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ON THE IMPOSSIBILITY TO CONSTRUCT DIAMETRICALLY CRITICAL GRAPHS BY EXTENSIONS

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The graphs considered in this paper are undirected, finite, without loops and multiply edges.

A graph G is called e-critical (v-critical) if its diameter is changed after removing any edge (any vertex and all edges incidental with it), respectively. These graphs were studied in [4], [8], [3]. An ω_d-graph is a graph of diameter $d \geqq 2$ and girth at least $d + 2$. An $\bar{\omega}_d$-graph is a graph of diameter $d \geqq 2$, girth at least $d + 3$ and minimum degree at least two. Obviously, every ω_d-graph is e-critical, (see [4]) and every $\bar{\omega}_d$-graph is v-critical, (see [3]).

Let G be a graph. Then $V(G)$ will denote the vertex set of G, $d(G)$ the diameter of G, $d_G(u, v)$ the distance between the vertices u, v of G, $N(z)$ the neighbourhood of a vertex z of G and $|A|$ the cardinality of a set A. Definitions of notions not included here can be found in [6].

Definition 1. *Let G and Q be vertex disjoint graphs and let R be a graph such that $V(R) = V(G) \cup V(Q)$ and G and Q are induced subgraphs of R. Then we say that the graph R is a Q-extension of G, or that the graph R is a connection of graphs G and Q and the related notation is $R = G \oplus Q$.*

In the sequel let $\mathfrak{A}$ be a certain class of graphs. Let $\mathfrak{P}$ and $\mathfrak{S}$ be finite sets of graphs not necessarily disjoint.

Definition 2. *We say that a graph G can be constructed from a set $\mathfrak{P}$ by extensions $\mathfrak{S}$ if either $G \in \mathfrak{P}$ or there exists a finite sequence of graphs $H_0, H_1, \dots, H_{n-1}, H_n$ such that $H_0 \in \mathfrak{P}$, $H_n = R$, $H_i \in \mathfrak{A}$ and H_{i+1} is a Q-extension of H_i for some graph $Q \in \mathfrak{S}$, where $1 \leqq i \leqq n - 1$.*

Analogously a class of graphs $\mathfrak{A}$ can be constructed from a set $\mathfrak{P}$ by extensions $\mathfrak{S}$ if every graph $R \in \mathfrak{A}$ can be constructed from $\mathfrak{P}$ by extensions $\mathfrak{S}$.

In [5] it is proved that ω_2-graphs and e-critical graphs of diameter $d \geqq 2$ cannost be constructed from a set $\mathfrak{P}$ by extensions $\mathfrak{S}$. We shall prove the same results for ω_d-graphs and $\bar{\omega}_d$-graphs, where $d \geqq 3$ and for v-critical graphs of diameter $d \geqq 2$. We note that many classes of graphs are constructed from a finite set of primitive graphs by using finite set of some operations, e.g. [6], [7], [9].

Now we describe two constructions of graphs that will be used later.

Example 1. Let $k \geqq 4$, $m \geqq 4$ be given integers. The following ω_2-graph $D = D(k, m)$ is constructed in [5]. The vertex set of D,

$$V(D) = \{v_0, v_1, \ldots, v_k\} \cup \{u_1, u_2, \ldots, u_{k-2}\} \cup \bigcup_{i=1}^{k} X_i, \quad \text{where } X_i = \{x_{i1}, x_{i2}, \ldots, x_{im}\}.$$

Let us put $X_{ij} = X_i - \{x_{ij}\}$, where $i = 1, 2, \ldots, k$; $j = 1, 2, \ldots, m$. The edge set of D is given by setting the neighbourhood to every vertex of D.

$$
\begin{aligned}
&N(v_0) = \{v_1, v_2, \ldots, v_k\}; \\
&N(v_i) = \{v_0, u_1, u_2, \ldots, u_{k-2}\} \cup X_i, \qquad \text{for } i = 1, 2; \\
&N(v_j) = \{v_0, u_{j-2}\} \cup X_j, \qquad \text{for } j = 3, 4, \ldots, k; \\
&N(u_i)\ \{v_1, v_2, v_{i+2}\} \cup \bigcup_{\substack{P=3 \\ P \neq i+2}}^{k} X_p, \qquad \text{for } i = 1, 2, \ldots, k-2; \\
&N(x_{1i}) = \{v_1\} \cup \{x_{2i}, x_{3i}, \ldots, x_{ki}\}, \qquad \text{for } i = 1, 2, \ldots, m; \\
&N(x_{2i}) = \{v_2, x_{1i}\} \cup \bigcup_{j=3}^{k} X_{ji}, \qquad \text{for } i = 1, 2, \ldots, m; \\
&N(x_{ji}) = \{v_j, x_{1i}\} \cup X_{2i} \cup \bigcup_{\substack{P=1 \\ P \neq j-2}}^{k-2} \{u_p\}, \qquad \text{for } j = 3, 4, \ldots, k;\ i = 1, \ldots, m;
\end{aligned}
$$

The sketch of this graph is in Fig. 1. It is clear that D is e-critical graph. The graph D is also v-critical one because after deleting the vertex:

$$
\begin{aligned}
&v_0 \text{ it would be } d(v_3, v_i) > 2, &&\text{for } i = 4, 5, \ldots, k; \\
&v_i \text{ it would be } d(v_0, x_{ij}) > 2, &&\text{for } i = 1, \ldots, k;\ j = 1, 2, \ldots, m; \\
&w_i \text{ it would be } d(v_{i+2}, x_{rs}) > 2, &&\text{for } i = 1, 2, \ldots, k-2; \\
&r = 3, 4, \ldots, k; \quad r \neq i+2; &&s = 1, 2, \ldots, m; \\
&x_{1i} \text{ it would be } d(x_{2i}, x_{ri}) > 2, &&\text{for } i = 1, 2, \ldots, m; \\
&r = 3, \ldots, k; && \\
&x_{2i} \text{ it would be } d(x_{1i}, z) > 2, &&\text{for every } z \in \bigcup_{j=3}^{k} X_{ji}, \quad i = 1, 2, \ldots, m; \\
&x_{ri} \text{ it would be } d(v_r, x_{1i}) > 2, &&\text{for } r = 3, 4, \ldots, k; \\
&i = 1, 2, \ldots, m; &&
\end{aligned}
$$

Example 2. Let $k \geqq 3$ be a prime number. We shall construct a regular graph $C = C(k)$ of degree $k + 1$, diameter three and girth six. Thus the graph C will be e-critical and v-critical one. The vertex set of C,

$$V(C) = U \cup V \cup \bigcup_{i=1}^{k} X_i \cup \bigcup_{i=1}^{k} Z_i, \qquad \text{where } U = \{u_0, u_1, \ldots, u_k\},$$

$$V = \{v_0, v_1, \ldots, v_k\}; \ X_i = \{x^i_1, x^i_2, \ldots, x^i_k\}; \ Z_i = \{z^i_1, z^i_2, \ldots, z^i_k\}.$$

It yields $|V(C)| = 2(k^2 + k + 1)$. The edge set of C is determined by setting the neighbourhood to every vertex:

$$
\begin{aligned}
N(u_0) &= \{u_1, u_2, \ldots, u_k\} \cup \{v_0\};\\
N(v_0) &= \{v_1, v_2, \ldots, v_k\} \cup \{u_0\};\\
N(u_i) &= X_i \text{ and } N(v_i) = Z_i, \qquad \text{for} \quad i = 1, 2, \ldots, k;
\end{aligned}
$$

In addition, let $r = 1, 2, \ldots, k$; $i = 1, 2, \ldots, k$ and let the arithmetical operations in indices be computed modulo k. Then we put

$$
\begin{aligned}
N(z_r^1) &= \{v_1\} \cup \{x_r^1, x_r^2, \ldots, x_r^k\},\\
N(z_r^2) &= \{v_2\} \cup \{x_r^1, x_{r+1}^2, \ldots, x_{r+k+1}^k\}.
\end{aligned}
$$

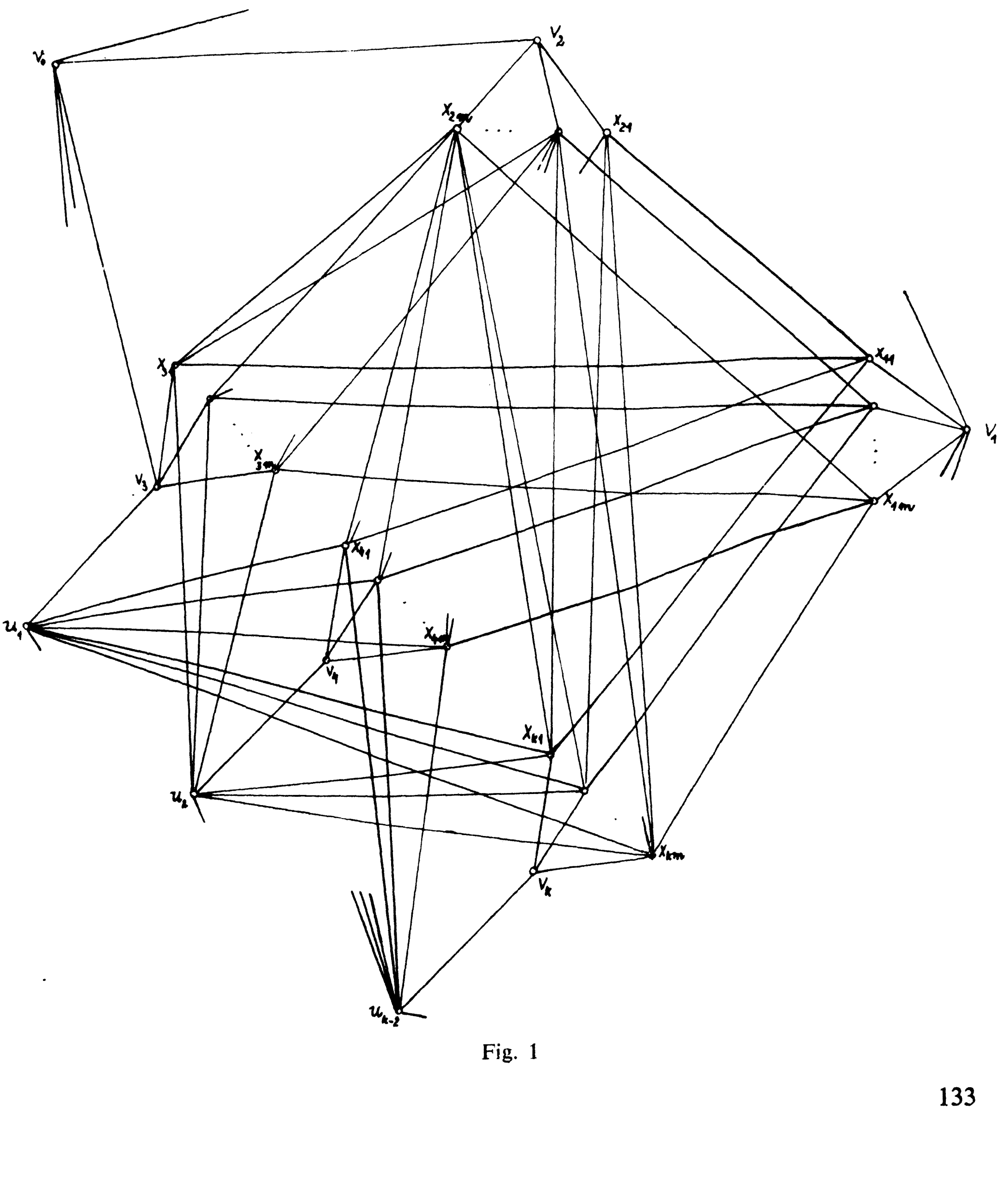

Fig. 1

Analogically in general form

$$N(z_r^i) = \{v_i\} \cup \{x_r^1, x_{r+(i-1)}^2, x_{r+2(i-1)}^3, \ldots, x_{r+(k-1)(i-1)}^k\},$$
$$N(x_r^i) = \{u_i\} \cup \{z_r^1, z_{r-(i-1)}^2, z_{r-2(i-1)}^3, \ldots, z_{r-(k-1)(i-1)}^k\}.$$

This construction is illustrated by the graph $C = C(3)$ in Fig. 2. If k is not a prime number, then the graph $C(k)$ contains a 4-angle. From the construction it follows that $C(k)$ is a regular graph of degree $k + 1$. One can verify that the diameter of $C(k)$ is equal to three and its girth is six by verifying separate cases.

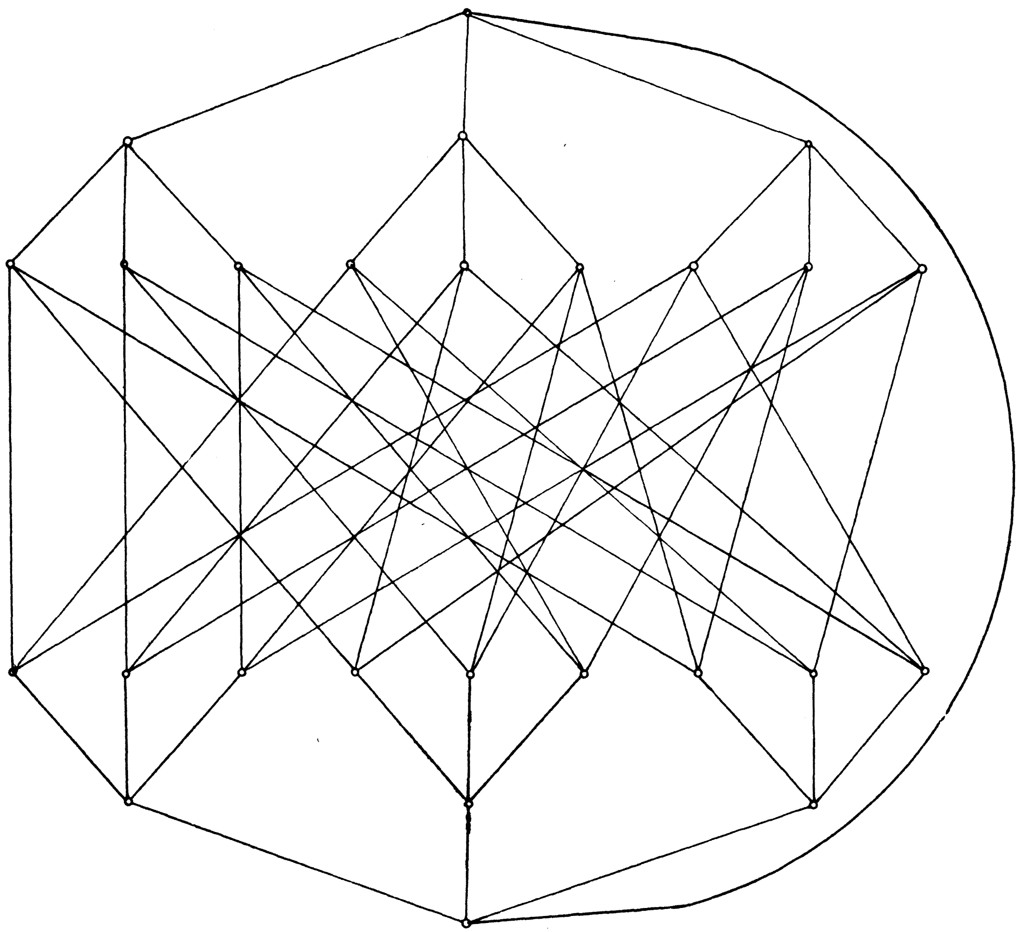

Fig. 2

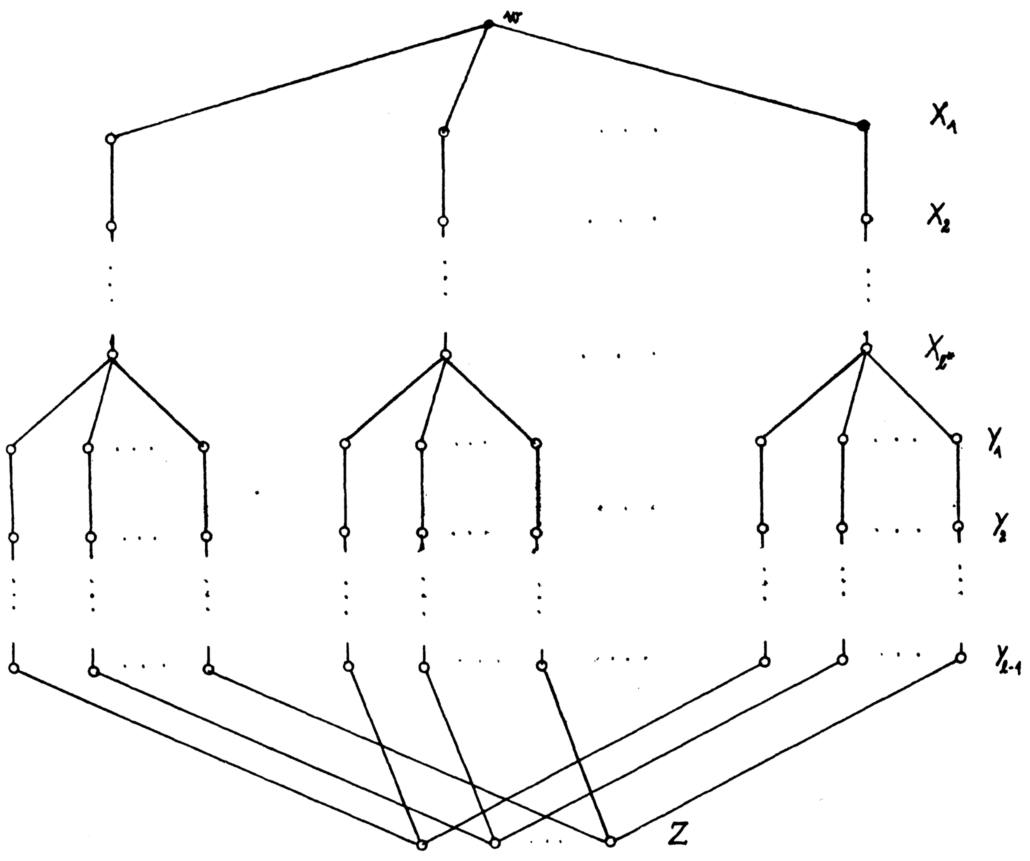

Fig. 3

Example 3. Let $d \geqq 4$, $m \geqq 2$, $k \geqq 2$ be given integers. Let us put $l = \binom{d}{2}$, $l^* = d - \left[\frac{d}{2}\right]$, where $[x]$ is the integer part of x. We construct an ω_d-graph $B = B(k, m)$ as follows: $V(B) = \{w\} \cup \bigcup_{i=1}^{l^*} X_i \cup \bigcup_{i=1}^{l-1} Y_i \cup Z$, where $Y_i = \{x_s^i\}_{s=1}^k$, $Z = \{z_i\}_{i=1}^m$, $Y_r = \bigcup_{s=1}^{k} Y^{r,s}$, for $Y^{r,s} = \{y_j^{r,s}\}_{j=1}^m$, $r = 1, 2, \dots, l-1$. The edge set of B is given again by the neighbourhoods. If $l \geqq 3$, then we put: $N(w) = X_1$;

$$
\begin{array}{lll}
N(x_j^1) & = \{w, x_j^2\}, & \text{for } j = 1, 2, \dots, k; \\
N(x_j^i) & = \{x_j^{i-1}, x_j^{i+1}\}, & \text{for } i = 2, 3, \dots, l^* - 1; j = 1, 2, \dots, k; \\
N(x_i^{l^*}) & = \{x_i^{l^*-1}\} \cup Y^{1,i}, & \text{for } i = 1, 2, \dots, k; \\
N(y_j^{1,i}) & = \{x_i^l, y_j^{2,i}\}, & \text{for } i = 1, 2, \dots, k; j = 1, \dots, m; \\
N(y_i^{r,s}) & = \{y_i^{r-1,s}, y_i^{r+1,s}\}, & \text{for } r = 2, \dots, l-2; s = 1, \dots, k; i = 1, \dots, m; \\
N(y)_i^{l-1,s} & = \{y_i^{l-2,s}, z_i\}, & \text{for } i = 1, 2, \dots, m; s = 1, 2, \dots, k;
\end{array}
$$

If $l = 2$, then we replace the last three formulas by the following one $N(y_i^{l-1,s}) = \{x_s^{l*}, z_i\}$, for $i = 1, \dots, m$; $s = 1, \dots, k$. $N(z_i) = \{y_i^{l-1,s}\}$, for $i = 1, 2, \dots, m$. This construction is illustrated in Fig. 3.

One can verify that the diameter of B is equal to d and the girth of B is equal to $2d$ for d even and $2d - 2$ for d odd. The graph B is an $\bar{\omega}_d$-graph because its minimum degree is two and its girth is at least $d + 3$. Thus B is also ω_d-graph, e-critical and v-critical graph.

Now we prove the following theorem.

Theorem 1. Let $\mathfrak{P}$, $\mathfrak{S}$ be finite sets of graphs. Let d be an integer. Then the class of e-critical graphs of diameter $d \geqq 2$, v-critical graphs of diameter $d \geqq 2$, ω_d-graphs for $d \geqq 2$ and $\bar{\omega}_d$-graphs, for $d \geqq 3$ cannot be constructed from the set $\mathfrak{P}$ by extensions $\mathfrak{S}$.

Proof. Let N be an integer greater than the number of vertices of any graph of the sets $\mathfrak{P}$ and $\mathfrak{S}$.

1. Let $d = 2$. The assertion of Theorem 1 can be proved quite analogously as in [5] by using v-critical graphs $D(k, m)$ for $k, m > N + 2$. We do not repeat it for the briefness.

2. Let $d = 3$ and let k be a prime number greater than N. We prove that $\bar{\omega}_3$-graph $C(k)$ cannot be constructed from $\mathfrak{P}$ by extensions $\mathfrak{S}$. Then the class of $\bar{\omega}_3$-graphs cannot be constructed from any set $\mathfrak{P}$ by some extensions $\mathfrak{S}$, since there exist an infinite number of prime numbers $k > N$.

If a graph $C = C(k)$ can be constructed from $\mathfrak{P}$ by extensions $\mathfrak{S}$, then $C = G \oplus Q$, where either $G \in \mathfrak{P}$, $Q \in \mathfrak{S}$ or $Q \in \mathfrak{S}$ and G is an $\bar{\omega}_3$-graph. The case $G \in \mathfrak{P}$, $Q \in \mathfrak{S}$ never occurs since $|V(C)| > 2N$. Therefore the second case occurs. If a vertex u of $C(k)$ belong to Q, then at most one vertex of the set $N(u)$ belongs to G, since $d(x, y) > 3$ for every $x, y \in N(u)$ in the graph $G - u$. (This fact follows from the construction of C.) So the graph Q contains at least $1 + (|N(u)| - 1)$ vertices, which is impossible as $|N(u)| = k + 1 > N > |V(Q)|$. Thus $\bar{\omega}_3$-graphs cannot be constructed from $\mathfrak{P}$ by extensions $\mathfrak{S}$. The mentioned proof also proves the same assertion for ω_3-graphs, e-critical and v-critical graphs of diameter 3, since we used only the properties of $C(k)$ diameter.

3. Let $d \geqq 4$ and let $k, m > N + 2$. The $\bar{\omega}_d$-graph $B(k, m)$ is not a connection $G \oplus Q$, where $G \in \mathfrak{P}$, $Q \in \mathfrak{S}$ because $|V(B)| > 2N$. Let $B = G \oplus Q$, where G is an $\bar{\omega}_d$-graph and $Q \in \mathfrak{S}$.

If $w \in V(Q)$, then at least $|X_1| - 1 = k - 1$ vertices would belong to $V(Q)$ since $d(x, y) > d$ for every $x, y \in N(w) = X_1$ in the graph $B - w$. This contradicts the fact $|V(Q)| < N < k - 2$. Therefore $w \in V(G)$ is valid.

The vertex $x_i^s \in V(G)$, where $1 \leqq i \leqq l^*$, $1 \leqq s \leqq k$, because in the reverse case there would be $d_G(w, z) > d$ for every $z \in Y^{l-1,s}$ and so the set $Y^{l-1,s}$ belongs to $V(Q)$ which is in contradiction with $|V(Q)| < N$.

The vertex $y_j^{r,s} \in V(G)$, where $1 \leqq r \leqq l-1$, $1 \leqq s \leqq k$, $1 \leqq j \leqq m$ since in the reverse case there would be $d_G(x_s^2, y_j^{l-1,s}) > d$ for $i \neq s$, $i = 1, 2, \ldots, k$ and hence the set $\{y_j^{l-1,s}\}_{\substack{i=1 \\ i \neq s}}^{k}$ belongs to $V(Q)$, which is impossible.

Finally, we have $z_i \in V(G)$ for $1 \leqq i \leqq m$ since in the reverse case there would be $d_G(y_i^{l-1,r}, y_i^{l-1,s}) > d$ for $r \neq s$, $1 \leqq r$, $s \leqq k$. Thus the graph $B(k, m) = G$ and then the graphs $B(k, m)$ for $k, m > N + 2$ cannot be constructed from $\mathfrak{P}$ by extensions $\mathfrak{S}$. Hence the class of $\bar{\omega}_d$-graphs, $d \geqq 4$ cannot be constructed from $\mathfrak{P}$ by extensions $\mathfrak{S}$. This proof also proves the same assertion for ω_d-graphs, e-critical and v-critical graphs of diameter $d \geqq 4$, since we used the property of diameter of B only. This completes the proof.

The Moore graphs can be defined as a graph of diameter $d \geqq 2$ and girth $2d + 1$, see [1]. So the Moore graphs of diameter two are $\bar{\omega}_2$-graphs. Three such graphs are known and the existence of Moore graphs of diameter two and degree 57 is possible.

The existence of regular graphs of diameter $d \geqq 2$ and girth $2d$ is studied in [2]. We only note that the next corollary follows from second part of the above proof.

Corollary 1. Let $\mathfrak{P}$ and $\mathfrak{S}$ be finite sets of graphs. Then the regular graphs of diameter three and of girth six cannot be constructed from the set $\mathfrak{P}$ by extensions $\mathfrak{S}$.

For constructive description of discussed classes of graphs it is necessary to study other type of operations, too.

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