Archivum Mathematicum

Ján Futák

Oscillation of solutions of a non-linear delay differential equation of the fourth order

Archivum Mathematicum, Vol. 11 (1975), No. 1, 25--29

Persistent URL: `http://dml.cz/dmlcz/104839`

ARCH. MATH. 1, SCRIPTA FAC. SCI. NAT. UJEP BRUNENSIS
XI: 25-30, 1975

OSCILLATION OF SOLUTIONS OF A NON-LINEAR DELAY DIFFERENTIAL EQUATION OF THE FOURTH ORDER

JÁN FUTÁK, Žilina

(Received January 29, 1974)

In the paper [1] there is investigated a delay differential equation of the 4^{th} order. In the papers [2], [3], [5] there are introduced some properties of solutions found for ordinary differential equations of the 3^{rd} and 4^{th} orders. This paper is a generalization of several results of [1], [2], [3] and [5].

In this paper we shall be concerned with the oscillation properties of solutions of a delay differential equation of the form:

$$y^{(4)} + p(t)\, y'' + q(t)\, y' + r(t)\, y + y(t) \sum_{i=1}^{n} Q_i(t)\, F_i(y[h_i(t)]) = g(t). \tag{1}$$

We shall suppose that the functions $p(t)$, $q(t)$, $r(t)$, $g(t)$, $Q_i(t)$, $h_i(t)$, $i = 1, 2, \ldots, n$ belong to the class $C_0(J)$, where $J \equiv \langle t_0, \infty)$ and n is a positive integer. Moreover we suppose that

$$\inf_{t \in J} [t - h_i(t)] \geqq d > 0, \qquad h_i(t) \to +\infty, \qquad t \to \infty,$$
$$F_i(z) \in C_0(-\infty, \infty), \qquad F_i(z) \geqq 0, \qquad i = 1, 2, \ldots, n.$$

A fundamental initial problem (next only initial problem) is understood to be the following problem (see [4] pg. 14): Let $\Phi(t)$ be a function defined and continuous on the initial set

$$E_{t_0} = \bigcup_{i=1}^{n} E^i_{t_0}, \qquad E^i_{t_0} = \langle \inf h_i(t), t_0 \rangle$$

and let $y_0^{(k)}$, $k = 1, 2, 3$ be arbitrary real numbers. We find such a solution $y(t)$ of (1) on J that fulfils initial conditions:

$$y(t_0) = \Phi(t_0) = y_0, \qquad y^{(k)}(t_0 + 0) = y_0^{(k)}, \qquad k = 1, 2, 3, \tag{2}$$
$$y(t) \equiv \Phi(t) \qquad \text{for } t \in E_{t_0}.$$

Existence and uniqueness of the solution of the initial problem (1), (2) is proved in paper [1].

Suppose next that $\int\limits_{t_0}^{\infty} | g(t) |\, dt < \infty$.

Lemma: *L et $q(t) \in C_1(J)$, $q(t) \geqq 0$ and let for $t \in J$ there hold:*
$2 - |p(t)| \geqq 0$, $2r(t) - |p(t)| - q'(t) - |g(t)| \geqq 0$, $Q_i(t) \geqq 0$, $i = 1, 2, \dots, n$.
If for the solution $y(t)$ of the initial problem (1), (2) *there holds*

$$H[y(t_0)] + \frac{1}{2}\int_{t_0}^{\infty} |g(t)|\, dt \leqq K < 0, \tag{3}$$

where $H[y(t)] = y(t)\, y''(t) - y'(t)\, y'''(t) + \frac{1}{2} q(t)\, y^2(t)$, then zero points of functions $y(t)$ and $y''(t)$ are interlaced.

Proof of this lemma is done analogously as in paper [1].

Theorem 1. *Let assumptions of Lemma be fulfilled and let for $t \in J$ furthermore there hold:*

$p(t) \in C_1(J)$, $p(t) \geqq 0$, $p'(t) \leqq 0$, $r(t) \geqq 0$, $Q_i(t) \geqq m > 0$, $i = 1, 2, \dots, n$ and let the functions $F_i(z)$, $i = 1, 2, \dots, n$ be increasing. Then every solution of the initial problem (1), (2) *fulfilling* (3) *is oscillatory on J.*

Proof: Let the solution $y(t)$ of the initial problem (1), (2) would be non-oscillatory. Then with regard to the functions $y(t)$, $y'(t)$ the following cases may occur:

1. $y(t)$ is non-oscillatory and $y'(t)$ is oscillatory on J; 2. $y(t) > 0$, $y'(t) \geqq 0$; 3. $y(t) > 0$, $y'(t) \leqq 0$; 4. $y(t) < 0$, $y'(t) \leqq 0$; 5. $y(t) < 0$, $y'(t) \geqq 0$, for $t \in \langle t_1, \infty)$ $t_1 \in J$.

We shall prove that none of the above-mentioned cases may occur:

1. If $y(t)$ is non-oscillatory and $y'(t)$ is oscillatory on J, then $y''(t)$ is oscillatory on J as well. This is a contradiction with the fact that $y(t)$ is non-oscillatory.

2. With regard to assumptions of theorem from (1) there follows:

$$y^{(4)}(t) \leqq |g(t)| - p(t)\, y''(t) - m y(t_1) \sum_{i=1}^{n} F_i(y[h_i(t_1)]).$$

After integrating this inequality from t_1 to $t(\geqq t_1)$ and arranging, we obtain:

$$y'''(t) \leqq y'''(t_1) + \int_{t_1}^{t} |g(s)|\, ds + p(t_1)\, y'(t_1) - p(t)\, y'(t) + \int_{t_1}^{t} p'(s)\, y'(s)\, ds -$$
$$- m y(t_1) \sum_{i=1}^{n} F_i(y[h_i(t_1)])\,(t - t_1),$$

from this inequality we have that $y'''(t) \to -\infty$ for $t \to \infty$. This is in contradiction with the fact that $y(t) > 0$.

3. If we multiply equation (1) by $y(t)$, use an inequality $\pm\, 2ab \leqq |a|\,(1 + b^2)$ and integrate from t_0 to $t(> t_1 \geqq t_0)$, we obtain:

$$(4) \qquad H[y(t)] \leqq H[y(t_0)] + \frac{1}{2}\int_{t_0}^{t} | g(s) | \, ds - \int_{t_0}^{t}\left(1 - \frac{1}{2} | p(s) |\right) y''^2(s) \, ds -$$
$$- \int_{t_0}^{t}\left[r(s) - \frac{1}{2} | p(s) | - \frac{1}{2} q'(s) - \frac{1}{2} | g(s) |\right] y^2(s) \, ds -$$
$$- \sum_{i=1}^{n} \int_{t_0}^{t} y^2(s) \, Q_i(s) \, F_i(y[h_i(s)]) \, ds.$$

From this inequality it follows that the function $H[y(t)]$ is bounded from above with a decreasing function. Furthermore from (3) it holds that $H[y(t_0)] < 0$ and therefore $H[y(t_1)] < 0$. It means that $H[y(t)] < 0$ for $t \in \langle t_1, \infty)$

Consider $y''(t)$ as follows:

a) Let $y''(t) \leqq 0$ for $t \in \langle t_2, \infty)$, $t_2 \geqq t_1$. But it means that $y(t)$ is concave and non-increasing. Therefore there exists such a point $t_3 \in \langle t_2, \infty)$, that $y(t_3) = 0$, which is a contradiction with $y(t) > 0$.

b) Let $y''(t) \geqq 0$ for $t \in \langle t_2, \infty)$, $t_2 \geqq t_1$. As $H[y(t)] < 0$ on $\langle t_1, \infty)$, and $y(t) > 0$, $y'(t) \leqq 0$, $y''(t) \geqq 0$, it must be $y'''(t) < 0$ for $t \in \langle t_2, \infty)$. With regard to signs of the functions $y(t)$, $y'(t)$, $y''(t)$, $q(t)$ and (4), (3) there holds $y(t)\, y'''(t) \leqq H[y(t)] < K$, so that $y'''(t) < \dfrac{K}{y(t)}$. Because $\lim_{t \to \infty} y(t) = c \geqq 0$, so for any $\varepsilon > 0$ there holds $y'''(t) < \dfrac{K}{c + \varepsilon}$ for t sufficiently large. Hence it follows that, $\lim_{t \to \infty} y''(t) = -\infty$, which is a contradiction.

c) Let $y''(t)$ be oscillatory for $t \in \langle t_2, \infty)$, $t_2 \geqq t_1$. With regard to lemma it means that the function $y(t)$ is also oscillatory for $t \in \langle t_2, \infty)$, which is in contradiction with $y(t) > 0$.

The cases 4, and 5, can be proved similarly as the cases 2, and 3.

Theorem 2. *Let the assumptions of lemma be fulfilled and let for $t \in J$ there hold:*

$$p(t) \in C_1(J), \quad p(t) \geqq 0, \quad p'(t) \leqq 0, \quad r(t) \geqq 0 \quad \text{and} \quad \int_{t_0}^{\infty} r(t) \, dt = +\infty.$$

Then every solution $y(t)$ of the initial problem (1), (2) *fulfilling* (3) *is oscillatory on J.*

Proof: Under assumption that the solution $y(t)$ of the initial problem (1), (2) is non-oscillatory, five cases may occur similarly as in the proof of theorem 1: 1. $y(t)$ is non-oscillatory and $y'(t)$ is oscillatory on J; 2. $y(t) > 0$, $y'(t) \geqq 0$; 3. $y(t) > 0$, $y'(t) \leqq 0$; 4. $y(t) < 0$, $y'(t) \leqq 0$; 5. $y(t) < 0$, $y'(t) \geqq 0$; for $t \in \langle t_1, \infty)$, $t_1 \in J$.

We shall only prove the case 2.

From the differential equation (1) there holds

$$y^{(4)}(t) \leqq |g(t)| - p(t)\,y''(t) - r(t)\,y(t),$$

from that after integrating from t_1 to $t(\geqq t_1)$ we have:

$$y'''(t) \leqq y'''(t_1) + \int_{t_1}^{t} |g(s)|\,\mathrm{d}s + p(t_1)\,y'(t_1) - y(t_1)\int_{t_1}^{t} r(s)\,\mathrm{d}s.$$

From the last inequality there holds $y'''(t) \to -\infty$ for $t \to \infty$. It is a contradiction with $y(t) > 0$ for $t \in \langle t_1, \infty)$. The case 4, can be proved similarly.

The cases 1, 3, 5, can be proved similarly as in theorem 1.

Theorem 3. *Let for $t \in J$ assumptions of lemma hold. Let instead of* (3) *there hold:*

$$H[y(t_0)] + \frac{1}{2}\int_{t_0}^{\infty} |g(t)|\,\mathrm{d}t \leqq 0. \tag{3'}$$

Suppose furthermore that $\int_{t_0}^{\infty} q(t)\,\mathrm{d}t = +\infty$. Then every solution $y(t)$ of the initial problem (1), (2) *fulfilling* (3') *is oscillatory on J.*

Proof: Let $y(t)$ be a non-oscillatory solution of the initial problem (1), (2). Then $y(t) \neq 0$ for $t \in \langle t_1, \infty)$, where $t_1 \in J$. Suppose that $y(t) > 0$ for $t \in \langle t_1, \infty)$. If we use assumptions of the theorem we obtain from (4) an inequality $H[y(t)] \leqq 0$, hence for $t \in \langle t_1, \infty)$ there follows:

$$\left[\frac{y''(t)}{y(t)}\right]' \leqq -\frac{1}{2}\,q(t).$$

By integrating the last inequality from t_1 to $t(\geqq t_1)$, we obtain

$$\frac{y''(t)}{y(t)} \leqq \frac{y''(t_1)}{y(t_1)} - \frac{1}{2}\int_{t_1}^{t} q(s)\,\mathrm{d}s, \tag{5}$$

from which there holds: $\lim_{t\to\infty} \dfrac{y''(t)}{y(t)} = -\infty$. It means that there exists $t_2 \in \langle t_1, \infty)$ such that $\dfrac{y''(t)}{y(t)} < 0$ for $t \in \langle t_2, \infty)$.

Because $y(t) > 0$, so $y''(t) < 0$ for $t \in \langle t_2, \infty)$. With regard to the function $y'(t)$ wo cases may occur:

a) There exists $t_3 \in \langle t_2, \infty)$ such that $y'(t_3) < 0$.

b) $y'(t) > 0$ for $t \in \langle t_2, \infty)$.

In the a) case it leads to existence $\lim_{t\to\infty} y(t) = -\infty$, which is a contradiction with $y(t) > 0$, $t \in \langle t_1, \infty)$.

In the b) case for any $t \in \langle t_2, \infty)$ there holds:

$$\frac{y''(t)}{y(t_2)} \leqq \frac{y''(t)}{y(t)}.$$

It is evident from the last inequality with regard to (5) that $y''(t) \to -\infty$ for $t \to \infty$. That is also a contradiction with $y(t) > 0$, $t \in \langle t_1, \infty)$.

The proof of the cases when $y(t) < 0$ for $t \in \langle t_1, \infty)$ can be easily done in a similar way.

REFERENCES

[1] Futák, J.: *On the properties solutions of nonlinear differential equations of the fourth order with delay*. Acta Fac. R. Nat. Univ. Comen., Math. 1974, 31.

[2] Futák, J., Šoltés, P.: *O nulových bodoch riešení lineárnej diferenciálnej rovnice 4. rádu*. Práce a štúdie VŠD č. 1, 1974.

[3] Lazer, A. C.: *The behavior of solutions of the differential equation* $y''' + p(x)\, y' + q(x)\, y = 0$. Pacific Journal of Math., Vol. 17, No. 3., 1966, 435—466.

[4] El'sgoľc, L. E., Norkin, S. B.: *Vvedenie v teoriju differenciaľnych uravnenij s otklonjajuščimsja argumentom*. Izdateľstvo „Nauka", Moskva 1971.

[5] Šoltés, P.: *A remark on the oscillatory behaviour of solutions of differential equations of order 3 and 4*. Archivum Math., Brno (v tlači).

J. Futák
010 88 Žilina, Marxa—Engelsa 25
Czechoslovakia