ such that

$$\varkappa(t) < (1 - \gamma_1^2)_1 \qquad \text{for } t \geqq T_0. \tag{29}$$

Denote $F(z) = \operatorname{Re} \bar{b}_0 z(z\bar{z} + b_0\bar{b}_0)^{-1}$ and define

$$\begin{aligned}
\Omega &= \{(t, z) : t \in R, z \in K\},\\
u(t, z) &= F(z) - \gamma_0,\\
v(t, z) &= T_0 - t,\\
\Omega^\circ &= \{(t, z) \in \Omega : \gamma_0 > F(z), t > T_0\},\\
U &= \{(t, z) \in \Omega : \gamma_0 = F(z), t \geqq T_0\},\\
V &= \{(t, z) \in \Omega : \gamma_0 \geqq F(z), t = T_0\}.
\end{aligned}$$

On the set U, the derivative $\dot{u}(t, z)$ with respect to (18) is

$$\dot{u}(t, z) = 2[1 - \gamma^2(t)] \operatorname{Re} b_0 g^{-1}(t) + \xi^{-2}[z(t)] (2\Phi[z(t)] + 4\Psi[z(t)]).$$

Using (22), (23) and (29), we get

$$\dot{u}(t, z) > 2[1 - \gamma^2(t)]^{\frac{1}{2}} ([1 - \gamma^2(t)]^{\frac{1}{2}} - [1 - \gamma_1^2]) \operatorname{Re} b_0 g^{-1}(t).$$

Since $(t, z) \in U$ implies $\gamma(t) = \gamma_0$, it is on U $\dot{u}(t, z) > 0$. Next it holds $\dot{v}(t, z) = -1 < 0$ on V. Hence, using the notations and results from Hartman's monograph [4, pp. 278–283], Ω^0 is a (u, v)-subset of Ω with respect to (18) and it is

$$\Omega_E^0 = \Omega_{SE}^0 = U - V = \{(t, z) \in \Omega : \gamma_0 = F(z), t > T_0\},$$

where Ω_E^0 (Ω_{SE}^0) is the subset of all egress points (strict egress points) of Ω^0. For any $T > T_0$ define

$$S = \{(t, z) \in \Omega : F(z) \leqq \gamma_0, t = T\}.$$

Then $S \cap \Omega_E^0 = \{(t, z) \in \Omega : F(z) = \gamma_0, t = T\}$ is a retract of Ω_E^0 (e.g. the mapping $(t, z) \to (T, z)$ is a retraction) but is not a retract of S. For if there exists a retraction $\Pi : S \to S \cap \Omega_E^0$, then there exists a continuous mapping of S into itself (e.g. the product of Π and the symmetry with respect to the centre of the circle $S \cap \Omega_E^0$) without fixed points, which is impossible.

Hence all the assumptions of Ważewski's principle are fulfilled so that there exists a solution $z(t)$ of (18) such that $(t, z(t)) \in \Omega^\circ$ for all $t \geqq T$.

Now we are going to prove that $z(t)$ fulfils

$$\lim_{t \to \infty} d[z(t), \operatorname{int} K_{-\varrho_0}] = 0. \tag{30}$$

Suppose by contradiction that (30) is not satisfied. Then there exists a $\gamma_2 \in R$, $-\varrho_0 < < \gamma_2 < \gamma_1$ and a sequence $\{t_n\}$, $t_n \to \infty$ such that $z(t_n) \,\bar{\in}\, \overline{\operatorname{int} K_{\gamma_2}}$, $n = 1, 2, \ldots$. Choose a $T_2 \geqq T$ so large that $\varkappa(t) < (1 - \gamma_2^2)^{\frac{1}{2}}$ for $t \geqq T_2$. Then there exists an n_0 such that $t_{n_0} > T_1$. But $z(t_{n_0}) \,\bar{\in}\, \overline{\operatorname{int} K_{\gamma_2}}$ and $z(t) \,\bar{\in}\, \overline{\operatorname{int} K_{\gamma_2}}$ for all $t \geqq t_{n_0}$ since

$$\gamma'(t) \geqq 2[1 - \gamma^2(t)]^{\frac{1}{2}} \left([1 - \gamma^2(t)]^{\frac{1}{2}} - (1 - \gamma_2^2)^{\frac{1}{2}}\right) \operatorname{Re} b_0 g^{-1}(t) > 0 \tag{31}$$

for all $t \geqq T_2$ for which $|\gamma(t)| < \gamma_2$. On the other hand $z(t) \in \operatorname{int} K_{\gamma_1}$ so that the corresponding values of $\gamma(t)$ satisfy

$$\gamma_2 < \gamma(t) < \gamma_1 \tag{32}$$

and for $t \geqq t_{n_0}$ is by (31)

$$\gamma'(t) \geqq 2(1 - \gamma_1^2)^{\frac{1}{2}} [(1 - \gamma_1^2)^{\frac{1}{2}} - (1 - \gamma_2^2)^{\frac{1}{2}}] \operatorname{Re} b_0 g^{-1}(t).$$

Integrating this inequality from t_{n_0} to t we receive in view of (8) $\gamma(t) \to \infty$ for $t \to \infty$, which contradicts (32). Thus (30) is satisfied and this implies (28) in the same way as in the proof of Theorem 5. This completes the proof.

7. Theorem. *Suppose in addition to the assumptions stated in Theorem* 5

$$\lim_{t \to \infty} \varkappa(t) = 0. \tag{33}$$

Then each solution $u(t)$ of the equation (1) *satisfying* (13) *has no zeros for $t \geqq t_1$ and*

$$\lim_{t \to \infty} [g(t)\, u'(t)\, u^{-1}(t) - f(t)] = a_0 + b_0 \tag{34}$$

and there exists a solution $v(t)$ with the property

$$\lim_{t \to \infty} [g(t)\, v'(t)\, v^{-1}(t) - f(t)] = a_0 - b_0. \tag{35}$$

Proof. The relations (34) and (35) follow from (14), (28), respectively, for (33) implies $\varkappa_0 = 0$ and $\varrho_0 = 1$ by (15).

8. Theorem. *Let us suppose* (4) *and let the functions* $A(t)$, $B(t)$ *defined by* (5) *satisfy*

$$\lim_{t\to\infty} A(t) = A_0, \lim_{t\to\infty} B(t) = B_0, A_0, B_0 \in K, A_0 \neq B_0. \tag{38}$$

Let for $a_0 = \frac{1}{2}(A_0 + B_0)$, $b_0 = \frac{1}{2}(A_0 - B_0)$ *the assumptions of Theorem* 7 *be fulfilled and let either*

$$\int_{t_0}^{\infty} |\mathrm{d}A(t)| < \infty \tag{39}$$

or

$$\int_{t_0}^{\infty} |\mathrm{d}B(t)| < \infty. \tag{40}$$

Then there exists a fundamental system of solutions u, v *of* (1) *such that*

$$u(t) \sim \exp \int^t (f + A) g^{-1}, \qquad v(t) \sim \exp \int^t (f + B) g^{-1}. \tag{41}$$

Proof. Let us suppose (40). Since all the assumptions of Theorem 7 are fulfilled, it follows in view of (16) that there exist solutions $w_1(t)$, $w_2(t)$ of (3) converging to A_0, B_0, respectively, for $t \to \infty$. Let $t_1 \geqq t_0$ be so large that $w_1(t)$, $w_2(t)$ are defined for $t \geqq t_1$ and $w_1(t) \neq B(t)$. From the identity

$$\log [w_1(t) - B(t)] = c_1 + \int_{t_1}^{t} \frac{\mathrm{d}(w_1 - B)}{w_1 - B}$$

it follows

$$\int_{t_1}^{t} \frac{w_1'}{w_1 - B} = \log [w_1(t) - B(t)] + \int_{t_1}^{t} \frac{\mathrm{d}B}{w_1 - B} - c_1 .$$

It is seen from (38) and (40) that the right hand side has a limit for $t \to \infty$ and this implies the convergence of the integral $\int_{t_1}^{\infty} w_1'(w_1 - B)^{-1}$.

On the other hand, in view of (6), $w_1'(w_1 - B)^{-1} = (A - w_1) g^{-1}$ and thus the integral $\int_{t_1}^{\infty} (A - w_1) g^{-1}$ is convergent, too. By Lemma 4 there exists a solution $x_1(t)$ of (1) such that $x_1(t) \neq 0$ for $t \geqq t_1$ and $w = g x_1' x_1^{-1} - f$. Hence there exists the limit

$$\lim_{t\to\infty} \int_{t_1}^{t} [(A + f) g^{-1} - x_1' x_1^{-1}]$$

and consequently

$$\lim_{t\to\infty}\left[-\log x_1(t) + \int_{t_1}^{t} (A + f)\, g^{-1}\right].$$

This implies

$$x_1(t) \sim c \exp \int_{t_1}^{t} (A + f)\, g^{-1}$$

for a suitable $c \in K$.

Now, let

$$x_2(t) = \exp \int_{t_1}^{t} (w_2 + f)\, g^{-1}$$

so that x_2 is by Lemma 4 a solution of (1). Let $W(t)$ denote the wronskian of x_1 and x_2; then there is a $c_1 \in K$ such that

$$W(t) = c_1 \exp - \int_{t_1}^{t} p.$$

Then there is

$$g(t)\, W(t)\, x_1^{-1}(t)\, x_2^{-1}(t) = g(t)\, [x_2'(t)\, x_2^{-1}(t) - x_1'(t)\, x_1^{-1}(t)] = w_2(t) - w_1(t) \to$$
$$\to B_0 - A_0 = c_2\,.$$

Hence,

$$x_2(t) \sim c_3 g(t)\, W(t)\, x_1^{-1}(t) \sim c_4 g(t) \exp \int_{t_1}^{t} (-p - (A + f)\, g^{-1}).$$

But with respect to (5) it is $-pg - A - f = B + f - g'$ so that

$$x_2(t) \sim c_5 g(t) \exp \int_{t_1}^{t} (B + f - g')\, g^{-1} = c_0 \exp \int_{t_1}^{t} (B + f)\, g^{-1}.$$

If (39) is satisfied we prove first the existence of c_0 and then the one of c. Putting $u(t) = c^{-1} x_1(t)$, $v(t) = c^{-1} x_2(t)$, (41) is fulfilled and the proof is complete.

9. Theorem. *Let $p(t)$, $q(t) \in C^0(J)$ and let*

(42) $$\lim_{t\to\infty} p(t) = p_0, \qquad \lim_{t\to\infty} q(t) = q_0\,.$$

Let either

(43) $$\int_{t_0}^{\infty} |\,\mathrm{d}p(t)\,| < \infty$$

or

$$(44) \qquad \int_{t_0}^{\infty} |\,\mathrm{d}q(t)\,| < \infty.$$

Define $\Delta^2 = p^2 - 4q$ *and suppose*

$$(45) \qquad \lim_{t \to \infty} \Delta(t) = \Lambda, \qquad \operatorname{Re} \Lambda^{\frac{1}{2}} > 0.$$

Then there exists a fundamental system of solutions of (1) *such that*

$$u(t) \sim \exp \int^{t} \frac{1}{2}(-p + \Delta^{\frac{1}{2}}), \qquad v(t) \sim \exp \int^{t} \frac{1}{2}(-p - \Delta^{\frac{1}{2}}).$$

Proof. The statement is a consequence of Theorem 8 for $g \equiv 1, f \equiv 0$. Actually, (42) and (45) imply (38) and (43), (44) imply (39), (40), respectively. Since $A = -p + \Delta^{\frac{1}{2}}$, $B = -p - \Delta^{\frac{1}{2}}$, (33) is satisfied, too, for it is $h = (\Delta - \Lambda)\,\Lambda^{-\frac{1}{2}}$, $k = (-p + p_0)\,\Lambda^{-\frac{1}{2}}$ and all the assumptions of Theorem 8 are fulfilled.

10. Theorem. *Let the assumptions* (8) *be fulfilled for some constants* a_0, $b_0 \in K$, $a_0 \neq b_0$. *Let*

$$(47) \qquad \int_{t_0}^{\infty} |\, b^2(t) - b_0^2 - [a(t) - a_0]^2 \,|\; |\, g^{-1}(t) \,|\, \mathrm{d}t < \infty,$$

$$(48) \qquad \int_{t_0}^{\infty} |\, a(t) - a_0 \,|\; |\, g^{-1}(t) \,|\, \mathrm{d}t < \infty.$$

Then each solution of (1) *having no zeros on the interval* $[t_1, \infty)$, $t_1 \geqq t_0$ *fulfils one of the conditions*

$$(49) \qquad \lim_{t \to \infty} [g(t)\, x'(t)\, x^{-1}(t) - f(t)] = a_0 + b_0,$$

$$\int_{t_1}^{\infty} |\, g(t)\, x'(t)\, x^{-1}(t) - f(t) - a_0 - b_0 \,|\, \operatorname{Re} b_0 g^{-1}(t)\, \mathrm{d}t < \infty,$$

$$(50) \qquad \lim_{t \to \infty} [g(t)\, x'(t)\, x^{-1}(t) - f(t)] = a_0 - b_0,$$

$$\int_{t_1}^{\infty} |\, g(t)\, x'(t)\, x^{-1}(t) - f(t) - a_0 + b_0 \,|\, \operatorname{Re} b_0 g^{-1}(t)\, \mathrm{d}t < \infty.$$

Proof. Let $x(t)$ be any solution of (1), $x(t) \neq 0$ for $t \geqq t_1$. Then $z(t) = g(t)\, x'(t)\, x^{-1}(t) - f(t) - a_0$ is a trajectory of (18) defined for $t \geqq t_1$. The point $z(t)$

of this trajectory pertains to a certain circle $K_{\gamma(t)}$ and $\gamma(t)$ is given by (19). From the basic inequality (24) and the fact that

$$(51) \qquad |\gamma(t)| \leqq 1$$

we get

$$(52) \qquad |\gamma'(t) - 2[1 - \gamma^2(t)] \operatorname{Re} b_0 g^{-1}(t)| \leqq \varkappa(t) \operatorname{Re} b_0 g^{-1}(t).$$

According to (47) and (48) it is

$$\int_{t_1}^{\infty} \varkappa(t) \operatorname{Re} b_0 g^{-1}(t) \, dt < \infty.$$

Integrating the inequality (52) from t_1 to t, we see that

$$(53) \qquad \int_{t_1}^{\infty} [1 - \gamma^2(t)] \operatorname{Re} b_0 g^{-1}(t) \, dt < \infty$$

since the assumption that this integral is divergent implies $\gamma(t) \to \infty$ and this contradicts (51). Hence

$$\int_{t_1}^{\infty} |\gamma'(t)| \, dt < \infty.$$

Therefore $\gamma(t)$ converges to a real number when $t \to \infty$ and with respect to (53) and (8) it is either $\lim_{t \to \infty} \gamma(t) = 1$ or $\lim_{t \to \infty} \gamma(t) = -1$. This means that

$$(54) \qquad \lim_{t \to \infty} z(t) = b_0$$

or

$$(55) \qquad \lim_{t \to \infty} z(t) = -b_0.$$

Since

$$\frac{1 \mp \gamma(t)}{\gamma(t)} = \frac{|z(t) \mp b_0|^2}{2 \operatorname{Re} \bar{b}_0 z(t)},$$

we get in view of (54) and (55)

$$(56) \quad |z(t) \pm b_0| \leqq c_1 \left(\frac{1 \pm \gamma(t)}{|\gamma(t)|} \right)^{\frac{1}{2}} = c_1 \left(\frac{1 - \gamma^2(t)}{|\gamma(t)| (1 \mp \gamma(t))} \right)^{\frac{1}{2}} \leqq c_2 [1 - \gamma^2(t)]^{\frac{1}{2}}$$

for suitable $c_1, c_2 \in R$ and large enough t. On the other hand if follows from (24)

$$\left| \frac{-\gamma(t) \gamma'(t)}{2[1 - \gamma^2(t)]^{\frac{1}{2}}} + \gamma(t) [1 - \gamma^2(t)]^{\frac{1}{2}} \operatorname{Re} b_0 g^{-1}(t) \right| \leqq \varkappa(t) \operatorname{Re} b_0 g^{-1}(t).$$

Using the same argument as in the proof of (53), we get

$$\left| \int_{t_1}^{\infty} \gamma(t) [1 - \gamma^2(t)]^{\frac{1}{2}} \operatorname{Re} b_0 g^{-1}(t) \, dt \right| < \infty,$$

which is equivalent to

$$\int_{t_1}^{\infty} [1 - \gamma^2(t)]^{\frac{1}{2}} \operatorname{Re} b_0 g^{-1}(t) \, dt < \infty$$

and this guarantees, respecting (56), the convergence of the integrals

$$\int_{t_1}^{\infty} | z(t) \mp b_0 | \operatorname{Re} b_0 g^{-1}(t) \, dt < \infty.$$

The relations (54), (55) and (57) imply in view of (17) and (16) the statement. The proof is complete.

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