Miroslav Bartušek

On the distribution of zeros of solutions of some types of differential equation $y'' = q(t)y$

Archivum Mathematicum, Vol. 11 (1975), No. 1, 1--24

Persistent URL: `http://dml.cz/dmlcz/104838`

ARCH. MATH. 1, SCRIPTA FAC. SCI. NAT. UJEP BRUNENSIS
XI:1–24, 1975

ON THE DISTRIBUTION OF ZEROS OF SOLUTIONS OF SOME TYPES OF DIFFERENTIAL EQUATION $y'' = q(t)\,y$

MIROSLAV BARTUŠEK, Brno

(Received January 30, 1974)

This work deals with the distribution of zeros of solutions and theirs derivatives of the differential equation

$$(q) \qquad y'' = q(t)\,y$$

for the following sets of the functions q:

1. $q(t) = \sum_{k=0}^{\infty} q_k t^k$, $|t| < R$.

2. $q(t) = \sum_{k=-2}^{\infty} q_{k+2} t^k$ where the power series $\sum_{k=0}^{\infty} q_k t^k$ converges for $|t| < R$.

3. The function q has the asymptotic expansion for $t \to \infty$ $\quad q(t) \sim \sum_{k=0}^{\infty} q_k t^{-k}$.

1.1. Consider a differential equation

$$(q) \qquad y'' = q(t)\,y, \qquad q \in C^{\circ}(a, b), \qquad a \geqq -\infty, \qquad b \leqq \infty,$$

where $C^n(a, b)$ (n being a non-negative integer) is the set of all continuous functions having continuous derivatives up to and including the order n on (a, b).

Let y_1, y_2 be two linearly independent solutions of (q). Then every continuous function α which fulfils the relation

$$\operatorname{tg} \alpha(t) = \frac{y_1(t)}{y_2(t)}, \qquad t \in (a, b),$$

on (a, b) exceptionally zeros of y_2 is called the first phase of the basis (y_1, y_2). Every first phase α of the basis (y_1, y_2) has the following properties (see [4] § 5):

$$(1) \quad \begin{cases} 1. \ \alpha \in C^3(a, b), \\ 2. \qquad \alpha'(t) = \dfrac{-W}{y_1^2 + y_2^2} \neq 0, \qquad t \in (a, b), \\ \quad \text{where } W \text{ is Wronskian of } (y_1, y_2). \end{cases}$$

(1)

3. α fulfils the non-linear differential equation
$$-\frac{1}{2}\frac{\alpha'''}{\alpha'}+\frac{3}{4}\left(\frac{\alpha''}{\alpha'}\right)^2-\alpha'^2=q(t), \qquad t\in(a,b).$$
4. If, in addition, the equation (q) is oscillatoric for $t\to b_-$ (i.e. every non-trivial solution has infinitely many zeros on every interval of the form $[t_0, b)$, $t_0\in(a,b)$), then $\lim\limits_{t\to b_-}\alpha(t)=\infty$ if α is increasing, $\lim\limits_{t\to b_-}\alpha(t)=-\infty$ if α is decreasing.

Let $q(t)<0$, $t\in(a,b)$. Every continuous function which fulfils the equation
$$\operatorname{tg}\beta(t)=\frac{y_1'(t)}{y_2'(t)}$$
on (a,b) exceptionally zeros of y_2' is called the second phase of the basis (y_1, y_2). The second phase β of the basis (y_1, y_2) has the following properties (see [4] § 5):

(2)

1. $\beta\in C^1(a,b)$,
2. $$\beta'(t)=\frac{Wq(t)}{y_1'^2(t)+y_2'^2(t)}\neq 0, \qquad t\in(a,b).$$
3. If α is the first phase of the basis (y_1, y_2), then
$$\beta(t)=\alpha(t)+\operatorname{arccotg}\frac{1}{2}\left(\frac{1}{\alpha'(t)}\right)', \qquad t\in(a,b)$$
is the second phase of the basis (y_1, y_2).
4. If $q\in C^2(a,b)$, then β fulfils the non-linear differential equation
$$-\frac{1}{2}\frac{\beta'''}{\beta'}+\frac{3}{4}\left(\frac{\beta'''}{\beta'}\right)^2-\beta'^2=q-\frac{1}{2}\frac{q''}{q}+\frac{3}{4}\left(\frac{q'}{q}\right)^2$$
on the interval (a,b).

The first (second) phase of the differential equation (q) is understood to be the first (second) phase of a basis of (q).

1.2. In our later considerations we shall need the following statements concerning first and second phases of q.

1. Let $t_0\in(a,b)$, α_0, $\alpha_0'\neq 0$, α_0'' be arbitrary numbers. There exists exactly one first phase α of (q) which fulfils Cauchy initial conditions (see [4] § 7.1.)
$$\alpha(t_0)=\alpha_0, \qquad \alpha'(t_0)=\alpha_0', \qquad \alpha''(t_0)=\alpha_0''.$$

2. Let $t_0\in(a,b)$, β_0, $\beta_0'\neq 0$, β_0'' be arbitrary numbers and let $q(t)<0$, $t\in(a,b)$. Let $q'(t_0)$ exist. Then there exists exactly one second phase β of (q) which fulfils Cauchy initial conditions (see [4] § 7.17)
$$\beta(t_0)=\beta_0, \qquad \beta'(t_0)=\beta_0', \qquad \beta''(t_0)=\beta_0''.$$

3. Let $\alpha(\beta)$ be the first (second) phase of the basis (y_1, y_2) and λ an arbitrary number (and $q(t) < 0$ on (a, b)). Then $\alpha + \lambda$ $(\beta + \lambda)$ is the first (second) phase of the basis $(\bar{y}_1, \bar{y}_2)$, where

$$\begin{aligned}\bar{y}_1 &= \quad y_1 \cos \lambda + y_2 \sin \lambda,\\ \bar{y}_2 &= -y_1 \sin \lambda + y_2 \cos \lambda.\end{aligned}$$

See [4] § 5.17.

1.3. Let y_1 be a non-trivial solution of (q) vanishing at $t \in (a, b)$ and y_2 a non-trivial one the derivative of which vanishes at t. Let n be a positive integer. If $\varphi_n(t)$ is the n-th zero of y_1 lying on the right of t, then φ_n is called the n-th central dispersion of the first kind. Let $q(t) < 0$ on (a, b). If $\psi_n(t)$, or $\chi_n(t)$, or $\omega_n(t)$ is the n-th zero of y_2', or y_1', or y_1, resp., lying on the right of t, then ψ_n, or χ_n, or ω_n is called the n-th central dispersion of the second, or third, or fourth kind, resp. In all the work we shall omit the word "central".

If (q) is an oscillatoric $(t \to a_+, t \to b_-)$ differential equation, then the dispersions have the following properties (see [4] § 13):

(3)

1. $\varphi_n \in C^3(a, b)$,
   $\psi_n, \chi_n, \omega_n \in C^1(a, b)$,
2. $\delta(t) > t$, $\delta'(t) > 0$, $t \in (a, b)$, $\lim\limits_{t \to a_+} \delta(t) = a$, $\lim\limits_{t \to b_-} \delta(t) = b$ where δ is any n-th dispersion of the k-th kind of (q), $k = 1, 2, 3, 4$, $n \geqq 1$,
3. $-\{\varphi_n, t\} + q(\varphi_n)\, \varphi_n'^2 = q(t)$, $t \in (a, b)$.
   If $q \in C^2(a, b)$, then

$$\begin{aligned}&-\{\psi_n, t\} + \hat{q}(\psi_n)\, \psi_n'^2 = \hat{q}(t),\\ &-\{\chi_n, t\} + \hat{q}(\chi_n)\, \chi_n'^2 = q(t),\\ &-\{\omega_n, t\} + q(\omega_n)\, \omega_n'^2 = \hat{q}(t),\ t \in (a, b),\end{aligned}$$

   where

$$\hat{q}(t) = q(t) - \frac{1}{2}\frac{q''(t)}{q(t)} + \frac{3}{4}\left(\frac{q'(t)}{q(t)}\right)^2,$$

$$\{\delta, t\} = \frac{1}{2}\frac{\delta'''(t)}{\delta'(t)} - \frac{3}{4}\left(\frac{\delta''(t)}{\delta'(t)}\right)^2.$$

2. Abel's relations.
   Let $\alpha(\beta)$ be the first (second) phase of an arbitrary basis (y_1, y_2) of (q) such such that $0 < \beta - \alpha < \pi$, $\alpha(\beta)$ increasing (α, β can be chosen in this way, see [4] § 13.7). Then

$$\begin{aligned}\alpha(\varphi_n(t)) &= \alpha(t) + n\pi,\\ \beta(\psi_n(t)) &= \beta(t) + n\pi,\\ \beta(\chi_n(t)) &= \alpha(t) + n\pi,\\ \alpha(\omega_n(t)) &= \beta(t) + (n - 1)\,\pi.\end{aligned}$$

1.4.1. Let $f(t)$ be a function, $t \in [a, \infty)$. If there exists a sequence $\{a_n\}_0^\infty$ such that

$$\lim_{t\to\infty} f(t) = a_0,$$

$$\lim_{t\to\infty} t^{n+1}\left(f(t) - a_0 - \frac{a_1}{t} - \dots - \frac{a_n}{t^n}\right) = a_{n+1},$$

then it is said that the function f has the asymptotic expansion for $t \to \infty$

$$f(t) \sim \sum_{n=0}^{\infty} a_n t^{-n}.$$

1.4.2. The basic theorems concerning asymptotic expansions can be found e.g. in [3] § 2.8. The following statement is valid.

Let $f(t) = \sum_{k=0}^{\infty} a_k t^{-k}$, $g(t) = \sum_{k=0}^{\infty} b_k t^{-k}$. Then for arbitrary constants c_1, c_2 the relations

$$c_1 f + c_2 g \sim \sum_{k=0}^{\infty} (c_1 a_k + c_2 b_k)\, t^{-k},$$

$$fg \sim \sum_{k=0}^{\infty} d_k t^{-k}, \qquad d_k = \sum_{n=0}^{k} a_n b_{k-n}$$

hold. If $a_0 \neq 0$, then

$$\frac{1}{f(t)} \sim \sum_{k=0}^{\infty} d_k t^{-k}, \qquad a_0 d_0 = 1, \qquad \sum_{n=0}^{k} a_n d_{k-n} = 0, \qquad (k > 0).$$

If $f'(t) \sim \sum_{k=0}^{\infty} d_k t^{-k}$, then $d_0 = d_1 = 0$, $d_k = -(k-1)\, a_k$ for $k \geqq 2$. Further,

$$\int_t^{\infty} f(t)\,\mathrm{d}t \sim \sum_{k=2}^{\infty} \frac{a_k}{k-1} t^{-k} \qquad \text{for } a_0 = a_1 = 0,$$

$\left|\int_t^{\infty} f(t)\,\mathrm{d}t\right| = \infty$, when at least one of the coefficients a_0, a_1 is different from zero.

2. Two auxiliary functions have the great importance in the theory of transformations of linear differential equations of the second order and in the theory of dispersions of these equations. They are the first and second phases of (q). They pervade all the theory and theirs properties determine a row of fundamental properties of transformation functions. In this paragraph we shall study the expression of the first and second phases of (q) by means of power and asymptotic series for some coefficients q.

2.1. Let a differential equation

$$y'' = q(t)\, y \tag{4}$$

be given, where

$$q(t) = \sum_{k=0}^{\infty} q_k t^k, \qquad |t| < R \leqq \infty,$$

(q_k are real numbers). Then an arbitrary (complex) solution has the form (see [9] § 3.3.2.)

$$y(t) = \sum_{k=0}^{\infty} c_k t^k, \qquad |t| < R.$$

The coefficients c_k can be found out by the substitution of this solution into the equation (4) and by the comparison of the coefficients at the same powers: c_0, c_1 are arbitrary constants

$$(5) \qquad c_n = \frac{1}{n(n-1)} \sum_{k=0}^{n-2} c_k q_{n-k-2}, \qquad n = 2, 3, 4, \dots$$

2.1.1. Let α be an arbitrary first phase of the equation (4). Then there exist two linearly independent solutions y_1, y_2 of (q) such that α is the first phase of the basis (y_1, y_2). We have:

$$(6) \qquad y_1(t) = \sum_{k=0}^{\infty} a_k t^k, \qquad y_2(t) = \sum_{k=0}^{\infty} b_k t^k, \qquad |t| < R.$$

The coefficients a_k, resp. b_k are given by the relation (5) where $c_k = a_k$, resp. $c_k = b_k$ $(k \geqq 2)$ and a_0, a_1, b_0, b_1 are suitable numbers.

Let W be Wronskian of the basis (y_1, y_2):

$$W = W(y_1, y_2) = a_0 b_1 - a_1 b_0 \neq 0.$$

According to (1) we have

$$(7) \qquad \alpha'(t) = \frac{-W}{y_1^2(t) + y_2^2(t)} = -W \sum_{n=0}^{\infty} \mathrm{d}_n t^n, \qquad |t| < R,$$

where

$$(8) \qquad \mathrm{d}_0 = \frac{1}{a_0^2 + b_0^2}, \qquad \mathrm{d}_n = -\mathrm{d}_0 \sum_{k=0}^{n-1} \mathrm{d}_k \sum_{i=0}^{n-k} (a_i a_{n-k-i} + b_i b_{n-k-i}).$$

The assertion about the radius of convergence of the power series in (7) follows from the following consideration:

According to (6) the function $f = y_1^2 + y_2^2$ is holomorphic in the circle $|t| < G$ and it has no roots in it (see [5] Al). It follows from this that the function $-\mathrm{W}/f$ ist holomorphic for $|t| < R$, too and it can be expanded to the power series which must be identical with $-W . \sum_{k=0}^{\infty} \mathrm{d}_n \mathrm{t}^n$.

By the integration of the relation (7) in the limits from zero to t and from the definition of α we get:

$$(9) \qquad \alpha(t) = \sum_{k=0}^{\infty} \alpha_k t^k, \qquad |t| < R,$$

where

$$
(10)\quad \begin{cases} \alpha_0 = \operatorname{arctg}\dfrac{a_0}{b_0} + m\pi & \text{for } b_0 \neq 0\,(\operatorname{arctg} 0 = 0), \\ \quad = \left(m + \dfrac{1}{2}\right)\pi & \text{for } b_0 = 0, \\ \alpha_n = -W\dfrac{d_{n-1}}{n}, & m \text{ is a suitable integer.} \end{cases}
$$

But we can obtain the direct recurrence relation for the coefficients α_k by virtue of α fulfiling the non-linear differential equation (1) 3. By the substitution of (9) into this equation and by the comparison of the coefficients at the same powers we obtain:

$$
(11)\quad \begin{cases} \alpha_0, \alpha_1 \neq 0, \alpha_2 \text{ are arbitrary numbers} \\ \alpha_{k+3} = \dfrac{1}{3\binom{k+3}{3}\alpha_1}\Big\{-3\sum\limits_{n=0}^{k-1}\binom{n+3}{3}(k-n+1)\,\alpha_{n+3}\alpha_{k-n+1} + \\ \quad + 3\sum\limits_{n=0}^{k}\binom{n+2}{2}\binom{k-n+2}{2}\alpha_{n+2}\alpha_{k-n+2} - \sum\limits_{n=0}^{k}\big[\sum\limits_{s=0}^{n}(s+1)(n-s+1)\times \\ \quad \times \alpha_{s+1}\alpha_{n-s+1}\big]\big[q_{k-n} + \sum\limits_{s=0}^{k-n}(s+1)(k-n-s+1)\,\alpha_{s+1}\alpha_{k-n-s+1}\big]\Big\} \end{cases}
$$

(according to 1.2. the numbers $\alpha_0, \alpha_1 \neq 0, \alpha_2$ can be chosen really arbitrary).

Theorem 1. *Let α be the first phase of a differential equation (4) determined by Cauchy initial conditions at the point $t = 0$*

$$\alpha(0) = \alpha_0, \qquad \alpha'(0) = \alpha_1, \qquad \alpha''(0) = 2\alpha_2,$$

where $\alpha_0, \alpha_1 \neq 0, \alpha_2$ are arbitrary numbers. Then

$$\alpha(t) = \sum_{k=0}^{\infty} \alpha_k t^k, \qquad |t| < R$$

and the coefficients α_k are given by the recurrence relation (11).

2.1.2. Let β be an arbitrary second phase of the equation (4). Suppose in this paragraph that $q(t) < 0$, $|t| < R$. We can examine the second phase β in the same way as the first phase α. According to (2) we have

$$
(12)\qquad \beta'(t) = \frac{Wq(t)}{y_1'^2(t) + y_2'^2(t)} = W\sum_{k=0}^{\infty} h_k t^k, \qquad |t| < R,
$$

where y_1, y_2 are suitable solutions of (4), linearly independent

$$y_1(t) = \sum_{k=0}^{\infty} a_k t^k, \qquad y_2(t) = \sum_{k=0}^{\infty} b_k t^k, \qquad |t| < R$$

and W is Wronskian of the basis (y_1, y_2). The statement about the radius of convergence R can be proved in the same way as for the first phase α. By integration of (12) we obtain that the following formula is valid

$$\beta(t) = \sum_{k=0}^{\infty} \beta_k t^k, \qquad |t| < R, \tag{13}$$

where

$$\left\{\begin{aligned} \beta_0 &= \operatorname{arctg}\frac{a_1}{b_1} + m\pi \qquad \text{for } b_1 \neq 0\, (\operatorname{arctg} 0 = 0),\\ &= \left(m + \frac{1}{2}\right)\pi \qquad \text{for } b_1 = 0,\\ \beta_k &= w\frac{h_{k-1}}{k}, \qquad k = 1, 2, 3, \ldots, \qquad h_0 = \frac{q_0}{a_1^2 + b_1^2},\\ h_k &= \frac{1}{a_1^2 + b_1^2}\Big[q_k - \sum_{s=0}^{k-1} h_s \sum_{l=0}^{k-s} (l + 1)(k - s - l + 1)\times\\ &\quad \times (a_{l+1}a_{k-s-l+1} + b_{l+1}b_{k-s-l+1})\Big], \qquad k = 1, 2, 3, \ldots\\ &m \text{ is a suitable integer.} \end{aligned}\right. \tag{14}$$

For β_k we can get the direct recurrence relation by the substitution of (13) into the differential equation (2) 4 and comparison of the coefficients at the same powers. We obtain

$$\left\{\begin{aligned} &\beta_0, \beta_1 \neq 0, \beta_2 \text{ are arbitrary numbers}\\ &\beta_{k+3} = \frac{2}{(k+3)(k+2)(k+1)\beta_1}\Big\{-3\sum_{n=0}^{k-1}\binom{n+3}{3}(k - n + 1)\beta_{n+3}\beta_{k-n+1} +\\ &\quad + 3\sum_{n=0}^{k}\binom{n+2}{2}\binom{k-n+2}{2}\beta_{n+2}\beta_{k-n+2} - \sum_{n=0}^{k}\Big[\sum_{s=0}^{n}(s + 1)(n - s + 1)\times\\ &\quad \times \beta_{s+1}\beta_{n-s+1}\Big]\Big[\bar{q}_{k-n} + \sum_{s=0}^{k-n}(s + 1)(k - n - s + 1)\beta_{s+1}\beta_{k-n-s+1}\Big]\Big\},\\ &\text{where}\\ &\sum_{k=0}^{\infty}\bar{q}_k t^k = q(t) - \frac{1}{2}\frac{q''(t)}{q(t)} + \frac{3}{4}\frac{q'^2(t)}{q^2(t)}. \end{aligned}\right. \tag{15}$$

It follows from 1.2. that the function β defined by means of (13) and (15) is really the second phase of (4). Hence we have proved the following

Theorem 2. *Let a differential equation (4) be given and let $q(t) < 0$, $|t| < R$. Let β be its second phase determined by Cauchy initial conditions at $t = 0$*

$$\beta(0) = \beta_0, \qquad \beta'(0) = \beta_0', \qquad \beta''(0) = 2\beta_2,$$

where β_0, $\beta_1 \neq 0$, β_2 are arbitrary numbers. Then

$$\beta(t) = \sum_{k=0}^{\infty} \beta_k t^k, \qquad |t| < R$$

and the coefficients β_k are given by means of the recurrence relation (15).

Remark 1. In our later considerations about dispersions of the differential equation (4) we shall need to know when the first phase α and the second phase β belong to the same basis. The following statement follows from (2) 3:

Let the first phase α be determined by Cauchy initial conditions $\alpha(0) = \alpha_0$, $\alpha'(0) = \alpha_1 \neq 0$, $\alpha''(0) = 2\alpha_2$ where α_0, α_1, α_2 are arbitrary numbers. Then the second phase β belonging to the same basis as α is determined by the following Cauchy initial conditions:

$$\beta(0) = \alpha_0 + \operatorname{arccotg}\left(-\frac{\alpha_2}{\alpha_1^2}\right),$$

$$\beta'(0) = \alpha_1\left(1 - \frac{-3\alpha_2^2 + \alpha_1^4 + q_0\alpha_1^2}{\alpha_1^4 + \alpha_2^2}\right),$$

$$\beta''(0) = \frac{1}{(\alpha_1^4 + \alpha_2^2)^2}\left(-q_1\alpha_1^7 - 2q_0^2\alpha_1^4\alpha_2 - q_1\alpha_1^3\alpha_2^2\right),$$

where arctg is an arbitrary branch of this function.

2.2. Consider a differential equation

(16) $$y'' = q(t)\,y,$$

where

$$q(t) = \frac{q_0}{t^2} + \frac{q_1}{t} + q_2 + q_3 t + \ldots, \qquad q_0 < -\frac{1}{4}$$

and

$$t^2 q(t) = \sum_{k=0}^{\infty} q_k t^k$$

converges for $|t| < R \leqq \infty$ (q_k are real numbers).

Our aim is to express the first and second phases of (16) by means of infinite power series in the real domain. The equation (16) is the equation with regular singular point at $t = 0$. Its fundamental equation is

$$\varrho^2 - \varrho - q_0 = 0$$

with the roots:

$$\varrho_{1,2} = \frac{1}{2}\left(1 \pm i\sqrt{-1 - 4q_0}\right).$$

Then (16) has a (complex) solution y of the form (see [9] § 3.3.5)

$$y(t) = t^{\varrho_1}[1 + \sum_{k=1}^{\infty} c_k t^k], \qquad 0 < t < R,$$

where the coefficients c_k can be expressed by the following formulae:

$$c_1 = \frac{q_1}{2\varrho_1}, \qquad c_k = \frac{q_k + \sum_{n=1}^{k-1} c_{k-n} q_n}{k(k + 2\varrho_1 - 1)}. \tag{17}$$

As the second solution which we get in the same way from the root q_2 (and which is linearly independent with y) is complex conjugate with y, we obtain two linearly independent real solutions of (16) by separation of real and imaginary parts of y. Then

$$(18)\quad \begin{cases} y_1(t) = \sqrt{t}[\sum_{k=0}^{\infty} a_k t^k \cos A(t) - \sum_{k=0}^{\infty} b_k t^k \sin A(t)] \\ y_2(t) = \sqrt{t}[\sum_{k=0}^{\infty} b_k t^k \cos A(t) + \sum_{k=0}^{\infty} a_k t^k \sin A(t)], \\ \text{where} \quad a_0 = 1, \ b_0 = 0, \ a_n = \operatorname{Re} c_n, \ b_n = \operatorname{Im} c_n, \ n = 1, 2, \ldots \\ 0 < t < R, \qquad A(t) = \frac{\sqrt{-1 - 4q_0}}{2} \ln t. \end{cases}$$

From here

$$y_1^2(t) + y_2^2(t) = t\,[(\sum_{k=0}^{\infty} a_k t^k)^2 + (\sum_{k=0}^{\infty} b_k t^k)^2], \qquad 0 < t < R, \tag{19}$$

$$\begin{aligned} y_1'^2(t) + y_2'^2(t) = \frac{1}{t}\Bigg\{&\Bigg[\sum_{k=0}^{\infty}\left(\frac{\sqrt{-1 - 4q_0}}{2} a_k + k b_k\right) t^k\Bigg]^2 + \\ &+ \Bigg[\sum_{k=0}^{\infty}\left(\frac{\sqrt{-1 - 4q_0}}{2} b_k - k a_k\right) t^k\Bigg]^2 + \sum_{k=0}^{\infty} a_k t^k \cdot \sum_{k=0}^{\infty}\left(k + \frac{1}{4}\right) a_k t^k + \\ &+ \sum_{k=0}^{\infty} b_k t^k \cdot \sum_{k=0}^{\infty}\left(k + \frac{1}{4}\right) b_k t^k\Bigg\}. \end{aligned} \tag{20}$$

2.2.1. If we take y_1, y_2 for the functions of complex independent variable, then $f(t) = (y_1^2(t) + y_2^2(t))/t$ (after defining it at $t = 0$ by the limit) is holomorphic and different from zero for $|t| < R$. It follows from this that the function $1/f$ is holomorphic for $|t| < R$, too and it can be expanded into the power series with the centre at $t = 0$. Thus (now we consider t as a real variable)

$$\bar{\alpha}'(t) = -\frac{W}{t} \sum_{k=0}^{\infty} d_k t^k, \qquad 0 < t < R \tag{21}$$

holds for the first phase $\bar{\alpha}$ of the basis (y_1, y_2) where the coefficients d_k are given by the relation

$$d_0 = 1, \qquad d_k = -\sum_{n=0}^{k-1} d_n \sum_{m=0}^{k-n} (a_m a_{k-n-m} + b_m b_{k-n-m}), \qquad k \geqq 1$$

and W is Wronskian of (y_1, y_2):

$$W = \lim_{t \to 0^+} (y_1 y_2' - y_1' y_2) = \frac{\sqrt{-1 - 4q_0}}{2}. \tag{22}$$

By the integration of (21) and the rotation of (y_1, y_2) according to 1.2. for suitable λ we obtain that there exists the first phase α of (16) such that

$$\alpha(t) = -W \ln t + \sum_{k=0}^{\infty} \alpha_k t^k, \qquad 0 < t < R, \tag{23}$$

where α_0 is an arbitrary number

$$\alpha_k = -\frac{W \,.\, d_k}{k}, \qquad k = 1, 2, 3, \ldots \tag{24}$$

Now we find the basis $(\bar{y}_1, \bar{y}_2)$ to which the first phase (23) belongs. To this purpose we determine the coefficient $\bar{\alpha}_0$ of the first phase (23) of the basis (y_1, y_2). For a suitable integer n and a suitable branch of the function arctg we have (by use of the definition $\bar{\alpha}$):

$$\bar{\alpha}_0 = \lim_{t \to 0^+} \sum_{k=0}^{\infty} \bar{\alpha}_k t^k = \lim_{t \to 0^+} \operatorname{arctg}(\operatorname{tg}(\bar{\alpha} + W \ln t)) =$$

$$= \lim_{t \to 0^+} \operatorname{arctg}\left(\frac{\operatorname{tg}\bar{\alpha} + \operatorname{tg}(W \ln t)}{1 - \operatorname{tg}\bar{\alpha} \,.\, \operatorname{tg}(W \ln t)}\right) = \lim_{t \to 0^+} \operatorname{arctg}\left(\frac{y_1 + y_2 \operatorname{tg}(W \ln t)}{y_2 - y_1 \operatorname{tg}(W \ln t)}\right) =$$

$$= \lim_{t \to 0^+} \operatorname{arctg} \frac{\sum_{k=0}^{\infty} a_k t^k}{\sum_{k=0}^{\infty} b_k t^k} = \frac{\pi}{2} + n\pi,$$

$\Big(t \in J = (0, \varepsilon)$ where $\varepsilon > 0$ is a suitable number such that $\bar{\alpha}(t) + W \ln t \neq (2k + 1)\dfrac{\pi}{2}$, $t \in J$, $k = 0, \pm 1, \pm 2, \ldots\Big)$.

Also we can see that the basis (y_1, y_2) has the countable set of the first phases (23) where $\bar{\alpha}_0 = (2k + 1)\dfrac{\pi}{2}$, $k = 0, \pm 1, \pm 2, \ldots$ It follows from this and from 1.2. that the first phase α given by (23), (24) is the first phase of the basis $(\bar{y}_1, \bar{y}_2)$ where

$$\begin{aligned} \bar{y}_1 &= y_1 \sin \alpha_0 - y_2 \cos \alpha_0, \\ \bar{y}_2 &= y_1 \cos \alpha_0 + y_2 \sin \alpha_0. \end{aligned} \tag{25}$$

We can get the direct recurrence relation for the coefficients α_k by means of the substitution of (23) into the differential equation (1) 3 and by the comparison of coefficients at the same powers. We obtain:

$$
(26)\quad \begin{cases}
\alpha_0 \text{ is an arbitrary number} \\
\alpha_1 = -\dfrac{Wq_1}{2q_0}, \qquad \alpha_2 = \dfrac{6W^2\alpha_1^2 + q_0\alpha_1^2 + W^2q_2}{4W(1+W^2)}, \\
\alpha_3 = \dfrac{2}{W(12W^2+27)}[\alpha_1\alpha_2(4q_0 + 24W^2) + q_2(W^2 - 2W\alpha_1) - 4W\alpha_1^3 + \\
\quad + q_1\alpha_1^2 - 4W\alpha_2 q_1], \\
\alpha_{k+4} = \dfrac{2}{W(k+4)[(k+3)(k+5) - 4q_0]}\left\{\sum_{n=0}^{k}\left[3\binom{n+3}{3}(k-n+1)\times\right.\right. \\
\quad \left.\times\alpha_{n+3}\alpha_{k-n+1} - 3\binom{n+2}{2}\binom{k-n+2}{2}\alpha_{n+2}\alpha_{k-n+2}\right] + \\
\quad + (2W^2 + q_0)\sum_{n=0}^{k+2}(n+1)(k-n+3)\alpha_{n+1}\alpha_{k-n+3} + (4W\alpha_1 + q_1)\times \\
\quad \times[-2W(k+3)\alpha_{k+3} + \sum_{n=0}^{k+1}(n+1)(k-n+2)\alpha_{n+1}\alpha_{k-n+2}] + \\
\quad + W^2q_{k+4} - 2W\alpha_1 q_{k+3} + \sum_{n=0}^{k}(-2W(n+2)\alpha_{n+2} + \sum_{s=0}^{n}(s+1)\times \\
\quad \times(n-s+1)\alpha_{s+1}\alpha_{n-s+1})(q_{k-n+2} - 2W(k-n+2)\alpha_{k-n+2} + \\
\quad \left. + \sum_{s=0}^{k-n}(s+1)(k-n-s+1)\alpha_{s+1}\alpha_{k-n-s+1})\right\}.
\end{cases}
$$

Theorem 3. *There exist two sets of the first phases of the differential equation* (16) *the elements of which are*

$$
(27)\quad \begin{aligned}
\alpha(t) &= -\frac{\sqrt{-1-4q_0}}{2}\ln t + \sum_{k=0}^{\infty}\alpha_k t^k \qquad &&(\textit{decreasing phases}) \\
\bar{\alpha}(t) &= -\alpha(t), \qquad 0 < t < R &&(\textit{increasing phases}),
\end{aligned}
$$

where the coefficients α_k are given by the recurrence relation (26) and $\alpha(\bar{\alpha})$ belongs to the basis $(\bar{y}_1, \bar{y}_2)$ $((-\bar{y}_1, \bar{y}_2))$ given by (25).

2.2.2. Let $\bar{\beta}$ be the second phase of the basis (18). Let $q(t) < 0$, $0 < t < R$. Then it follows from (2) 2 and (20) that

$$
(28)\qquad \bar{\beta}'(t) = \frac{Wq(t)}{y_1'^2(t) + y_2'^2(t)} = \frac{W}{t}\sum_{k=0}^{\infty}d_k t^k, \qquad 0 < t < R,
$$

where W is Wronskian of (y_1, y_2) and the coefficients d_k are given as follows: $d_0 = -1$,

$$
d_k = -\frac{1}{q_0}\{q_k - \sum_{n=0}^{k-1}d_{k-n}\sum_{m=0}^{n}[(-q_0 + (n-m)(m+1))(a_m a_{n-m} + b_m b_{n-m}) + W(n-2m)(a_m b_{n-m} - a_{n-m}b_m)]\}.
$$

By the integration and by the rotation of (y_1, y_2) according to (2) 4 for suitable λ we obtain that there exists the second phase of (16) such that

$$\beta(t) = -W \ln t + \sum_{k=0}^{\infty} \beta_k t^k, \qquad 0 < t < R, \tag{29}$$

where β_0 is an arbitrary number

$$\beta_k = \frac{Wd_k}{k}, \qquad k = 1, 2, 3, \ldots$$

and β belongs to the basis $(\bar{y}_1, \bar{y}_2)$ where

$$\begin{aligned} \bar{y}_1 &= y_1 \,.\, \cos \bar{\bar{\beta}} + y_2 \sin \bar{\bar{\beta}}, \qquad \bar{\bar{\beta}} = \beta_0 - \operatorname{arctg} \frac{1}{\sqrt{-1 - 4q_0}}, \\ \bar{y}_2 &= -y_1 \,.\, \sin \bar{\bar{\beta}} + y_2 \cos \bar{\bar{\beta}}, \qquad \operatorname{arctg} 0 = 0. \end{aligned} \tag{30}$$

For the coefficients β_k we could obtain the direct recurrence relation by substitution of (28) into the differential equation (2) 4 and by comparison coefficients at the same powers.

Theorem 4. *Let (16) be a differential equation such that $q(t) < 0$, $0 < t < R$. Then there exist two sets of the second phases of (16) $\{\beta\}$, $\{\bar{\beta}\}$ such that*

$$\begin{aligned} \beta(t) &= -\frac{\sqrt{-1 - 4q_0}}{2} \ln t + \sum_{k=0}^{\infty} \beta_k t^k \qquad &&\textit{(decreasing phases)}, \\ \bar{\beta}\,(t) &= -\beta(t), \qquad 0 < t < R &&\textit{(increasing phases)}, \end{aligned}$$

where the coefficients β_k are given by (29), (28), (18) and (17). The second phase $\beta(\bar{\beta})$ belongs to the basis $(\bar{y}_1, \bar{y}_2)$ $((-\bar{y}_1, \bar{y}_2))$ given by (30).

2.2.3. **Remark 2.** Let α and β are the first and second phases of (16) given by (23) and (29), resp. Then α and β belong to the same basis $(\bar{\bar{y}}_1, \bar{\bar{y}}_2)$ iff

$$\alpha_0 - \beta_0 = \frac{\pi}{2} + \operatorname{arctg}\left(\frac{1}{\sqrt{-1 - 4q_0}}\right),$$

where arctg is an arbitrary branch of this function.

Remark 3. Some other differential equations can be transformed into the equation (16). For example

$$Y'' = Q(x)\, Y, \qquad Q(x) = \sum_{k=2}^{\infty} \frac{q_{k-2}}{x^k}, \qquad x > R > 0, \qquad q_0 < -\frac{1}{4},$$

investigated in a neighbourhood of $x = \infty$. The substitution

$$x = \frac{1}{t}, \qquad y(t) = Y(x)/x$$

transforms this differential equation into

$$y'' = q(t)\,y, \qquad q(t) = \frac{1}{t^2}\sum_{k=0}^{\infty} q_k t^k,$$

which is the equation (16).

2.3. Consider a differential equation

$$(31)\quad \left\{\begin{array}{l} y'' = q(t)\,y, \\ \text{where the function } q \text{ has the asymptotic expansion for } t \to \infty \\ q(t) \sim \sum_{k=0}^{\infty} q_k t^{-k}, \qquad q_0 < 0. \end{array}\right.$$

Then (31) has two linearly independent solutions $\bar{y}_1, \bar{y}_2$ which have the following (complex conjugate) asymptotic expansions (see [3] § 2.9):

$$(32)\qquad \bar{y}_1 \sim e^{\lambda t} t^{\sigma} \sum_{k=0}^{\infty} A_k t^{-k}, \qquad \bar{y}_2 \sim e^{-\lambda t} t^{\bar{\sigma}} \sum_{k=0}^{\infty} \bar{A}_k t^{-k}.$$

The function $\bar{y}_1'$, resp. $\bar{y}_2'$ has also the asymptotic expansion that can be obtained by means of the differentiation of (32).

The coefficients λ, σ, A_k we get by the substitution of (32) into (31) and by comparison of coefficients at the same powers. We have:

$$(33)\quad \left\{\begin{array}{l} \lambda = i\sqrt{-q_0}, \qquad \sigma = -\dfrac{q_1}{2\sqrt{-q_0}}\, i, \\ (i \text{ is the imaginary unit}),\ A_0 \neq 0 \text{ is an arbitrary number but we put } A_0 = 1 \\ A_1 = \dfrac{1}{2\lambda}(\sigma(\sigma - 1) - q_2), \\ A_k = \dfrac{1}{2\lambda k}[-\sum_{n=0}^{k-1} A_n q_{k-n+1} + A_{k-1}(\sigma(\sigma - 1) - 2\sigma(k-1) + k(k-1))]. \end{array}\right.$$

Let us define: $a_k = \operatorname{Re}(A_k)$, $b_k = \operatorname{Im}(A_k)$.

Then by separation of the real and imaginary parts of $\bar{y}_1$ we obtain the asymptotic expansions of two linearly independent real solutions of (31):

$$(34)\quad \begin{aligned} y_1(t) &\sim \sum_{k=0}^{\infty} a_k t^{-k} \cos Z - \sum_{k=0}^{\infty} b_k t^{-k} \sin Z, \\ y_2(t) &\sim \sum_{k=0}^{\infty} b_k t^{-k} \cos Z + \sum_{k=0}^{\infty} a_k t^{-k} \sin Z, \\ Z &= \sqrt{-q_0}\,t - \frac{q_1}{2\sqrt{-q_0}} \ln t. \end{aligned}$$

Wronskian of these two solutions is

$$(35)\qquad W(y_1, y_2) = \lim_{t\to\infty} (y_1 y_2' - y_2 y_1') = \sqrt{-q_0}.$$

It follows from (34) and 1.4.2. that

$$
(36)\quad \begin{cases}
y_1^2(t) + y_2^2(t) \sim \sum\limits_{k=0}^{\infty} \sum\limits_{n=0}^{k} (a_n a_{k-n} + b_n b_{k-n})\, t^{-k}, \\
y_1'^2(t) + y_2'^2(t) \sim \sum\limits_{k=0}^{\infty} \mathrm{d}_k t^{-k}, \\
\text{where } d_0 = -q_0,\ d_1 = W\overline{W}, \\
d_k = 2W(Wa_k + \overline{W}a_{k-1} - (k-1)\, b_{k-1}) + \sum\limits_{n=1}^{k-1} [(Wa_n + \overline{W}a_{n-1} - \\
\quad - (n-1)\, b_{n-1})\,(Wa_{k-n} + \overline{W}a_{k-n-1} - (k-n-1)\, b_{k-n-1}) + \\
\quad + (Wb_n + \overline{W}b_{n-1} + (n-1)\, a_{n-1})\,(Wb_{k-n} + \overline{W}b_{k-n-1} + \\
\quad + (k-n-1)\, a_{k-n-1})], \\
\overline{W} = -\dfrac{q_1}{2\sqrt{-q_0}}, \qquad k = 2, 3, \ldots
\end{cases}
$$

holds.

2.3.1. Let $\bar{\alpha}$ be the first phase of the basis (y_1, y_2). Then according to (1) 3, (36) and 1.4.2 we have

$$\bar{\alpha}'(t) = \frac{-W}{y_1^2(t) + y_2^2(t)} \sim -W \sum_{k=0}^{\infty} c_k t^{-k},$$

where the coefficients c_k are given by the relations

$$
(37)\qquad \begin{aligned}
c_0 &= 1, \\
c_k &= -\sum_{n=1}^{k} c_{k-n} \sum_{s=0}^{n} (a_s a_{n-s} + b_s b_{n-s}).
\end{aligned}
$$

By the integration from t to ∞ we have (by use of 1.4.2.)

$$\bar{\alpha}(t) + W(t + c_1 \ln t) - \lim_{t\to\infty} (\bar{\alpha}(t) + Wt + Wc_1 \ln t) \sim$$

$$\sim W \,.\, \sum_{k=1}^{\infty} \frac{c_{k+1}}{k} t^{-k},$$

$$(38)\qquad \bar{\alpha}(t) \sim \bar{\alpha}_0 - Wt - c_1 W \ln t + W \sum_{k=1}^{\infty} \frac{c_{k+1}}{k} t^{-k},$$

where $\bar{\alpha}_0$ is a suitable constant which we find out now.

For a suitable integer n and for a suitable branch of the function arctg we have:

$$\bar{\alpha}_0 = \lim_{t\to\infty} (\bar{\alpha}(t) + Wt + Wc_1 \ln t) = \lim_{t\to\infty} \operatorname{arctg} \operatorname{tg} (\bar{\alpha}(t) + (Wt + Wc_1 \ln t)) =$$

$$= \lim_{t\to\infty} \operatorname{arctg} \left(\frac{\operatorname{tg} \bar{\alpha}(t) + \operatorname{tg}(Wt + Wc_1 \ln t)}{1 - \operatorname{tg} \bar{\alpha}(t) \,.\, \operatorname{tg}(Wt + Wc_1 \ln t)} \right) = \lim_{t\to\infty} \operatorname{arctg}$$

$$\left(\frac{\bar{y}_1 + \bar{y}_2 \operatorname{tg}(Wt + Wc_1 \ln t)}{\bar{y}_2 - \bar{y}_1 \operatorname{tg}(Wt + Wc_1 \ln t)}\right) = \operatorname{arctg} \lim_{t \to \infty} \frac{\sum_{k=0}^{\infty} a_k t^{-k}}{\sum_{k=0}^{\infty} b_k t^{-k}} = \frac{\pi}{2} + n\pi,$$

$\left(t \in J = (\varepsilon, \infty)\right.$, where $\varepsilon > 0$ is a suitable number such that $\bar{\alpha}(t) + Wt + Wc_1 \ln t \neq$ $\left.\neq \frac{\pi}{2} + k\pi,\ t \in J,\ k = 0, \pm 1, \pm 2, \ldots\right)$.

Thus we can see that a countable set of the first phases (38) belongs to the basis (y_1, y_2) where $\bar{\alpha}_0 = \frac{\pi}{2} + n\pi$, $n = 0, \pm 1, \pm 2, \ldots$ By means of the rotation of the basis (y_1, y_2) according to 1.2. we get that the equation (31) has the first phase α such that

$$\alpha(t) \sim -Wt - c_1 W \ln t + \sum_{k=0}^{\infty} \alpha_k t^{-k}$$

holds where α_0 is an arbitrary number and

$$\alpha_k = W \cdot \frac{c_{k+1}}{k}. \tag{39}$$

This first phase belongs to the basis (z_1, z_2):

$$\begin{aligned} z_1 &= y_1 \sin \alpha_0 - y_2 \cos \alpha_0, \\ z_2 &= y_1 \cos \alpha_0 + y_2 \sin \alpha_0. \end{aligned} \tag{40}$$

Theorem 5. *There exist two sets of the first phases of the differential equation (31) the elements of which have the following asymptotic expansions:*

$$\begin{aligned} &\alpha(t) \sim \sqrt{-q_0}\, t - \frac{q_1}{2\sqrt{-q_0}} \ln t - \sum_{k=0}^{\infty} \alpha_k t^{-k} && \textit{(increasing phases)}, \\ &\bar{\alpha}(t) = -\alpha(t) && \textit{(decreasing phases)}. \end{aligned} \tag{41}$$

The coefficients α_k are given by (33), (37), (39) and $\alpha(\bar{\alpha})$ belongs to the basis $(-z_1, z_2)$ $((z_1, z_2))$ given by (40).

Remark 4. We can get the direct recurrence relation for the coefficients a_k by means of substitution of (41) into the equation (1) 3. Asymptotic expansions of α', α'', α''' we obtain by the differentiation of (41). We may do it because asymptotic expansions of α', α'', α''' exist as it follows from (1) 2, (31), (34), 1.4.3.

2.3.2. Let $\bar{\beta}$ be the second phase of the basis (y_1, y_2). Then according to (2) 2, (36) and 1.4.2.

$$\bar{\beta}'(t) = \frac{Wq(t)}{y_1'^2 + y_2'^2} \sim W \sum_{k=0}^{\infty} c_k t^{-k}$$

holds where

$$c_0 = -1, \qquad c_k = \frac{1}{d_0}\left(q_k - \sum_{n=0}^{k-1} c_n d_{k-n}\right). \tag{42}$$

We obtain in the same way as for the first phase $\bar{\alpha}$ of (31) that

$$\bar{\beta}(t) \sim \bar{\beta}_0 - Wt + c_1 W \ln t - W \sum_{k=1}^{\infty} \frac{c_{k+1}}{k} t^{-k} \tag{43}$$

holds where $\bar{\beta}_0$ is a suitable number which we can find out analogously as $\bar{\alpha}_0$ in 2.3.1. The result is as follows: $\bar{\beta}_0 = k\pi$, $k = 0, \pm 1, \pm 2, \ldots$

Finally, by means of the rotation of the basis (y_1, y_2) according to 1.2. for a suitable λ we can see that the differential equation (31) has the second phase of the basis (z_1, z_2) where

$$\beta(t) \sim c_1 W \ln t - Wt + \sum_{k=0}^{\infty} \beta_k t^{-k}, \tag{44}$$

$$\beta_0 \text{ is an arbitrary number}, \qquad \beta_k = -W \frac{c_{k+1}}{k}$$

and

$$\begin{aligned} Z_1 &= y_1 \cos \beta_0 + y_2 \sin \beta_0, \\ Z_2 &= -y_1 \sin \beta_0 + y_2 \cos \beta_0. \end{aligned} \tag{45}$$

Theorem 6. *There exist two sets $\{\beta\}$, $\{\bar{\beta}\}$ of the second phases of the differential equation (31) with the following asymptotic expansions of their elements:*

$$\begin{aligned} \beta(t) &\sim \sqrt{-q_0}\, t - \frac{q_1}{2\sqrt{-q_0}} \ln t - \sum_{k=0}^{\infty} \beta_k t^{-k} && (\textit{increasing phases}), \\ \bar{\beta}(t) &= -\beta(t) && (\textit{decreasing phases}). \end{aligned} \tag{46}$$

The coefficients β_k are given by (44), (42), (36) and (33). The second phase $\beta(\bar{\beta})$ belongs to the basis $(-z_1, z_2)$ $((z_1, z_2))$ given by (45).

Remark 5. In case that $q \in C^2[t_0, \infty)$ for a suitable number t_0, the direct recurrence relation for the coefficients β_k from Theorem 6 can be obtained by substitution of (46) into the equation (2) 4.

2.3.3. **Remark 6.** Let α and β be the first and second phases of the differential equation (31) given by the relations (41) and (44). These phases belong to the same basis if, and only if

$$\alpha_0 - \beta_0 = \frac{\pi}{2} + k\pi$$

holds, where k is a suitable integer.

3. Consider an oscillatory ($t \to a_+$, $t \to b_-$) differential equation

$$y'' = q(t)\, y, \tag{4}$$

where

$$q(t) = \sum_{k=0}^{\infty} q_k t^k, \qquad |t| < R \leqq \infty, \qquad a \leqq -R, \qquad b \geqq R.$$

In this paragraph we shall study the expression of the n-th dispersion of the k-th kind ($k = 1, 2, 3, 4$) of the differential equation (4) by means of power series. If we discuss the dispersions of the 2-nd, 3-rd or 4-th kind we shall suppose that $q(t) < 0$ for $|t| < R$.

Let α and β be the first and second phase of (4), resp. Then according to Theorems 1 and 2

$$\alpha(t) = \sum_{k=0}^{\infty} \alpha_k t^k, \qquad \beta(t) = \sum_{k=0}^{\infty} \beta_k t^k, \qquad |t| < R \tag{47}$$

holds. Let us choose the initial conditions in the following way:

$$\begin{gathered} \alpha_0 = \alpha_2 = 0, \qquad \alpha_1 = 1, \\ \beta_0 = \frac{\pi}{2}, \qquad \beta_1 = -q_0, \qquad \beta_2 = -q_1. \end{gathered} \tag{48}$$

According to Remark 1 α and β belong to the same basis, are increasing, and $0 < \beta(t) - \alpha(t) < \pi$ for $|t| < R$. Thus we can use them for solving the Abel's relations (3) 4.

Let γ and δ be arbitrary functions from α, β:

$$\gamma(t) = \sum_{k=0}^{\infty} \gamma_k t^k, \qquad \delta(t) = \sum_{k=0}^{\infty} \delta_k t^k, \qquad |t| < R. \tag{49}$$

As γ is increasing, there exists the inverse function γ^{-1} and we have (see [8] pp. 235, 255)

$$\gamma^{-1}(t) = \sum_{k=1}^{\infty} a_k (t - \gamma_0)^k, \qquad |t - \gamma_0| < R_1, \qquad a_1 = \frac{1}{\gamma_1},$$

$$a_k = \frac{(-1)^{k-1}}{k!\,\gamma_1^{2k-1}} \sum_{\lambda_1, \lambda_2, \dots} (-1)^{\lambda_1} \frac{(2k-2-\lambda_1)!}{\lambda_2!\,\lambda_3!\dots} \gamma_1^{\lambda_1} \gamma_2^{\lambda_2} \dots,$$

where the sum goes through all non-negative integers $\lambda_1, \lambda_2, \dots$ which fulfil two following equations:

$$\begin{aligned} \lambda_1 + \lambda_2 + \dots \qquad &= k - 1, \\ \lambda_2 + 2\lambda_2 + 3\lambda_3 + \dots &= 2k - 2. \end{aligned}$$

Let n be an integer such that

$$\lim_{t \to -R} \delta(t) < -n\pi. \tag{51}$$

It follows from this that there exists a number t_0, $|t_0| < R$ such that

(52) $$\delta(t_0) = -n\pi + \gamma_0$$

holds. Then $\delta(t) + n\pi - \gamma_0 = \sum_{k=1}^{\infty} b_k(t - t_0)^k$, $|t - t_0| < R - t_0$,

(53) $$b_k = \frac{\delta^{(k)}(t_0)}{k!} = \sum_{l=0}^{\infty} \binom{l+k}{k} \delta_{l+k} t_0^l.$$

According to [8] § 153 we have

(54) $$\begin{aligned} f(t) &= \gamma^{-1}(\delta(t) + n\pi) = \sum_{k=1}^{\infty} a_k\Big(\sum_{l=1}^{\infty} b_l(t - t_0)^l\Big)^k, \\ f(t) &= \sum_{k=1}^{\infty} f_k(t - t_0)^k, \qquad |t - t_0| < R_2, \end{aligned}$$

where R_2 is a suitable number. The coefficients f_k are given by

(55) $$f_k = \sum_{l=1}^{k} a_l \sum_{\substack{i_1 + i_2 + \ldots + i_l = k \\ i_s \neq 0}} \prod_{s=1}^{l} b_{i_s}.$$

As to the meaning of the function f the following statements are valid.

a) If $\gamma = \delta = \alpha$, then f is the n-th dispersion of the first kind of (4).

b) If $\gamma = \delta = \beta$, then f is the n-th dispersion of the second kind of (4).

c) If $\gamma = \alpha$, $\delta = \beta$, then f is the $(n + 1)$-st dispersion of the fourth kind of (4).

d) If $\gamma = \beta$, $\delta = \alpha$, then f is the n-th dispersion of the third kind of (4).

Let $\bar{R}$, $\bar{R} < R$ be an arbitrary positive number. Then there exist numbers M_1, M_2 such that

$$\left|\frac{\gamma_k}{\gamma_1}\right| \leqq \frac{M}{\bar{R}^{k-2}}, \qquad |\delta_k| \leqq \frac{M_2}{(\bar{R} - t_0)^k}$$

holds. It follows from [8] § 153, 155 that the function f is given by means of the power series $\sum_{k=1}^{\infty} f_k(t - t_0)^k$ on the interval $|t - t_0| < R_2$ where

(56) $$R_2 \geqq \frac{|\gamma_1| \bar{R}(\bar{R} - t_0)}{|\gamma_1| \bar{R} + 2M_2(1 + 2M_1\bar{R})}.$$

The coefficients f_k from (54) are given by the relations (55), (53), (52), (50) and (48). But for f_k we can get the direct recurrence relation by the substitution of (54) into the differential equation (3) 3. The result is as follows:

$$
(57)\left\{\begin{aligned}
&t_0, f_1, f_2 \text{ must be calculated by the above-mentioned way}\\
&f_{k+3} = \frac{-2}{(k+3)(k+2)(k+1)f_1}\left\{3\cdot\sum_{n=0}^{k-1}\binom{n+3}{3}(k-n+1)f_{n+3}f_{k-n+1} -\right.\\
&\quad - 3\sum_{n=0}^{k}\binom{n+2}{2}\binom{k-n+2}{2}f_{n+2}f_{k-n+2} +\\
&\quad + \sum_{n=0}^{k}\Big[\sum_{s=0}^{n}(s+1)(n-s+1)f_{s+1}f_{n-s+1}\Big]\cdot\bar{q}_{k-n} - \sum_{n=0}^{k}a_n \times\\
&\quad \times \sum_{s=0}^{k-n}\Big[\sum_{l=0}^{s}(l+1)(s-l+1)f_{l+1}f_{s-l+1}\Big]\times\\
&\quad \left.\times\Big[\sum_{l=0}^{k-n-s}(l+1)(k-n-s-l+1)f_{l+1}f_{k-n-s-l+1}\Big]\right\},\\
&\qquad\qquad a_k = \sum_{l=1}^{k}\bar{\bar{q}}_l\cdot\sum_{i_1+i_2+\ldots+i_l=k}\prod_{s=1}^{l}f_{i_s}.\\
&\qquad\qquad\qquad i_s \geqq 1,\\
&\text{where} \qquad \bar{q}_k = \bar{\bar{q}}_k = q_k \qquad \text{for the } n\text{-th dispersion of the 1st kind,}\\
&\sum_{k=0}^{\infty}\bar{q}_k t^k = \sum_{k=0}^{\infty}\bar{\bar{q}}_k t^k = \hat{q}(t) \quad \text{for the } n\text{-th dispersion of the 2nd kind,}\\
&\bar{q}_k = q_k,\ \sum_{k=0}^{\infty}\bar{\bar{q}}_k t^k = \hat{q}(t) \quad \text{for the } n\text{-th dispersion of the 3rd kind,}\\
&\sum_{k=0}^{\infty}\bar{q}_k t^k = \hat{q}(t),\ \bar{\bar{q}}_k = q_k \quad \text{for the } n\text{-th dispersion of the 4th kind,}\\
&\text{Here:}\\
&\qquad \hat{q}(t) = q(t) - \frac{1}{2}\frac{q''(t)}{q(t)} + \frac{3}{4}\left(\frac{q'(t)}{q(t)}\right)^2.
\end{aligned}\right.
$$

Theorem 7. *Let n be a positive integer. Let γ be the n-th (n + 1-st) dispersion of the k-th (4-th) kind of the differential equation* (4), $k = 1, 2, 3$ *and let* $q(t) < 0$, $|t| < R$ *if γ is the dispersion of the 2-nd, 3-rd or 4-th kind. Let* $\lim_{t\to -R}\alpha(t) < -n\pi$ *if γ is the dispersion of the 1-st or 3-rd kind, and* $\lim_{t\to -R}\beta(t) < -n\pi$ *if γ is the dispersion of the 2-nd or 4-th kind.*

Here α and β are the first and second phases of (4) *given by* (47). *Then*

$$\gamma(t) = \sum_{k=1}^{\infty} f_k(t - t_0)^k, \qquad |t - t_0| < R_2.$$

An estimation for the radius of convergence R_2 is given by (56) *and the coefficients f_k by means of the relation* (57).

Remark 7. If the differential equation (4) is oscillatory for $t \to -R$, then the assumption about the phases α and β in Theorem 7 is fulfiled because

$$\lim_{t \to -R} \alpha(t) = \lim_{t \to -R} \beta(t) = -\infty.$$

4. In this paragraph we shall study problems concerning the numerical processing of dispersions. By using the electronic computers it is very useful to use the Abel's relations (3) 4 which bind dispersions with phases of (q).

4.1. It is very useful to use the following method for the solution of Abel's relations (this method is called "*bisection*"). We make it clear briefly for the solution of the equation

$$\alpha(\varphi_n(t)) = \alpha(t) + n\pi. \tag{58}$$

Here φ_n is the n-th dispersion of the 1-st kind and α is the first phase of (4). Suppose that $q \in C^\circ(a, c)$, $a \leqq -R$, $c \geqq R$ and that the equation (4) is oscillatory ($t \to a_+$, $t \to c_-$). It follows from Theorem 1 that

$$\alpha(t) = \sum_{k=0}^{\infty} \alpha_k t^k, \qquad |t| < R$$

holds where the coefficients α_k are given by the recurrence relations (11). Let α be increasing.

Let t_0 be a number such that

$$\lim_{t \to R} \alpha(t) > \alpha(t_0) + n\pi. \tag{59}$$

(This condition is fulfiled if $c = R$). Our task is to find $\varphi_n(t_0)$ from (58). As α is increasing, it follows from (59) that there exists a number $b < R$ such that $t_0 < \varphi(t_0) \leqq b$. Let us define the sequences of numbers $\{\bar{t}_k\}_0^\infty$ and intervals $\{J_k\}_0^\infty$, $J_k = [a_k, b_k]$ such that

$$\bar{t}_k = \frac{a_k + b_k}{2}, \qquad a_0 = t_0, \qquad b_0 = b.$$

If $\alpha(\bar{t}_k) - \alpha(t_0) - n\pi > 0$, then $a_{k+1} = a_k$, $b_{k+1} = \bar{t}_k$. In the opposite case: $a_{k+1} = \bar{t}_k$, $b_{k+1} = b_k$. We can see that $\lim_{k \to \infty} \bar{t}_k = \varphi_n(t_0)$ holds and we have an estimation:

$$|\varphi_n(t_0) - t_k| \leqq \frac{b - t_0}{2^{k+1}}. \tag{60}$$

Thus we can take approximately $\varphi_n(t_0) = \bar{t}_k$ for a suitable large k.

This method can be used also for computation of the dispersions of an arbitrary kind and not only for the equation (4) but for those, too for which we are able to computate the first and second phases with sufficient accuracy. It is e.g. for the equation (16) on the interval $0 < t < R$ and for (31) in some neighbourhood of the point $t = \infty$. We must use corresponding Abel's relations (3) 4 and the first and second phases belonging to the same basis and fulfiling $0 < \beta - \alpha < \pi$ (see Remarks 2 and 6), of course.

The estimation b from (60) must be found out from the relation $\alpha(b) > \alpha(t) + n\pi$. If $R < \infty$, then we can take: $b = R$.

Remark 8. For computation of dispersions from Abel's relations we can use also the other methods used for solution of transcendental equations, e.g. Newton method, Regula Falsi method and others. The advantage of the method used in § 4.1. consists in its simplicity, quick convergence, very good estimation of error and in the fact that the approximation $\bar{t}_k$ is not weighted with errors of the previous approximations $\bar{t}_i$, $i < k$.

4.2. When computing dispersions by means of the method in § 4.1. we can see that it is necessary to find out values of infinite series. In the fact we must use only a limited number of terms of these series and the coefficients are weighted with errors which are the result of theirs computing by means of the recurrence relations. Now we shall derive some estimations of errors of the method. The estimation will be done for the k-th dispersion of the 1-st kind φ_k.

Let α be the first phase of (q) and $\bar{\alpha}$ its approximation. Let $\bar{\bar{\varphi}}_k$ be the exact solution of the equation $\bar{\alpha}(\bar{\bar{\varphi}}_k) = \bar{\alpha}(t) + k\pi$ and $\bar{\varphi}_k$ an approximation of $\bar{\bar{\varphi}}_k$. Suppose that

$$\begin{aligned} &|\alpha - \bar{\alpha}| \leqq \delta && \text{for } t = t_0 \text{ and } t = \bar{\varphi}_k, \\ &|\bar{\bar{\varphi}}_k - \bar{\varphi}_k| \leqq \delta_1 && \text{for } t = t_0. \end{aligned} \tag{61}$$

The constant δ_1 can be find out easily from (60). From this and from (58) we have successively:

$$|\bar{\alpha}(\bar{\bar{\varphi}}_k) - \alpha(\varphi_k)| \leqq \delta,$$
$$|\alpha(\bar{\bar{\varphi}}_k) - \alpha(\varphi_k)| \leqq 2\delta,$$
$$|\alpha'(\xi)|\,|\bar{\bar{\varphi}}_k - \varphi_k| \leqq 2\delta,$$

$$|\bar{\varphi}_k - \varphi_k| \leqq \delta_1 + \frac{2\delta}{|\alpha'(\xi)|}, \qquad \xi \in (\bar{\bar{\varphi}}_k, \varphi_k), \quad \text{resp.} \quad \xi \in (\varphi_k, \bar{\bar{\varphi}}_k), \tag{62}$$

(all functions are taken in the point $t = t_0$). Thus we obtain the estimation for the k-th dispersion of the 1-st kind. As the numbers δ, δ_1 can be chosen arbitrary small and the function $1/|\alpha'|$ can be estimated (see § 4.3) the difference $|\bar{\varphi}_k - \varphi_k|$ can be done arbitrary small, too. The same estimation is valid also for the other dispersions. We must only add the assumption $|\beta - \bar{\beta}| \leqq \delta$ if we use the second phase in Abel's relation. Further, for the dispersions of the 2-nd and 3-rd kind it is necessary to replace the function α' by β' in the relation (62).

4.3. This paragraph is devoted to the estimation of the derivatives of the 1-st and 2-nd phases which we need for using of (62).

4.3.1. Consider a differential equation (4). First of all we estimate the numbers c_n from (5). Let us define:

$$\bar{q}(t) = \sum_{k=0}^{\infty} | q_k | t^k.$$

The radius of convergence of this power series is R. Let $n_0 \geqq 2$ be the smallest integer such that

$$R_1^2 \bar{q}(R_1)/(n_0(n_0 - 1)) \leqq 1$$

holds where R_1, $t < R_1 < R$ is an arbitrary, but standing number. Let

$$M = \max_{k<n_0} c_k R_1^k. \tag{63}$$

We shall prove by induction that the following formula is valid:

$$| c_n | \leqq M \,.\, R_1^{-n}. \tag{64}$$

For $k = 0, 1, \ldots, n_0 - 1$ the statement follows from (63). Let it be valid for $k < n$. Then

$$| c_n | = \frac{\sum_{k=0}^{n-2} c_k \, q_{n-k-2}}{n(n-1)} \leqq \frac{M \,.\, \sum_{k=0}^{n-2} R_1^{-k} | q_{n-k-2} |}{n(n-1)} =$$

$$= \frac{M}{n(n-1)} R_1^{2-n} \sum_{k=0}^{n-2} | q_{n-k-2} | R_1^{n-k-2} \leqq \frac{M}{n(n-1)} R_1^{-n+2} \bar{q}(R_1) \leqq M \,.\, R_1^{-n}.$$

Thus we can see that (64) is valid (the estimation (64) could be obtained directly from the fact that the radius of convergence of the series $\sum_{k=0}^{\infty} c_k t^k$ is R. But here we have the concrete specification of M).

Now consider the first phase α. Then (see (7) and the denotation from 2.1.1.)

$$\left| \frac{1}{\alpha'(t)} \right| = \frac{y_1^2 + y_2^2}{| W |} = \frac{1}{| W |} \sum_{n=0}^{\infty} \sum_{k=0}^{n} (a_k a_{n-k} + b_k b_{n-k}) t^n \leqq$$

$$\leqq \frac{M_2}{| W |} \sum_{n=0}^{\infty} \frac{(n+1)}{R_1^n} t^n = \frac{M_2}{| W |} \left[\sum_{n=0}^{\infty} \frac{t^{n+1}}{R_1^n} \right]' = \frac{M_2}{| W |} \left[\frac{R_1 t}{R_1 - t} \right]',$$

$$\left| \frac{1}{\alpha'(t)} \right| \leqq \frac{M_2}{| W |} \left(\frac{R_1}{R_1 - t} \right)^2, \qquad R_1 > t. \tag{65}$$

Here $M_2 = 2 \max (M^2, M_1^2)$ where M, M_1 are constants from (64) for $c_n = a_n$ and $c_n = b_n$. W is Wronskian of the basis to which the first phase α belongs. Let β be the second phase of the same basis as α. Then

$$\left|\frac{1}{\beta'(t)}\right| = \frac{1}{|W|\,|q(t)|}\sum_{n=0}^{\infty}\sum_{k=0}^{n}(k+1)(n-k+1)(a_{k+1}a_{n-k+1}+b_{k+1}b_{n-k+1})\,t^n \leqq$$

$$\leqq \frac{M_2}{|W|\,|q(t)|}\sum_{n=0}^{\infty}\frac{t^n}{R_1^{n+2}}(n+1)(n+2)(n+3) = \frac{M_2}{|W|\,|q(t)|}\left(\sum_{n=0}^{\infty}\frac{t^{n+3}}{R_1^{n+3}}\right)''' =$$

$$= \frac{M_2}{|W|\,|q(t)|}\left[\frac{t^3}{R_1-t}\right]''' = \frac{6M_2}{|W|\,|q(t)|}\frac{R_1^2}{(R_1-t)^4}, \qquad R_1 > t.$$

4.3.2. Consider a differential equation (16). We shall use the denotation from § 2.2. First of all we estimate the coefficients a_k, b_k from (18). We have:

$$|k(k+2\varrho_1-1)| > k^2, \qquad |a_k| \leqq |c_k|, \qquad |b_k| \leqq |c_k|,$$

$$|c_k| < \frac{|q_k| + \sum_{n=1}^{k-1}|c_{k-n}|\cdot|q_n|}{k^2}.$$

Thus we can see that we have the similar situation as in the preceding paragraph. So we can similarly prove that

$$|a_k|,\ |b_k| \leqq \frac{M}{R_1^k}, \qquad t < R_1 < R, \tag{66}$$

holds where R_1 is an arbitrary but fixed number, $M = \max_{k<n_0}|c_k|\,R_1^{-k}$ and n_0 is the smallest integer fulfiling the inequality

$$\frac{\bar{q}(R_1)}{n_0^2} \leqq 1, \qquad \bar{q}(t) = \sum_{k=0}^{\infty}|q_k|\,t^k.$$

Similarly as in the preceding paragraph we can obtain the following estimation for the first and second phases of the basis (18):

$$\left|\frac{1}{\alpha'(t)}\right| \leqq \frac{2M^2|t|}{|W|}\frac{R_1^2}{(R_1-t)^2}, \qquad R > R_1 > t,$$

$$\left|\frac{1}{\beta'(t)}\right| \leqq \frac{2M^2R_1^2}{|t|\,|q(t)|\,|W|(R_1-t)^2}\left(-q_0 + \frac{R_1}{R_1-t} + \frac{R_1^2}{(R_1-t)^2}\right).$$

These relations are valid not only for the basis (18) but for (29), too because $y_1^2 + y_2^2 = \bar{y}_1^2 + \bar{y}_2^2$ and $y_1'^2 + y_2'^2 = \bar{y}_1'^2 + \bar{y}_2'^2$ hold. W is Wronskian of the corresponding basis.

4.4. For computation of the dispersions of the differential equation (4) we can use the results of § 3, too, because the dispersions are given by means of power series. But it seems that it is better to use the method in § 4.1. as:

– the recurrence formula (57) for the coefficients of the dispersions is essentially more complicated than ones for phases (11) and (15) and so they are weighted with greater error.
– the estimation for the radius of convergence of the series in (54), given by means of (56) is insufficient.
– the way of computation from § 4.1. can be used for the essentially wider set of the differential equations (q).

REFERENCES

[1] Bartušek M.: Connection between Asymptotic Properties and Zeros of Solutions of $y'' = q(t)\,y$. Arch. Math., **3**, VIII, 1972.
[2] Bartušek M.: About Relations among Dispersions of an Oscillatory Differential Equation $y'' = q(t)\,y$. Acta Univ. Palackianae Olom. **41**, 1973, pp. 55–61.
[3] Bellman R.: Teorija ustojčivosti rešenij differencialnych uravněnij. Moskva 1954.
[4] Borůvka O.: Lineare Differentialtransformationen 2. Ordnung. VEB Berlin 1967.
[5] Hille E.: Lecture on Ordinary Differential Equations. London, 1969.
[6] Neuman F.: A Role of Abel's Equation in the Stability Theory of Differential Equations. Aequationes Mathematicae. **6**, 1971, pp. 66—70.
[7] Neuman F.: Distribution of Zeros of Solutions of $y'' = q(t)\,y$ in Relation to their Behaviour in Large. Acta Math. Acad. Scientiarum Hungaricae. **8** (1973) 177–185
[8] Petr K.: Počet differenciální. Praha 1923.
[9] Sansone G.: Obyknovennyje differencialnyje urovněnija I, II. Moskva 1953.

M. Bartušek
662 95 Brno, Janáčkovo nám. 2a
Czechoslovakia