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# ON ZEROS OF SOLUTIONS OF THE DIFFERENTIAL EQUATION $(p(t)y')' + f(t, y, y') = 0$

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1. Consider a differential equation

$$(1)\qquad \begin{cases} (p(t)\,y')' + f(t, y, y') = 0, \\ \text{where } p(t) \in C^{\circ}[a, \infty),\ p(t) > 0 \text{ on } [a, \infty), \\ f(t, y, v) \text{ is continuous on } D = \{(t, y, v)\colon t \in [a, \infty), \\ -\infty < y, v < \infty\},\ f(t, y, v)\,y > 0 \text{ for } y \neq 0. \end{cases}$$

We do not suppose the uniqueness of the Cauchy initial problem for the equation (1). In all the work we shall omit the trivial solution $y(t) \equiv 0$ from our considerations.

A solution $y$ of (1) is called oscillatory if there exists a sequence of numbers $\{t_k\}_1^\infty$ such that $a \leqq t_k < t_{k+1}$, $y(t_k) = 0$, $y(t) \neq 0$ for $t \in (t_k, t_{k+1})$, $k = 1, 2, \ldots$, $\lim\limits_{t\to\infty} t_k = \infty$.

Let $y$ be an oscillatory solution of (1) and $\{t_k\}_1^\infty$ the sequence of its zeros. Then there exists one and only one sequence of numbers $\{\tau_k\}_1^\infty$ called the sequence of extremants of $y$, such that $t_k < \tau_k < t_{k+1}$, $y'(\tau_k) = 0$ holds (see [1] or Lemma 1 in the present work).

The work [1] deals with some asymptotic properties of the sequence $\{\Delta_k\}_1^\infty$, $\Delta_k = t_{k+1} - t_k$. It was shown that under the assumptions

$$|f(t, y, y')| \geqq q(t)\,|y|, \qquad (t, y, y') \in D,$$
$$\lim_{t\to\infty} q(t) = \infty, \qquad p(t) \leqq M = \text{const.} < \infty$$

or

$$|f(t, y, y')| \leqq q(t)\,|y|, \qquad (t, y, y') \in D,$$
$$\lim_{t\to\infty} q(t) = 0, \qquad p(t) \geqq M = \text{const.} > 0$$

the relation $\lim\limits_{k\to\infty} \Delta_k = 0$ or $\lim\limits_{k\to\infty} \Delta_k = \infty$ holds, respectively. In the present work it will be shown that these assumptions can be reduced if $y$ is such that

$$0 < M_1 \leqq |y(\tau_k)| \leqq M_2 < \infty, \qquad k = 1, 2, \ldots$$

holds where $\{\tau_k\}_1^\infty$ is the sequence of extremants of $y$.

The following lemma was proved in [1] and it is necessary for our later considerations.

**Lemma 1.** *Let $y$ be an arbitrary solution of* (1) *and $t_1 < t_2$ its consecutive zeros* ($y(t) \neq 0$ *on* $(t_1, t_2)$). *Then $t_1$ and $t_2$ are the simple zeros of $y$, there exists one and only one number $\tau$ such that $t_1 < \tau < t_2$, $y'(\tau) = 0$ holds and the function* $\operatorname{sgn} y \,.\, p(t)\, y'$ *is decreasing on* $(t_1, t_2)$.

**2. Lemma 2.** *Let $t_1$ be an arbitrary zero of an oscillatory solution $y$ of* (1) *and $\tau$ the first extremant of $y$ lying on the right of $t$. Then*

$$p(t_1)\,|\,y'(t_1)\,|\,(\tau - t_1) > |\,y(\tau)\,| \min_{t_1 \leqq t \leqq \tau} p(t),$$

$$f(t, y(t), y'(t))\, y'(t) > 0, \qquad t \in (t_1, \tau).$$

Proof. We will prove the statement e.g. for $y(t) > 0$, $t \in (t_1, \tau]$. For $y(t) < 0$, $t \in (t_1, \tau]$ the proof is similar. We have from (1):

$$[p(t)\, y'(t)]' < 0, \qquad t \in (t_1, \tau],$$

$$y'(t) - \frac{p(t_1)\, y'(t_1)}{p(t)} < 0.$$

From this by integration in the limits from $t_1$ to $\tau$ we get:

$$0 > y(\tau) - y(t_1) - p(t_1)\, y'(t_1) \int_{t_1}^{\tau} \frac{\mathrm{d}t}{p(t)} \geqq y(\tau) - p(t_1)\, y'(t_1) \frac{\tau - t_1}{\min\limits_{t_1 \leqq t \leqq \tau} p(t)},$$

(because of $y'(t_1) > 0$) and this is the first part of the statement. As $y'(t)$ does not change the sign on $(t_1, \tau)$ and $y'(t_1) > 0$ we have $y'(t) > 0$, $t \in (t_1, \tau)$. But according to (1) $f(t, y(t), y'(t)) > 0$ and so the statement of the lemma is proved.

**Theorem 1.** *Let $y$ be an oscillatory solution of* (1) *such that* $|\,y(\tau_k)\,| \leqq M_1 = \text{const.} < \infty$, $k = 1, 2, \ldots$ *holds where $\{\tau_k\}_1^\infty$ is the sequence of its extremants. Let a continuous function $f^*(t, y)$ exist with the following properties: $f^*$ is defined on* $D_1 = \{(t, y)\colon t \in [a, \infty),\ 0 \leqq y < \infty\}$, *$f^*$ is non-decreasing with respect to $y$,*

$$|\,f(t, y, y')\,| \leqq f^*(t, |\,y\,|), \qquad (t, y, y') \in D,$$

$$\lim_{t \to \infty} f^*(t, M) = 0 \qquad \textit{for } 0 < M = \text{const.} < \infty.$$

Then

*a) If there exists a constant $M_2$ such that $p(t) \leqq M_2 < \infty$ holds, then* $\lim\limits_{t \to \infty} p(t)\, y'(t) = 0$.

*b) If there exist positive constants $M_3$, $M_4$ such that* $|\,y(\tau_k)\,| \geqq M_3$, $k = 1, 2, \ldots$, $p(t) \geqq M_4 > 0$ *hold, then* $\lim\limits_{k \to \infty} \Delta_k = \infty$.

Proof. Let $\{t_k\}_1^\infty$ is the sequence of the zeros of $y$, $t_k < \tau_k < t_{k+1}$.

a) By multiplying the equation (1) by $-2y'p$ and by the integration we obtain ($J_k = [t_k, \tau_k]$):

$$[p(t_k)\,y'(t_k)]^2 = 2\int_{t_k}^{\tau_k} p(t)\,f(t, y, y')\,y'(t)\,\mathrm{d}t = 2\int_{t_k}^{\tau_k} p(t)\,|\,f(t, y, y')\,|\;|\,y'(t)\,|\,\mathrm{d}t,$$

(we must use Lemma 2, too). From this

$$[p(t_k)\,y'(t_k)]^2 \leqq 2M_2 \int_{t_k}^{\tau_k} f^*(t, M_1)\,|\,y'(t)\,|\,\mathrm{d}t \leqq 2M_2 \times$$

$$\times \max_{t\in J_k} f^*(t, M_1)\,|\,y(\tau_k)\,| \leqq 2M_2M_1 \max_{t\in J_k} f^*(t, M_1) \underset{k\to\infty}{\to} 0.$$

So the statement of the theorem is proved in this case.

b) It follows by integration of (1) that

$$p(t)\,|\,y'(t)\,| = \left|\int_t^{\tau_k} |\,f(t, y(t), y'(t))\,|\,\mathrm{d}t\right|, \qquad t\in[t_k, t_{k+1}] \tag{2}$$

holds. From this for $t = t_k$ and according to Lemma 2 we have

$$\Delta_k > (\tau_k - t_k) > \frac{|\,y(\tau_k)\,|}{p(t_k)\,|\,y'(t_k)\,|} \min_{t\in J_k} p(t) \geqq M_3M_4 \times$$

$$\times\left[\int_{t_k}^{\tau_k} |\,f(t, y(t), y'(t))\,|\,\mathrm{d}t\right]^{-1} \geqq M_3M_4\left[\int_{t_k}^{\tau_k} f^*(t, M_1)\,\mathrm{d}t\right]^{-1} \geqq$$

$$> M_3M_4\Delta_k^{-1}[\max_{t\in J_k} f^*(t, M_1)]^{-1}.$$

Thus

$$\Delta_k^2 > M_3M_4[\max_{t\in J_k} f^*(t, M_1)]^{-1} \underset{k\to\infty}{\longrightarrow} \infty$$

and the theorem is proved.

**Theorem 2.** *Let $y$ be an oscillatory solution of* (1) *and $\{\tau_k\}_1^\infty$ the sequence of its extremants. Let $f^*(t, y)$ be a continuous function on $D_1 = \{(t, y): t\in[a, \infty), 0 \leqq y < \infty\}$ such that $f^*$ is non-decreasing with respect to $y$ for an arbitrary $t\in[a, \infty)$,*

$$|\,f(t, y, v)\,| \geqq f^*(t, |\,y\,|) > 0,\ (t, y, v)\in D,$$

$\lim_{t\to\infty} f^*(t, M) = \infty$ *for an arbitrary constant $M$*, $0 < M < \infty$.

*Let* $0 < M_3 = \text{const.} \leqq |y(\tau_k)| \leqq M_1 = \text{const.} < \infty$, $k = 1, 2, \ldots$

*a) If there exist constants* $M_2, M_4$ *such that* $0 < M_4 \leqq p(t) \leqq M_2 < \infty$ *holds, then the function* $y'$ *is unbounded on* $[a, \infty)$.

*b) If there exists a constant* $M_2$ *such that* $p(t) \leqq M_2 < \infty$ *holds, then*

$$\lim_{k \to \infty} \Delta_k = 0.$$

Proof. Let $\{t_k\}_1^\infty$ be the sequence of the zeros of $y$, $t_k < \tau_k < t_{k+1}$. It follows from Lemma 1 that the arch of the curve $|y(t)|$ for $t \in [t_k, t_{k+1}]$ do not lay under the line segments connecting the points $[t_k, 0]$, $[\tau_k, |y(\tau_k)|]$ and $[\tau_k, |y(\tau_k)|]$, $[t_{k+1}, 0]$. Thus

$$\text{(3)} \qquad \begin{cases} |y(t)| \geqq |y(\tau_k)| \dfrac{t_{k+1} - t}{t_{k+1} - \tau_k}, & t \in [\tau_k, t_{k+1}], \\[2ex] |y(t)| \geqq |y(\tau_k)| \dfrac{t - t_k}{\tau_k - t_k}, & t \in [t_k, \tau_k]. \end{cases}$$

At first we prove the statement b).

b) By integration of (2) (in the limits from $t_k$ to $\tau_k$) and by use of (3) we have:

$$|y(\tau_k)| = \int_{t_k}^{\tau_k} \frac{1}{p(t)} \int_t^{\tau_k} |f(t, y(t), y'(t))| \, dt \, dt \geqq$$

$$\geqq M_2^{-1} \int_{t_k}^{\tau_k} \int_t^{\tau_k} f^*(s, |y(s)|) \, ds \, dt \geqq M_2^{-1} \int_{t_k + \frac{\tau_k - t_k}{2}}^{\tau_k} \int_t^{\tau_k} \cdot$$

$$f^*\left(s, |y(\tau_k)| \frac{s - t_k}{\tau_k - t_k}\right) ds \, dt \geqq M_2^{-1} \int_{t_k + \frac{\tau_k - t_k}{2}}^{\tau_k} \int_t^{\tau_k} f^*\left(s, \frac{M_3}{2}\right) ds \, dt \geqq$$

$$\geqq M_2^{-1} \min_{t_k \leqq s \leqq \tau_k} f^*\left(s, \frac{M_3}{2}\right) \frac{(\tau_k - t_k)^2}{8}.$$

From this

$$\text{(4)} \qquad \tau_k - t_k \leqq \sqrt{8 M_1 M_2} \left( \min_{t_k \leqq t \leqq \tau_k} f^*\left(t, \frac{M_3}{2}\right)\right)^{-\frac{1}{2}} \xrightarrow[k \to \infty]{} 0.$$

By the same way the following relation can be proved:

$$\text{(5)} \qquad t_{k+1} - \tau_k \leqq \sqrt{8 M_2 M_1} \left( \min_{\tau_k \leqq t \leqq t_{k+1}} f^*\left(t, \frac{M_3}{2}\right)\right)^{-\frac{1}{2}} \xrightarrow[k \to \infty]{} 0.$$

The statement of the theorem follows directly from (4) and (5).

a) According to Lemma 2 and the proved part of the theorem the following relations hold:

$$|y'(t_k)| > \frac{|y(\tau_k)|}{p(t_k)\Delta_k} \min_{t_k \leqq t \leqq \tau_k} p(t) \geqq \frac{M_3 M_4}{M_2 \Delta_k} \xrightarrow[k\to\infty]{} \infty.$$

So the theorem is proved.

**Remark 1.** How we can see from the proof, the Theorem 2 is also valid if we suppose that $f^*(t, y)$ is non-decreasing with respect to $y$ only in the region $D_2 = \{(t, y): t \in [a, \infty), 0 \leqq y \leqq M_1\}$ instead of in $D_1$.

**Theorem 3.** *Consider a differential equation*

$$y'' + q(t) f(y) h(y') = 0 \tag{6}$$

*where* $g \in C^\circ[a, \infty), f \in C^\circ(-\infty, \infty), h \in C^\circ(-\infty, \infty), q(t) > 0$ *for* $t \in [a, \infty), f(y)\, y > 0$ *for* $y \neq 0$.

*Let* $y$ *be its oscillatory solution and* $\{\tau_k\}_1^\infty$ *the sequence of the extremants of* $y$.

*a) Let* $\lim_{t\to\infty} q(t) = 0$, $0 < h(v) \leqq M < \infty$ *for* $-\infty < v < \infty$, and $|y(\tau_k)| \leqq M_2 < \infty, k = 1, 2, \ldots$ *Then*

$$\lim_{t\to\infty} y'(t) = 0.$$

*If, in addition,* $0 < M_1 \leqq |y(\tau_k)|$, $k = 1, 2, \ldots$ *then*

$$\lim_{k\to\infty} \Delta_k = \infty.$$

*b) Let* $\lim_{t\to\infty} q(t) = \infty$, $0 < M \leqq h(v)$ *for* $-\infty < v < \infty$ *and* $0 < M_1 \leqq |y(\tau_k)| \leqq M_2 < \infty$, $k = 1, 2, \ldots$. *Then the derivative* $y'$ *of* $y$ *is unbounded on* $[a, \infty)$ *and*

$$\lim_{k\to\infty} \Delta_k = 0.$$

Proof. The statement of the theorem follows directly from Theorems 1 and 2 and Remark 1 for

$$f^*(t, y) = Mq(t) \max_{|u| \leqq y} |f(u)|$$

and

$$f^*(t, y) = Mq(t) \min_{y \leqq |u| \leqq M_2} |f(u)|,$$

respectively.

**Remark 2.** When proving his Theorem, author of [2] proved the second part of Theorem 3b) (that the derivative $y'$ is unbounded) for the differential equation (6), $h \equiv 1$, but under many other assumptions on the functions $q$ and $f$.

# REFERENCES

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