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AN ABSTRACT MODEL OF LINEAR DIFFERENTIAL TRANSFORMATIONS

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Owing to the Sturm separation theorem the exact distribution of zeros for integrals of 2nd order linear differential equations in the real domain can be described by means of certain functions.

The study of these functions is in a close connection with transformations $Y(t) = m(t)\, y(\alpha(t))$ of solutions y and Y of the equations $y'' = q(t)\, y$ and $Y'' = Q(t)Y$.

These both problems meet in the 3rd order non-linear differential equations of the form

$$(q, Q) \qquad -\{\alpha, t\} + q(\alpha)\, \alpha'^2 = Q(t).$$

The corresponding algebraic behaviour has been described in [1], particularly in great detail for both-sided oscillatory equations in $\boldsymbol{R}$.

In this paper we present an abstract model which is directly applicable to the just mentioned case. The other possible applications can be expected in two directions, first for the general case of 2nd order transformations and secondly for the transformations of nth order lineare quations.

GENERAL SYMBOLS

$\boldsymbol{R}$	the set of all real numbers
$\boldsymbol{Z}$	the set of all integers
$f: M \twoheadrightarrow N$	surjection of M onto N, here $\operatorname{Dom} f = M$, $\operatorname{Im} f = N$
$f: M \rightarrowtail N$	bijection of M onto N
f/A	restriction of a map $f: M \to N$ to the subset $A \subseteq M$
M/f	decomposition of M corresponding to the surjection $f: M \to N$
$\mathrm{U}A_\xi$	disjoint union of sets A_ξ

[1]) This work was partially written during author's stay at the Istituto matematico "Ulisse Dini", Università degli studi, Firenze.

$B/_r A$ — decomposition of a set B with respect to a group A of permutations operating on B, the right-sided cosets decomposition of a group B with respect to a subgroup A included

${}^{\mathfrak{n}}A$ — normalizator of a subgroup A in a group B defined as the set of all $x \in B$ such that $x^{-1} Ax = A$

${}^{\mathfrak{z}}A$ — centre of a group A

${}^{\mathrm{z}}A$ — centralizator of subgroup A in a group B defined as the set of all $x \in B$ such that $xa = ax$ for all $a \in A$

C_J^3 — the set of all real functions having the continuous 3rd order derivative in an open interval $J \subseteq \boldsymbol{R}$

${}^{\mathfrak{i}}A$ — invertor of a subgroup A in a group B defined as the set of all $x \in B$ such that $xa = a^{-1}x$ for all $a \in A$

$]a, b[$ — open interval $a < x < b$ on $\boldsymbol{R}$

$[a, b[$ — semi-open interval $a \leqq x < b$ on $\boldsymbol{R}$

PARTICULAR SYMBOLS

$\mathfrak{A}_q$ — conjugate subgroup to an invariant subgroup $\mathfrak{A}$ in $\mathfrak{Z}$ defined by $\mathfrak{A}_q = \alpha^{-1}\mathfrak{A}\alpha$ for $\alpha \in \langle e, q\rangle$

${}^*\zeta$ — for $\zeta \in \langle O, q\rangle$ homomorphism $\alpha \to h$ of $\langle q, q\rangle$ onto $\mathscr{H}$ defined by the equation $\zeta\alpha = h\zeta$, particularly $\vartheta = {}^*\zeta$ for $\zeta = \Gamma\iota$

$\pi\langle q, q\rangle$ — for $\zeta \in \langle 0, q\rangle$ semi-centre of the group $\langle q, q\rangle$ defined as the inverse image of the centre ${}^{\mathfrak{z}}\mathscr{H}$ under ${}^*\zeta$

${}^*\alpha$ — automorphism $\beta \to \alpha^{-1}\beta\alpha$ of the group $\mathfrak{P}$ generated by $\alpha \in \mathfrak{P}$

${}^*u$ — for $u \in \langle q\rangle$ relation in $\langle q, q\rangle \times \mathscr{K}$ defined by the equation $u \,\square\, \alpha = ku$ for $\alpha \in \langle q, q\rangle$ and $k \in \mathscr{K}$, particularly homomorphism $\alpha \to k$ of $\langle q, q\rangle$ into $\mathscr{K}$

$\{\alpha, t\}$ — Schwarz's derivative equal to $\dfrac{1}{2}\dfrac{\alpha'''}{\alpha'} - \dfrac{3}{4}\dfrac{\alpha''^2}{\alpha'^2}$

(q) — differential equation $y'' = q(t)\, y$ with continuous coefficient $q(t)$ in some open interval $J \subseteq \boldsymbol{R}$

(q, Q) — differential equation $-\{\alpha, t\} + q(\alpha)\,\alpha'^2 = Q(t)$ with continuous $q(t)$ and $Q(t)$ in some open intervals

$[y, z]$ — ordered couple of linearly independent integrals of a differential equation (q).

Looking for the abstract gist of Borůvka's theory [1] of 2nd order linear differential transformations we can generalize the main lines of that machinery to the following model.

1. PHASES AND CARRIERS

Let $\mathfrak{P}$ be a group with the unit ι and elements $\alpha, \beta, \ldots$ etc. We suppose that $\mathfrak{F}$ is a subgroup the normalizator ${}^{n}\mathfrak{F}$ in $\mathfrak{P}$ of which fulfils the relation ${}^{n}\mathfrak{F} = \mathfrak{F}$.

Recall that the normalizator ${}^{n}\mathfrak{A}$ of a subgroup $\mathfrak{A}$ in $\mathfrak{P}$ is the set of all elements $\beta \in \mathfrak{P}$ fulfilling $\beta^{-1}\mathfrak{A}\beta = \mathfrak{A}$. We can see that the normalizator is the maximal subgroup in $\mathfrak{P}$ in which $\mathfrak{A}$ is still normal.

The right-sided decomposition $\mathfrak{P}/_{r}\mathfrak{F}$ to cosets is put in one-to-one correspondence with some set of symbols q. It is only a technical detail if we suppose $q \neq 0$ for every q and if we denote by e the symbol associated with $\mathfrak{F}$ itself.

The elements of $\mathfrak{P}$ are called phases, the symbols q are called carriers.

We introduce the notation $\mathfrak{F}\alpha = \langle e, q \rangle$ for the coset associated with the carrier q. The property ${}^{n}\mathfrak{F} = \mathfrak{F}$ allows the notation $\alpha^{-1}\mathfrak{F}A = \langle q, Q \rangle$ for $\alpha \in \langle e, q \rangle$ and $A \in \langle e, Q \rangle$. Then, in fact, there is one-to-one correspondence between the ordered pairs of carriers and the sets $\alpha^{-1}\mathfrak{F}A$.

Note the relations

(i) $$\langle q, Q \rangle \langle Q, \tilde{q} \rangle = \langle q, \tilde{q} \rangle$$

(ii) $$\langle q, Q \rangle^{-1} = \langle Q, q \rangle$$

(iii) $$\langle q, Q \rangle = \{\alpha \in \mathfrak{P} \mid \alpha^{-1}\langle q, q \rangle \alpha = \langle Q, Q \rangle\}.$$

2. SEMI-PHASES

Let $\mathscr{M}$ be a set of elements $\zeta, Z, \ldots$ etc. and $\Gamma : \mathfrak{P} \to \mathscr{M}$ be a map from $\mathfrak{P}$ onto $\mathscr{M}$. We suppose that a group $\mathscr{H}$ (with the unit i) of permutations operates on $\mathscr{M}$ and that a multiplication $\mathscr{M}\mathfrak{P} \subseteq \mathscr{M}$ is defined which is associative with respect to $\mathscr{H}$ and Γ, i. e. there holds

(i) $$h(\zeta\alpha) = (h\zeta)\,\alpha$$

and

(ii) $$\Gamma(\beta\alpha) = (\Gamma\beta)\,\alpha$$

for all $h \in \mathscr{H}$, $\zeta \in \mathscr{M}$ and $\alpha, \beta \in \mathfrak{P}$.

Moreover, we suppose that $\mathfrak{F}$ is the maximal subgroup for which there holds $\Gamma\mathfrak{F} = \mathscr{H}\Gamma\iota$.

The maximality property of $\mathfrak{F}$ is equivalent to the inclusion $\mathfrak{Z} \subseteq \mathfrak{F}$, where $\mathfrak{Z} = \Gamma^{-1}(\Gamma\iota)$ is an invariant subgroup in $\mathfrak{F}$. The elements of $\mathscr{M}$ are called semi-phases.

It holds

(iii) $\mathfrak{P}/\Gamma = \mathfrak{P}/_{r}\mathfrak{Z}$, i.e. the decomposition of $\mathfrak{P}$ corresponding to the map Γ is identical with the right-cosets decomposition of $\mathfrak{P}$ with respect to $\mathfrak{Z}$,

(iv) the formula $\Gamma\alpha = h\Gamma\iota$ defines a homomorphism $\vartheta : \mathfrak{F} \xrightarrow{\text{onto}} \mathscr{H}$ with the kernel $\mathfrak{Z}$,

(v) Γ maps one-to-one the cosets of $\mathfrak{P}/_r\mathfrak{F}$ to the classes $\mathscr{H}\zeta$ of the decomposition $\mathscr{M}/_r\mathscr{H}$ – if we denote in this way the decompositions corresponding to some group of permutations.

This last assertion allows the notation $\langle O, q\rangle = \Gamma\langle e, q\rangle$ so that $\mathscr{M} = \mathrm{U}\langle O, q\rangle$.

Inside any coset $\langle e, q\rangle$ we have one-to-one map $\langle e, q\rangle/_r\mathfrak{Z} \xrightarrow{\Gamma} \langle O, q\rangle$.

(vi) The equation $\zeta\alpha = hZ$ reads:

a) for each $\zeta \in \langle O, q\rangle$, $Z \in \langle O, Q\rangle$ and $\alpha \in \langle q, Q\rangle$ there exists $h \in \mathscr{H}$ such that the equation holds,

b) if for some $\zeta \in \langle O, q\rangle$, $Z \in \langle O, Q\rangle$, $\alpha \in \mathfrak{P}$ and $h \in \mathscr{H}$ the equation holds, then $\alpha \in \langle q, Q\rangle$.

Note that for every $\zeta \in \mathscr{M}$ there is $\zeta_\iota = \zeta$ and the multiplication is $\mathscr{M}\mathfrak{P} = \mathscr{M}$.

3. BASES

Let $\mathscr{B}$ be a set of elements $u, v, \ldots$ etc. and a map $\Delta : \mathscr{B} \xrightarrow{\text{onto}} \mathscr{M}$ be defined. We suppose that a group $\mathscr{K}$ (with the unit j) of permutations operates on $\mathscr{B}$ and that there exists a homomorphism $\Theta : \mathscr{K} \xrightarrow{\text{onto}} \mathscr{H}$ with the kernel $\operatorname{Ker}\Theta = \mathscr{R}$ such that $\mathscr{B}/\Delta = \mathscr{B}/_r\mathscr{R}$, where on the left there is the decomposition of $\mathscr{B}$ corresponding to the map Δ and on the right there is the right-cosets decomposition of $\mathscr{B}$ with respect to $\mathscr{R}$.

Let us suppose that $\Delta ku = h\Delta u$, where $h = \Theta k$, holds for any $k \in \mathscr{K}$ and $u \in \mathscr{B}$.

Moreover, we suppose the existence of a multiplication $\mathscr{B} \,\square\, \mathfrak{P} \subseteq \mathscr{B}$ which is associative with respect to Δ, i.e. it holds $\Delta(u \,\square\, \alpha) = (\Delta u)\,\alpha$ for every $\alpha \in \mathfrak{P}$ and $u \in \mathscr{B}$.

The elements of $\mathscr{B}$ are called bases. Then there holds

(i) Δ maps one-to-one $\mathscr{B}/_r\mathscr{K}$ onto $\mathscr{M}/_r\mathscr{H}$.

We can introduce the notation $\langle q\rangle = \Delta^{-1}\langle O, q\rangle$. Inside every class $\langle q\rangle$ we have $\langle q\rangle/_r\mathscr{R} \xrightarrow{\Delta} \langle O, q\rangle$, a one-to-one map,

(ii) the equation $u \,\square\, \alpha = kU$ is to be read as follows:

a) for every $u \in \langle q\rangle$, $U \in \langle Q\rangle$, $\alpha \in \langle q, Q\rangle$ there exists $k \in \mathscr{K}$ such that the equation holds,

b) if for some $u \in \langle q\rangle$, $U \in \langle Q\rangle$, $\alpha \in \mathfrak{P}$ and $k \in \mathscr{K}$ the equation holds, then $\alpha \in \langle q, Q\rangle$.

4. DISPERSIONS

The elements of the group $\langle q, q\rangle$ are also called dispersions of the carrier q.

(i) For $\alpha \in \langle e, q\rangle$ the group $\mathfrak{Z}_q \stackrel{\text{def}}{=} \alpha^{-1}\mathfrak{Z}\alpha \subseteq \langle q, q\rangle$ does not depend on α but on q only and is invariant in $\langle q, q\rangle$. We call the group $\mathfrak{Z}_q$ nucleus of the carrier q.

The elements φ of $\mathfrak{Z}_q$ are also called nuclear dispersions of q. They can be characterized by the relation $\zeta\varphi = \zeta$ for every $\zeta \in \langle O, q\rangle$.

(ii) For every $\alpha \in \langle e, q\rangle$ the group ${}^{\mathfrak{z}}\langle q, q\rangle \stackrel{\text{def}}{=} \alpha^{-1}\,{}^{\mathfrak{z}}\mathfrak{F}\alpha \subseteq \langle q, q\rangle$ is the centre of the group $\langle q, q\rangle$. The representation does not depend on α but only on q.

Of course, the central dispersions $\varphi \in {}^{\mathfrak{z}}\langle q,, q\rangle$ are characterized by the relation $\beta\varphi = \varphi\beta$ for every $\beta \in \langle q, q\rangle$.

(iii) A dispersion $\varphi \in \langle q, q\rangle$ is called semi-central if for every $\beta \in \langle q, q\rangle$ and some (and then for every) $\zeta \in \langle O, q\rangle$ there holds $\zeta\varphi\beta = \zeta\beta\varphi$.

The set ${}^{\pi}\langle q, q\rangle$ of all semi-central dispersions corresponding to the carrier q is an invariant subgroup in $\langle q, q\rangle$. We call it semi-centre corresponding to q.

For every $\alpha \in \langle e, q\rangle$ we have ${}^{\pi}\langle q, q\rangle = \alpha^{-1}\,{}^{\pi}\mathfrak{F}\alpha$. We can prove that $\varphi \in {}^{\pi}\langle q, q\rangle$ iff for some (and equivalently for every) $\zeta \in \langle O, q\rangle$ there exists $h \in {}^{\mathfrak{z}}\mathscr{H}$ (centre of $\mathscr{H}$) such that $\zeta\varphi = h\zeta$.

Another characterization is the following one: it is $\varphi \in {}^{\pi}\langle q, q\rangle$ iff for every $\beta \in \in \langle q, q\rangle$ there exists $\gamma \in \mathfrak{Z}_q$ such that $\varphi\beta = \gamma\beta\varphi$.

Note that there holds

$$\mathfrak{Z}_q, {}^{\mathfrak{z}}\langle q, q\rangle \subseteq \mathfrak{Z}_q\,{}^{\mathfrak{z}}\langle q, q\rangle \subseteq {}^{\pi}\langle q, q\rangle \subseteq \langle q, q\rangle.$$

5. HOMOMORPHISMS $*\zeta$

For each fixed $\zeta \in \langle O, q\rangle$ the equation $\zeta\beta = h\zeta$ defines a homomorphism $*\zeta : \langle q, q\rangle \xrightarrow{\text{onto}} \mathscr{H}$ with the kernel $\mathfrak{Z}_q$. Thus we have the isomorphism $\langle q, q\rangle/\mathfrak{Z}_q \cong \mathscr{H}$.

For every fixed $\zeta \in \langle O, q\rangle$ there holds $*\zeta^{-1}({}^{\mathfrak{z}}\mathscr{H}) = {}^{\pi}\langle q, q\rangle$ and thus we have the isomorphism ${}^{\pi}\langle q, q\rangle/\mathfrak{Z}_q \cong {}^{\mathfrak{z}}\mathscr{H}$. In particular there is $\mathfrak{Z}_q = {}^{\pi}\langle q, q\rangle$ iff ${}^{\mathfrak{z}}\mathscr{H} = \{i\}$.

In the following table there are special subroups of $\mathscr{H}$ in the first row and their inverse images in the second row:

$\subseteq \mathscr{H}$	$\{i\}$	$*\zeta\,{}^{\mathfrak{z}}\langle q, q\rangle$	${}^{\mathfrak{z}}\mathscr{H}$	$\mathscr{H}$
$*\zeta^{-1}$	$\mathfrak{Z}_q$	$\mathfrak{Z}_q\,{}^{\mathfrak{z}}\langle q, q\rangle$	${}^{\pi}\langle q, q\rangle$	$\langle q, q\rangle$

For the classification the position of $*\zeta\,{}^{s}\langle q, q\rangle$ in $\{i\} \subseteq *\zeta\,{}^{s}\langle q, q\rangle \subseteq {}^{s}\mathscr{H}$ is of some importance.

We have four cases:

I. $$\{i\} \subset *\zeta\,{}^{s}\langle q, q\rangle \subset {}^{s}\mathscr{H}$$

II. $$\{i\} = *\zeta\,{}^{s}\langle q, q\rangle \subset {}^{s}\mathscr{H}$$

III. $$\{i\} \subset *\zeta\,{}^{s}\langle q, q\rangle = {}^{s}\mathscr{H}$$

IV. $$\{i\} = *\zeta\,{}^{s}\langle q, q\rangle = {}^{s}\mathscr{H}$$

These cases can be characterized more directly by the following table:

I. $$\mathfrak{Z}_q \subset \mathfrak{Z}_q\,{}^{s}\langle q, q\rangle \subset {}^{\pi}\langle q, q\rangle$$

II. $${}^{s}\langle q, q\rangle \subseteq \mathfrak{Z}_q \subset {}^{\pi}\langle q, q\rangle$$

III. $$\mathfrak{Z}_q \subset \mathfrak{Z}_q\,{}^{s}\langle q, q\rangle = {}^{\pi}\langle q, q\rangle$$

IV. $${}^{s}\langle q, q\rangle \subseteq \mathfrak{Z}_q = {}^{\pi}\langle q, q\rangle$$

Note that we have the isomorphism $\mathfrak{Z}_q\,{}^{s}\langle q, q\rangle/\mathfrak{Z}_q \cong *\zeta\,{}^{s}\langle q, q\rangle$.

6. ABELIAN RELATIONS

Let $\mathfrak{A}$ be an invariant subgroup in $\mathfrak{F}$. Then the dispersions $\eta \in \mathfrak{A}$, $\varphi \in \mathfrak{A}_q \overset{\text{def}}{=} \beta^{-1}\mathfrak{A}\beta$ for $\beta \in \langle e, q\rangle$ and the phases α of any coset of the decomposition $\langle e, q\rangle/_{r}\,{}^{Z}\mathfrak{A} \cap \mathfrak{F}$ fulfil the Abelian relation $\alpha\varphi = \eta\alpha$ in the following sense: $*\alpha$ is an isomorphism of $\mathfrak{A}$ onto $\mathfrak{A}_q$ which is independent on α. Here $*\alpha$ is the map $\gamma \to \alpha^{-1}\gamma\alpha$.
In particular the Abelian relations are fulfilled by

(i) the dispersions $\eta \in \mathfrak{Z}$, $\varphi \in \mathfrak{Z}_q$ and α in each class of $\langle e, q\rangle/_{r}\,{}^{Z}\mathfrak{Z} \cap \mathfrak{F}$,

(ii) the dispersions $\eta \in {}^{s}\mathfrak{F}$, $\varphi \in {}^{s}\langle q, q\rangle$ and $\alpha \in \langle e, q\rangle$,

(iii) the dispersions $\eta \in {}^{\pi}\mathfrak{F}$, $\varphi \in {}^{\pi}\langle q, q\rangle$ and α in each class of $\langle e, q\rangle/_{r}\,{}^{Z\pi}\mathfrak{F} \cap \mathfrak{F}$.

7. RELATION $*u$.

For every fixed $u \in \langle q\rangle$ the equation $u \,\square\, \beta = ku$ defines a relation $*u$ in $\langle q, q\rangle \times \mathscr{K}$.

(i) For each $\beta \in \mathfrak{Z}_q$, $u \in \langle q\rangle$ there exists $k \in \mathscr{R}$ such that $u \,\square\, \beta = ku$. On the contrary, if $u \,\square\, \beta = ku$ holds for some $\beta \in \mathfrak{P}$, $u \in \langle q\rangle$ and $k \in \mathscr{R}$, then $\beta \in \mathfrak{Z}_q$.

(ii) For each $\beta \in \mathfrak{Z}_q\,{}^{s}\langle q, q\rangle$, $u \in \langle q\rangle$ there exists $k \in \Theta^{-1}\vartheta\,{}^{s}\mathfrak{F}$ such that $u \,\square\, \beta = ku$. Conversely if $u \,\square\, \beta = ku$ holds for some $\beta \in \mathfrak{P}$, $u \in \langle q\rangle$ and $k \in \Theta^{-1}\vartheta\,{}^{s}\mathfrak{F}$, then $\beta \in \mathfrak{Z}_q\,{}^{s}\langle q, q\rangle$.

(iii) For every $\beta \in {}^{\pi}\langle q, q\rangle$, $u \in \langle q\rangle$ there exists $k \in \Theta^{-1}{}^{\natural}\mathscr{H}$ such that $u \mathbin{\square} \beta = ku$. Similarly if for some $\beta \in \mathfrak{P}$, $u \in \langle q\rangle$ and $k \in \Theta^{-1}{}^{\natural}\mathscr{H}$ there is $u \mathbin{\square} \beta = ku$, then $\beta \in {}^{\pi}\langle q, q\rangle$.

It is evident that to obtain a map $^*u$ instead of a relation, some additional axioms about the multiplication $\mathscr{B} \mathbin{\square} \mathfrak{P} \subseteq \mathscr{B}$ are needed—see later.

8. MAPPINGS EQUIVALENT TO Γ.

The set of all carriers defines univocally the decompositions $\mathscr{B} = \bigcup_{Q} \langle Q\rangle$ and $\mathscr{M} = \bigcup_{Q} \langle O, Q\rangle$. But there are many decompositions to right cosets of $\mathfrak{P}$ and each of them can play the same role like $\mathfrak{P}/_r\mathfrak{F}$. To every fixed carrier q we can assign the decomposition $\mathfrak{P}/_r\langle q, q\rangle$. Then we have one-to-one correspondence between the set of all carriers and the set of decompositions.

The decomposition $\mathfrak{P}/_r\langle q, q\rangle$ depends on q but does not depend on a fixed $\zeta \in \in \langle O, q\rangle$. If we retain the symbol ζ also for the map $\zeta : \mathfrak{P} \xrightarrow{\text{onto}} \mathscr{M}$, where for each $\alpha \in \mathfrak{P}$ we have $\alpha \to \zeta\alpha$, then the map ζ has essentially the same properties like Γ. That means the multiplication $\mathscr{M}\mathfrak{P} \subseteq \mathscr{M}$ is associative with respect to ζ and there exists the maximal subgroup $\langle q, q\rangle$ for which $\zeta\langle q, q\rangle = \mathscr{H}\zeta$.

Then we have $\mathfrak{P}/\zeta = \mathfrak{P}/_r\mathfrak{Z}_q$, $\zeta\alpha = h\zeta$ defines the homomorphism $^*\zeta : \langle q, q\rangle \xrightarrow{\text{onto}} \mathscr{H}$ with the kernel $\mathfrak{Z}_q$, ζ maps one-to-one the cosets of $\mathfrak{P}/_r\langle q, q\rangle$ to the classes of $\mathscr{M}/_r\mathscr{H}$ and there holds $\zeta\langle q, Q\rangle = \langle O, Q\rangle$.

We can see that there exist exactly card $\mathscr{H}$ mappings ζ for $\zeta \in \langle O, q\rangle$ which give the same decomposition $\mathfrak{P}/_r\langle q, q\rangle$.

Let us note that the group $\mathscr{H}$ has a very special property: if $h_1\zeta = h_2\zeta$ holds for some fixed $\zeta \in \mathscr{M}$ and $h_1, h_2 \in \mathscr{H}$, then $h_1 = h_2$.

9. ADDITIONAL SUPPOSITIONS

Henceforth we suppose that the multiplication $\mathscr{B} \mathbin{\square} \mathfrak{P} \subseteq \mathscr{B}$ is associative with respect to $\mathscr{K}$ and $\mathfrak{P}$, i.e. there holds $k(u \mathbin{\square} \alpha) = (ku) \mathbin{\square} \alpha$ and $u \mathbin{\square} (\beta\alpha) = (u \mathbin{\square} \beta) \mathbin{\square} \alpha$ for every $k \in \mathscr{K}$, $u \in \mathscr{B}$ and $\alpha, \beta \in \mathfrak{P}$.

Moreover, we suppose that the group $\mathscr{K}$ has, like the group $\mathscr{H}$, a very special property: if for some fixed $u \in \mathscr{B}$ and any $k_1, k_2 \in \mathscr{K}$ there is $k_1 u = k_2 u$, then $k_1 = k_2$.

Then we can see that the multiplication is $\mathscr{B} \mathbin{\square} \mathfrak{P} = \mathscr{B}$ and for every $u \in \langle q\rangle$ the equation $u \mathbin{\square} \alpha = ku$ defines the homomorphism $^*u : \langle q, q\rangle \xrightarrow{\text{into}} \mathscr{K}$.

We suppose finally that for every $u \in \mathscr{B}$ the group Im $*u$ is independent of u and equals $\mathscr{K}'$. This is equivalent to the affirmation that $\mathscr{K}'$ is invariant in $\mathscr{K}$.

Let $\mathscr{L}$ be a subgroup in $\mathscr{K}$ with the properties $\mathscr{L}\mathscr{K}' = \mathscr{K}$ and $\mathscr{L} \cap \mathscr{K}' = \{j\}$.

The multiplication $\mathscr{B} \,\square\, \mathfrak{P} = \mathscr{B}$ is naturally related with the sets $\mathscr{B}'(u) = \{u \,\square\, \alpha \mid \alpha \in \mathfrak{P}\}$ for every fixed $u \in \mathscr{B}$. Then for each $k \in \mathscr{K}$ and $u \in \mathscr{B}$ we have $k\mathscr{B}'(u) = \mathscr{B}'(ku)$.

It holds $\mathscr{B}'(v) = \mathscr{B}'(u)$ iff $v \in \mathscr{B}'(u)$ and thus $\{\mathscr{B}'(u)\}_{u \in \mathscr{B}}$ is a decomposition of $\mathscr{B}$.

Let us recall that $\langle q \rangle = \mathscr{K}u$ for every fixed $u \in \langle q \rangle$.

Note that $\mathscr{B}'(u) = u \,\square\, \mathfrak{P}$ and $\mathscr{B} = \mathscr{K}u \,\square\, \mathfrak{P}$ for every fixed $u \in \mathscr{B}$.

We deal with two decompositions $\mathscr{B} = \bigcup_{q} \langle q \rangle = \bigcup_{u \in \mathscr{B}} \mathscr{K}u$ and $\mathscr{B} = \bigcup_{u \in \mathscr{B}} u \,\square\, \mathfrak{P}$.

For any fixed $u \in \langle q \rangle$ the intersection $(\mathscr{K}u) \cap (u \,\square\, \mathfrak{P})$ consists of those elements v which satisfy the equation $kv = v \,\square\, \alpha$. Hence $\alpha \in \langle q, q \rangle$ and $k \in \mathscr{K}'$. We can see that this intersection is $\mathscr{K}'u = u \,\square\, \langle q, q \rangle$.

By the way, $u \,\square\, \mathfrak{P} = \bigcup_{Q} u \,\square\, \langle q, Q \rangle$ where $u \in \langle q \rangle$ is fixed. Similarly $\langle q \rangle = \mathscr{K}u = \bigcup_{\mathscr{K}'k \in \mathscr{K}/\mathscr{K}'} (\mathscr{K}'k)\, u$ where the disjoint union ranges over the cosets $\mathscr{K}'k$ of the factorgroup.

Hence $\mathscr{B} = \bigcup_{\mathscr{K}'k \in \mathscr{K}/\mathscr{K}'} ((\mathscr{K}'k)\, u) \,\square\, \mathfrak{P}$ and we can see that the number of all classes $v \,\square\, \mathfrak{P} = \mathscr{B}'(v)$ equals the number of the cosets of the factorgroup $\mathscr{K}/\mathscr{K}'$ or of the elements of $\mathscr{L}$, since $\mathscr{L}$ and $\mathscr{K}/\mathscr{K}'$ are isomorph.

10. PSEUDONORMS

Let G be a group (with the unit 1) such that there exists a monomorphism $\mu : \mathscr{K}/\mathscr{K}' \to G$.

For every $\tilde{k} \in \mathscr{K}'k$ we define the pseudonorm $|\tilde{k}|$ by the formula $|\tilde{k}| = \mu(\mathscr{K}'k)$. Then the map $\tilde{k} \to |\tilde{k}|$ is a homomorphism of $\mathscr{K}$ into G with the kernel $\mathscr{K}'$. Two permutations $k_1, k_2 \in \mathscr{K}$ have the same pseudonorm if and only if they are in the same coset of $\mathscr{K}/\mathscr{K}'$. Particularly the elements $k \in \mathscr{K}'$ have the pseudonorm $|k| = 1$.

For every $v \in ((\mathscr{K}'k)\, u) \,\square\, \mathfrak{P}$ we define (with respect to some fixed $u \in \mathscr{B}$) the pseudonorm $|v|$ setting $|v| = |k|$.

We can see that two bases $v_1, v_2 \in \mathscr{B}$ have the same pseudonorm if and only if they are in the same class of the decomposition $\{((\mathscr{K}'k)\, u) \,\square\, \mathfrak{P}\}_{\mathscr{K}'k \in \mathscr{K}/\mathscr{K}'}$. Particularly all the bases $v \in (\mathscr{K}'u) \,\square\, \mathfrak{P}$ have the pseudonorm $|v| = 1$.

The bases and the permutations with the pseudonorm equal to 1 can be called unimodular ones.

Note that for every $v \in \mathscr{B}$ and $k \in \mathscr{K}$ there holds $|kv| = |k||v|$.

The group Im μ of pseudonorms is isomorph with the group $\mathcal{L}$. I.e. the elements of $\mathcal{L}$ can be chosen as the representatives of all pseudonorms.

As a very simple application of pseudonorms we introduce the following assertion: the equation $v \mathbin{\square} \alpha = k\tilde{v}$ holds iff $|v| = |k|\,|\tilde{v}|$. Particularly the bases $v, \tilde{v}$ are transformable by the formula $\tilde{v} = v \mathbin{\square} \alpha$ iff $|\tilde{v}| \approx |v|$.

11. HOMOMORPHISMS $*u$

Let $u \in \langle q \rangle$ be fixed. It is important that the following diagram is commutative ($\zeta = \Delta u$)

$$\begin{tikzcd} \langle q, q \rangle \arrow[r, "*\zeta"] \arrow[d, "*u"'] & \mathcal{H} \\ \mathcal{K}' \arrow[ur, "\Theta"'] & \end{tikzcd}$$

There holds $*\zeta = \Theta\, *u$ on $\langle q, q \rangle$ and hence

(i) $$\Theta \text{ is an epimorphism of } \mathcal{K}' \text{ onto } \mathcal{H}$$

(ii) $$\mathrm{Ker}\, *u \subseteq \mathfrak{Z}_q$$

(iii) $$*u\mathfrak{Z}_q \subseteq \mathcal{R} (= \mathrm{Ker}\, \Theta).$$

We can see that

1° $v = u \mathbin{\square} \gamma \Rightarrow *u = *v*\gamma$ and $\mathrm{Ker}\, *v = *\gamma\, \mathrm{Ker}\, *u$ ($*\gamma$ is the map $\alpha \to \gamma^{-1}\alpha\gamma$),
2° $v = ku \Rightarrow *u = k^{-1} *vk$ and $\mathrm{Ker}\, *v = \mathrm{Ker}\, *u$,
3° $*v = *u$ iff $v \in ({}^{z}\mathcal{K}')\, u$, where ${}^{Z}\mathcal{K}'$ is the centralizator of $\mathcal{K}'$ in $\mathcal{K}$. Recall that ${}^{Z}\mathcal{K}'$ is the set of all $k \in \mathcal{K}$ such that $kk' = k'k$ for each $k' \in \mathcal{K}'$. Note that for the centre ${}^{\mathfrak{z}}\mathcal{K}$ of $\mathcal{K}$ there holds ${}^{\mathfrak{z}}\mathcal{K} \subseteq {}^{Z}\mathcal{K}'$.

In the following table there are special subgroups of $\langle q, q \rangle$ in the first row and their images under $*u$ are in the second row:

$\subseteq \langle q, q \rangle$	Ker $*u$	$\mathfrak{Z}_q$	$\mathfrak{Z}_q\, {}^{\mathfrak{z}}\langle q, q \rangle$	${}^{\pi}\langle q, q \rangle$	$\langle q, q \rangle$
$*u$	$\{j\}$	$\mathcal{K}' \cap \mathcal{R}$	$\mathcal{K}' \cap (\Theta^{-1} *\zeta\, {}^{\mathfrak{z}}\langle q, q \rangle)$	$\mathcal{K}' \cap \Theta^{-1}\, {}^{\mathfrak{z}}\mathcal{H}$	$\mathcal{K}'$

Note that $\mathcal{L} \subseteq \mathcal{R}$, $\mathcal{L}$ is invariant in $\mathcal{K}$, there holds $\mathcal{R}\mathcal{K}' = \mathcal{K}$ and we have the isomorphisms

(iv) $$\mathfrak{Z}_q / \mathrm{Ker}\, *u \cong \mathcal{K}' \cap \mathcal{R} \cong \mathcal{R}/\mathcal{L}.$$

The equation $*\zeta = \Theta\, *u$ implies the following relations

(v) $$\operatorname{Ker} *u \subseteq \mathfrak{Z}_q = *u^{-1}(\mathcal{K}' \cap \mathcal{R}),$$

(vi) $${}^{\mathfrak{z}}\langle q, q\rangle \subseteq *u^{-1}(\mathcal{K}' \cap {}^{\mathfrak{z}}\mathcal{K}) \subseteq *u^{-1\,\mathfrak{z}}\mathcal{K}' \subseteq {}^{\pi}\langle q, q\rangle = *u^{-1\,\pi}\mathcal{K}',$$

where

$${}^{\pi}\mathcal{K}' = (\Theta/\mathcal{K}')^{-1\,\mathfrak{z}}\mathcal{H},$$

(vii) $$\mathcal{K}' \cap {}^{\mathfrak{z}}\mathcal{K} \subseteq {}^{\mathfrak{z}}\mathcal{K}',$$

(viii) $$\mathcal{K}' \cap \mathcal{R},\ \ {}^{\mathfrak{z}}\mathcal{K}' \subseteq {}^{\pi}\mathcal{K}' = \mathcal{K}' \cap {}^{\pi}\mathcal{K}$$

where

$${}^{\pi}\mathcal{K} = \Theta^{-1\,\mathfrak{z}}\mathcal{H},$$

(ix) $$\mathcal{R},\ {}^{\mathfrak{z}}\mathcal{K} \subseteq {}^{\pi}\mathcal{K},$$

(x) $$\mathcal{K}/\mathcal{R} \cong \mathcal{H} \cong \mathcal{K}'/(\mathcal{K}' \cap \mathcal{R}).$$

12. FINAL REMARKS

We have seen that the central dispersions behave well with respect to the Abelian relations in the following sense: for $\varphi \in {}^{\mathfrak{z}}\langle q, q\rangle$, $\eta \in {}^{\mathfrak{z}}\mathfrak{F}$ and all $\alpha \in \langle e, q\rangle$ there holds the relation $\alpha\varphi = \eta\alpha$, i.e. the map $\eta \to \alpha^{-1}\eta\alpha$ is an isomorphism of ${}^{\mathfrak{z}}\mathfrak{F}$ onto ${}^{\mathfrak{z}}\langle q, q\rangle$ which does not depend on $\alpha \in \langle e, q\rangle$.

We have also seen that the nuclear dispersions behave well under the homomorphism $*u$ in the following sense: for any fixed $u \in \langle q\rangle$ and $k \in \mathcal{K}' \cap \mathcal{R}$ the equation $u \mathbin{\square} \varphi = ku$ is fulfilled exactly by one class of the factorgroup $\mathfrak{Z}_q/\operatorname{Ker} *u$. Particularly the dispersions $\varphi \in \operatorname{Ker} *u$ form the solution of the equation $u \mathbin{\square} \varphi = u$ independently on $u \in \langle q\rangle$.

If $u \mathbin{\square} \alpha = \hat{k}u$ holds, where $\hat{k} \in \mathcal{K}' \cap {}^{\mathfrak{z}}\mathcal{K}$, then the equation $v \mathbin{\square} \alpha = \hat{k}v$ holds for every basis $v = ku$, where k ranges over $\mathcal{K}$.

Vice versa, if the equation $v \mathbin{\square} \alpha = \hat{k}v$ holds for every basis $v \in \langle q\rangle$, then $\hat{k} \in \mathcal{K}' \cap {}^{\mathfrak{z}}\mathcal{K}$.

Hence the group $*u^{-1}(\mathcal{K}' \cap {}^{\mathfrak{z}}\mathcal{K}) \subseteq \langle q, q\rangle$ is independent of $u \in \langle q\rangle$, i.e. the dispersions $\hat{\alpha} \in *u^{-1}(\mathcal{K}' \cap {}^{\mathfrak{z}}\mathcal{K})$ are characterized by the property that for each such $\hat{\alpha}$ there exists a $\hat{k} \in \mathcal{K}' \cap {}^{\mathfrak{z}}\mathcal{K}$ such that $u \mathbin{\square} \hat{\alpha} = \hat{k}u$ holds for every $u \in \langle q\rangle$.

Certainly, particular cases like

(i) $$\mathcal{K}' \cap {}^{\mathfrak{z}}\mathcal{K} = \mathcal{K}' \cap \mathcal{R}, \text{ or}$$

(ii) $${}^{\mathfrak{z}}\mathcal{K} = \mathcal{R} \quad\text{and}\quad {}^{\mathfrak{z}}\mathcal{H} = \{i\}, \text{ or}$$

(iii) $$\mathfrak{Z}_q = {}^{\mathfrak{z}}\langle q, q\rangle = {}^{\pi}\langle q, q\rangle$$

can be important.

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