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PARTITIONS AND CONGRUENCES IN ALGEBRAS IV. ASSOCIABLE SYSTEMS

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We recall briefly some definitions and results with which it is possible to get acquainted in the introductory paragraphs of the paper [11]. The notion of partition on a set was studied in many papers, e.g. [1, 3, 4, 6, 7, 8, 9, 10, 13, 14]. A partition in a set G is a partition on a subset of the set G. The elements of the partition are called blocks and the union $\cup A$ of all blocks of the partition A is called the domain of the partition A. The set $P(G)$ of all partitions in the set G is in a one-to-one correspondence with the set of all symmetric and transitive binary relations (ST-relations) in G. The papers [2, 5, 11, 12] deal with the partitions „in", to a smaller extent also [3, 4]. Under the congruence in a universal algebra (G, Ω) we understand the stable ST-relation in the algebra (G, Ω). By the symbol $\mathscr{K}(G)$ we denote the lattice of all congruences in (G, Ω), symbols $\vee_{\mathscr{K}}$, $\bigvee_{\mathscr{K}}$ mean the supremum in the lattice $\mathscr{K}(G)$. As a rule, however, we write simply $\vee$, $\bigvee$ instead of $\vee_P$, $\bigvee_P$ for the supremum in the lattice $P(G)$.

4.0 In the paper [8] the concept of associable system of partitions on a set was introduced and in [6] it was generalized for the partitions in a set. In [14] the term of "absolutely permutable system of relations of equivalence" was used for the same concept; see also [7]. This concept represents a generalization of the permutability of partitions in a set for a system $\{A_1, A_2\}$ of two partitions in a set G is associable if and only if A_1, A_2 commute [6] Lemma 1.2, see also 4.3. In this paper we consider a system of congruences in an algebra G associable if the corresponding system of partitions in G is associable. Many theorems in the sequel use and generalize results of the paper [6] on partitions and apply them to the congruences in algebras, especially in Ω-groups. Main general results for the partitions are included in Theorems 4.17, 4.19 and 4.22.

4.1 Definition. A system $\{A_\iota : \iota \in \Gamma\}$ of partitions in a set G is called *associable* if it satisfies: For any system $\mathfrak{A} = \{x_\iota : \iota \in \Gamma\}$ of elements of the set G fulfilling $x^\alpha(\bigvee_{\iota \in \Gamma} A_\iota)\, x^\beta$ $(\alpha, \beta \in \Gamma)$ there holds one of the following conditions:

(4.1,1) $x \in G$ exists such that $x^\iota A_\iota x$, $\iota \in \Gamma$

(4.1,2) $\alpha \in \Gamma$ and $A_\alpha^1 \in A_\alpha$ exist such that $\mathfrak{A} \subseteqq A_\alpha^1$ and if $A_\alpha^1 \cap \cup A_\beta \neq \emptyset$ for some $\beta \in \Gamma$, then $A_\alpha^1 \in A_\beta$. ([6], Definition 1.2)

4.2 Definition. A system of congruences $\{A_\iota : \iota \in \Gamma\}$ in an algebra G is called *associable* if the system of partitions $\{A_\iota : \iota \in \Gamma\}$ in the set G is associable.

4.3 *A system $\{A_1, A_2\}$ of partitions in a set G is associable if and only if the partitions A_1, A_2 commute.* ([6], Lemma 1.2)

Proof. Let the system $\{A_1, A_2\}$ be associable, $x^1 A_2 A_1 x^2$. Then $x^1(A_1 \vee A_2)\, x^2$. If the condition (4.1,1) is satisfied, then $x \in G$ exists such that $x^1 A_1 x A_2 x^2$, hence $x^1 A_1 A_2 x^2$. If (4.1,2) holds, there exists $\alpha \in \{1, 2\}$ and a block $A_\alpha^1 \in A_\alpha$ such that $x^1, x^2 \in A_\alpha^1$. For any $\iota \in \{1, 2\}$ there is $A_\alpha^1 \cap \cup A_\iota \neq \emptyset$ since x^1 or $x^2 \in A_\alpha^1 \cap \cup A_\iota$. Thus $A_\alpha^1 \in A_\iota (\iota = 1, 2)$, i.e. $(x^1, x^2) \in A_1 \cap A_2 \subseteqq A_1 A_2$. Hence $A_2 A_1 \subseteqq A_1 A_2$. The reverse inclusion can be proved symmetrically. Hence the required equality.

Conversely, let the partitions A_1, A_2 commute and let $x^1(A_1 \vee A_2)\, x^2$. By [12] 3.1.1(1) $A_1 \vee A_2 = A_1 A_2 \cup A_1 \cup A_2$, thus $x^1 A_1 A_2 x^2$ or $x^1 A_1 x^2$ or $x^1 A_2 x^2$. In the first case $x \in G$ exists such that $x^1 A_1 x A_2 x^2$, consequently (4.1,1) holds. In the second case $x^1, x^2 \in A_1^1$ for some block (say $\alpha = 1$) $A_1^1 \in A_1$. If for $i = 2$ there exists an element $x \in A_1^1 \cap \bigcup A_2$, then $x^2 A_1 x A_2 y$ for some $y \in G$; from the permutability of the partitions A_1, A_2 it follows $x^2 A_2 A_1 y$, thus $x^2 A_2 x^2$. Hence $x^1 A_1 x^2 A_2 x^2$ and therefore the condition (4.1,1) is satisfied. The last case $x^1 A_2 x^2$ is symmetric to the preceding one.

4.4 *A system $\{A_\iota : \iota \in \Gamma\}$ of partitions on a set G is associable if and only if for any system $\{x^\iota : \iota \in \Gamma\}$ of elements of G with $x^\alpha \bigvee_{\iota \in \Gamma} A_\iota x^\beta\ (\alpha, \beta \in \Gamma)$ there holds* (4.1,1). *In particular, a system of two partitions on the set G is associable if and only if these partitions commute.*

Proof follows immediately from definition 4.2; the second part from 4.3.

4.5 *If $\{A_\iota : \iota \in \Gamma\}$ is an associable system of partitions in a set G, $\emptyset \neq \Lambda \subseteqq \Gamma$, then the system $\{A_\iota : \iota \in \Lambda\}$ is associable as well. In particular, any two partitions of an associable system commute.*
([6], Theorem 2.1)

4.6 We shall study more in detail the structure of an associable system of partitions. Let $A = \{A_\iota : \iota \in \Gamma\}$ be an associable system of partitions in a set G. Let $G_0 = \bigcap_{\iota \in \Gamma} \cup A$. Then the following propositions 4.6.1 to 4.6.6 hold.

4.6.1 *If a system $\mathfrak{A} = \{x^\iota : \iota \in \Gamma\}$ of elements of G satisfying $x^\alpha \bigvee_{\iota \in \Gamma} A_\iota x^\beta\ (\alpha, \beta \in \Gamma)$ has the property* (4.1,1), *then $\mathfrak{A}$ is a subset of some block $V \in \bigvee_{\iota \in \Gamma} A_\iota$ and the blocks of every A_ι cover the set V.*

Remark. Such a system $\mathfrak{A}$ and block $V \in \bigvee_{\in \Gamma} A$ will be called *a system of the 1st kind* and *a block of the 1st kind.*

Proof. The first assertion is clear. Further let $y \in V$, $\iota \in \Gamma$. Then there exists $x \in G$ such that $x^{\iota} A_{\iota} x_1 A_{\alpha_1} x_2 \ldots x_n A_{\alpha_n} y$ for some $\alpha_1, \ldots, \alpha_n \in \Gamma$, $x_1, \ldots, x_n \in G$, otherwise written $x^{\iota} A_{\iota} A_{\alpha_1} \ldots A_{\alpha_n} y$. From the permutability of partitions of the system A (4.3 and 4.5) it follows that $x^{\iota} A_{\alpha_1} \ldots A_{\alpha_n} A_{\iota} y$, thus $y \in \cup A_{\iota}$.

4.6.2 *Let V be a block of the* 1*st kind,* $\mathfrak{B} = \{y^{\iota} : \iota \in \Gamma\} \subseteqq V$. *Then* $\mathfrak{B}$ *is a system of the* 1*st kind.*

Proof. If $\mathfrak{B}$ has the property (4.1,2), there exist $\alpha \in \Gamma$ and $A_{\alpha}^1 \in A_{\alpha}$ such that $\mathfrak{B} \subseteqq \subseteqq A_{\alpha}^1$. By 4.6.1 $A_{\alpha}^1 \cap \cup A_{\iota} \neq \emptyset$, $\iota \in \Gamma$, thus $A_{\alpha}^1 \in A_{\iota}$, $\iota \in \Gamma$. Hence any element $x \in A_{\alpha}^1$ satisfies the condition (4.1,1), consequently $\mathfrak{B}$ is a system of the 1st kind.

4.6.3 *Let V be a block of the* 1*st kind, $a, b \in V$.* Then *$aA_{\alpha}A_{\beta}b$ for all $\alpha, \beta \in \Gamma$, $\alpha \neq \beta$. In other words: for $\alpha, \beta \in \Gamma$, $\alpha \neq \beta$, every block of the partition A_{α} contained in V is incident with every block of the partition A_{β} contained in V.*

Proof. By 4.6.2 the system $\mathfrak{B} = \{y^{\iota} : \iota \in \Gamma\} \subseteqq V$ where $y^{\alpha} = a$, $y^{\beta} = b$ is of the 1st kind, there exists then $x \in G$ such that $y^{\alpha} A_{\alpha} x$, $y^{\beta} A_{\beta} x$, thus $aA_{\alpha}A_{\beta}b$.

4.6.4 *Let V be a block of the* 1*st kind. Then V is a block of every partition $A_{\alpha}A_{\beta}$, $\alpha, \beta \in \Gamma$, $\alpha \neq \beta$.*

Proof. $A_{\alpha}A_{\beta}$ is a partition according to [12] 3.1 as A_{α}, A_{β} are permutable partitions by 4.3 and 4.5. The assertion follows from 4.6.3 and from the relation $A_{\alpha}A_{\beta} \leqq \bigvee_{\iota \in \Gamma} A_{\iota}$.

4.6.5 $G_0 = \bigcap_{\iota \in \Gamma} \cup A_{\iota}$ *is union of all blocks of the* 1*st kind.*

Proof. By 4.6.1 every block of the 1st kind is a subset of G_0. Conversely, let $x \in G_0$. Then $\{x^{\iota} : \iota \in \Gamma\}$, where $x^{\iota} = x$, $\iota \in \Gamma$, is a system of the first kind, thus the block of the partition $\bigvee_{\iota \in \Gamma} A_{\iota}$ containing x is of the 1st kind (and is contained in G_0).

4.6.6 *If $V \in \bigvee_{\iota \in \Gamma} A_{\iota}$ is not a block of the* 1*st kind (i.e. if V is not a subset of G_0), then it is a block of every A_{ι} the domain of which intersects V.*

Proof. If V is not a block of the 1st kind, any system $\mathfrak{B} = \{x^{\iota} : \iota \in \Gamma\} \subseteqq V$ satisfies (4. 1, 2) (by 4.6.2). Then there exist $\alpha \in \Gamma$ and $A_{\alpha}^1 \in A_{\alpha}$ such that $\mathfrak{B} \subseteqq A_{\alpha}^1$ and there is $A_{\alpha}^1 \subseteqq V$. If $A_{\alpha}^1 \cap \cup A_{\iota} \neq \emptyset$ holds for some $\iota \in \Gamma$, then $A_{\alpha}^1 \in A_{\iota}$. Thus $V = A_{\alpha}^1 \in A_{\iota}$ for all such $\iota \in \Gamma$.

4.7 Now let G be an Ω-group and $\{A_{\iota} : \iota \in \Gamma\}$ be an associable system of congruences in G. The following propositions 4.7.1 to 4.7.5 hold.

4.7.1 $(\bigvee_{\iota \in \Gamma} A_{\iota})(O) = \sum_{\iota \in \Gamma} A_{\iota}(O) = A_{\alpha}(O) + A_{\beta}(O) \subseteqq G_0$ *for any $\alpha, \beta \in \Gamma$, $\alpha \neq \beta$.*

Proof. The block $V = (\bigvee_{\iota \in \Gamma} A_{\iota})(O)$ is of the 1st kind since the system $\{x^{\iota} = O : \iota \in \Gamma\}$ is of the 1st kind. By 4.6.5 $V \subseteqq G_0$. Whenever $\alpha \neq \beta$ ($\alpha, \beta \in \Gamma$), by

4.6.3 $A_\alpha(O)$ is incident with every block of the congruence A_β contained in V and these blocks of the partition A_β cover V (by 4.6.1). Thus $V = A_\alpha(O) + A_\beta(O)$.

4.7.2 *If for some $\alpha, \beta \in \Gamma$, $\alpha \neq \beta$ the domains $\cup A_\alpha$, $\cup A\beta$ are incident outside G_0, then $A_\iota(O) \subseteq A_\alpha(O) = A_\beta(O)$ for all $\iota \in \Gamma$.*

Proof. Let $x \in \cup A_\alpha \cap \cup A_\beta \cap (G \setminus G_0)$. Then $x \in \cup (\bigvee_{\iota \in \Gamma} A_\iota)$, hence $x^\alpha \bigvee_{\iota \in \Gamma} A_\iota x^\beta$ $(\alpha, \beta \in \Gamma)$ holds for the system $\{x^\iota = x : \iota \in \Gamma\}$. By 4.6.5 this system is not of the 1st kind, therefore the condition (4.1,2) is satisfied for some index $\gamma \in \Gamma$ and some block $A_\gamma^1 \in A_\gamma$. Since $x \in A_\gamma^1 \cap \cup A_\alpha$, $x \in A_\gamma^1 \cap \cup A_\beta$, there holds $A_\gamma^1 \in A_\alpha$, $A_\gamma^1 \in A_\beta$, consequently $A_\alpha(O) = A_\gamma(O) = A_\beta(O)$.

The inclusion $A_\iota(O) \subseteq A_\alpha(O)$ $(\iota \in \Gamma)$ can be obtained from 4.7.1 in the following way: $A_\alpha(O) = A_\alpha(O) + A_\beta(O) = A_\alpha(O) + A_\iota(O)$ for all $\iota \in \Gamma$, $\iota \neq \alpha$, then $A_\iota(O) \subseteq$ $\subseteq A_\alpha(O)$.

4.7.3 $G_0 / \sum_{\iota \in \Gamma} A_\iota(O) = \bigvee_{\mathscr{K}, \iota \in \Gamma} (A_\iota \sqcap G_0) = \bigvee_{P, \iota \in \Gamma} (A_\iota \sqcap G_0)\ (= \mathfrak{Y})$. *The partition $\bigvee_{P, \iota \in \Gamma} A_\iota$ as a set of its blocks contains the set of blocks $\mathfrak{Y}$.*

As for the symbol $\sqcap$ see [3] I 2.3: $A \sqcap G_0 = \{A^1 \cap G_0 : A^1 \in A,\ A^1 \cap G_0 \neq \emptyset\}$

Proof. The first equality follows from 4.7.1 (the system $\{A_\iota \sqcap G_0 : \iota \in \Gamma\}$ is associable), the second from the fact that $A_\iota \sqcap G_0$ are congruences on the Ω-group G_0 (for congruences on an algebra there is namely $\bigvee_{\mathscr{K}} = \bigvee_P$). The last assertion follows from 4.7.1 for $A_\iota(O) \subseteq G_0$, $\iota \in \Gamma$.

4.7.4 *It holds the following equality between the sets of blocks $\bigvee_{P, \iota \in \Gamma} A_\iota = G_0 / \sum_{\iota \in \Gamma} A_\iota(O) \cup$ $\cup \bigcup_{\iota \in \Gamma} [A_\iota \setminus (A_\iota \sqcap G_0)]$.*

Proof follows from 4.7.3 and 4.6.6.

4.8 Definition. We shall say that the subset $H \subseteq G$ *respects* a partition A in G if there holds: $A^1 \in A$, $A^1 \cap H \neq \emptyset \Rightarrow A^1 \subseteq H$.

4.8.1(a) *If $\{A_i : \iota \in \Gamma\}$ is an associable system of partitions in a set G, then each oj the sets $\cup A_\alpha$, $\alpha \in \Gamma$, $G_0 = \bigcap_{\iota \in \Gamma} \cup A_i$ respects each of the partitions A_β, $\beta \in \Gamma$, $\bigvee_{\iota \in \Gamma} A_i$.*

(b) *If A is a congruence in an Ω-group G, H a subgroup of the additive group G, then H respects the partition A if and only if $A(O) \subseteq H$.*

Proof. (a) Let $V \in \bigvee_{\iota \in \Gamma} A$, let A_α^1 be a block of a partition A_α for which $A_\alpha^1 \subseteq V$ and let $G_0 = \bigcup_{\iota \in \Gamma} \cup A_i$. By 4.6.6 it holds: $V \cap G_0 = \emptyset \Rightarrow V = A_\alpha^1 \subseteq \cup A_\alpha$. Let $V \cap G_0 \neq \emptyset$. If V is a block of the 1st kind, there is $V \subseteq G_0 \subseteq \cup A_\alpha$ (4.6.5). If V is not a block of the 1st kind, then by 4.6.6 $V = A_\alpha^1$, therefore again $V = A_\alpha^1 \subseteq \cup A_\alpha$.

We have proved that $\cup A_\alpha$ respects the partition $\bigvee_{\iota \in \Gamma} A_\iota$. Hence it already follows the assertion (a).

(b) is evident.

4.8.2 Definition. Let $A = \{A_\iota : \iota \in \Gamma\}$ be a system of partitions in a set G, $\emptyset \neq H \subseteqq G$. Under $A \sqcap H$ we understand the system $\{A_\iota \sqcap H : \iota \in \Gamma\}$.

4.9 *A system $A = \{A_\iota : \iota \in \Gamma\}$ of partitions in a set G is associable if and only if for $G_0 = \bigcup_{\iota \in \Gamma} \cup A_\iota$ there holds:*

(4.9,1) *$A \sqcap G_0$ is an associable system of partitions (on G_0):*

(4.9,2) *G_0 respects the partition A_ι, $\iota \in \Gamma$;*

(4.9,3) α, $\beta \in \Gamma$, $A^1_\alpha \in A_\alpha$, $A^1_\beta \in A_\beta$, $A^1_\alpha \cap A^1_\beta \cap (G \setminus G_0) \neq \emptyset \Rightarrow A^1_\alpha = A^1_\beta$.

([6] Lemma 2.2)

Remark. 1. In the Theorem it is possible to write instead of "for $G_0 = \bigcap_{\iota \in \Gamma} \cup A_\iota$ there holds" "there exists a subset $G_0 \subseteqq G$ such that it holds" together with the fact that the requirement "on G_0" in (4.9.1) will not be in the parantheses.

This latter formulation is used in [6] Lemma 2.2. From the proof of this lemma it follows that the formulation given in 4.9 can be used equivalently and that for G_0 the union of all blocks of the 1st kind of the partition $\bigvee_{\iota \in \Gamma} A_\iota$ can be taken. By 4.6.5 there is $G_0 = \bigcap_{\iota \in \Gamma} \cup A_\iota$.

2. 4.3 follows immediately from 4.9.

4.10 In the following theorems 4.10.1 and 4.10.2 which can be obtained by an easy modification of Theorem 4.9 for congruences in an Ω-group G, appropriate variants regarding G_0 as in Remark to 4.9 can be applied.

4.10.1 *A system $A = \{A_i : \iota \in \Gamma\}$ of congruences in an Ω-group G is associable if and only if there holds for $G_0 = \bigcap_{\iota \in \Gamma} \cup A_\iota$*

(4.10,1) *$A \sqcap G_0$ is an associable system of congruences (on the Ω-group G_0);*

(4.10,2) $A_\iota(O) \subseteqq G_0$, $\iota \in \Gamma$,

(4.10,3) α, $\beta \in \Gamma$. $(\cup A_\alpha \cap \cup A_\beta) \setminus G_0 \neq \emptyset \Rightarrow A_\alpha(O) = A_\beta(O)$.

4.10.2 *A system $\{A_\iota : \iota \in \Gamma\}$ of congruences in an Ω-group G is associable if and only if for $G_0 = \bigcap_{\iota \in \Gamma} \cup A_\iota$ there holds:*

(4.10,4) *$A_\iota(O)$ is an ideal of G_0, $\iota \in \Gamma$;*

(4.10,5) *$\{G_0/A_\iota(O) : \iota \in \Gamma\}$ is an associable system of congruences (on the* Ω-group G_0);

(4.10,6) α, $\beta \in \Gamma$, $(\cup A_\alpha \cap \cup A_\beta) \setminus G_0 \neq \emptyset \Rightarrow A_\alpha(O) = A_\beta(O)$.

4.11 Let a system $\{A_\iota : \iota \in \Gamma\}$ consist of two congruences in an Ω-group G ($\Gamma = \{1, 2\}$). Then the condition (4.10,1) is always satisfied because two congruences on an Ω-group commute and then they are associable by 4.4. Condition (4.10,3) is satisfied trivially. Hence, by 4.10.1 we obtain the following statement.

Congruences A_1, A_2 in an Ω-group form an associable system if and only if $A_1(O) \cup A_2(O) \subseteqq \cup A_1 \cap \cup A_2$, which is by 3.9 [12] equivalent to the permutability of congruences A_1, A_2. Thus we have recovered Theorem 4.3 for congruences in an Ω-group.

4.11.0 If we use Theorem 4.9 for a pair of congruences in an algebra, then taking regard to Theorem 4.3 we obtain the following propositions 4.11.1 and 4.11.2:

4.11.1 *Congruences A_1, A_2 in an algebra G commute if and only if, for the subalgebra $G_0 = \cup A_1 \cap \cup A_2$ there holds:*

(4.11,1) *Congruences $A_1 \sqcap G_0$, $A_2 \sqcap G_0$ (on G_0) commute;*
(4.11,2) *G_0 respects the partitions A_1, A_2.*

4.11.2 *Let G be an algebra of a variety in which congruences "on" commute. Then it holds: Congruences A_1, A_2 in the algebra G commute if and only if the set $\cup A_1 \cap \bigcup A_2$ respects the partitions A_1, A_2.*

E.g., the class of all Ω-groups and the class of all relatively complemented lattices (see 0.4 [11]) fulfil the assumptions of Theorem 4.11.2.

4.12 An example of a system of congruences on an Ω-group (that is of a system of congruences commuting in pairs) which is not associable.

Let card $\Gamma \geqq 3$, $G = \tilde{\sum\limits_{\iota \in \Gamma}} Z_\iota$, card $Z_\iota \geqq 2$, Z_ι being an arbitrary Ω-group ($\iota \in \Gamma$). For any $\iota \in \Gamma$, the set $A_\iota(O) = \{(\ldots O, a_\iota, O, \ldots) : a_\iota \in Z_\iota\}$ is an ideal in G, $A_\iota = G/A_\iota(O)$ is a congruence on the Ω-group G. The system $\{A_\iota : \iota \in \Gamma\}$ is not associable. Suppose the contrary. For $\alpha \in \Gamma$ let there be $x^\alpha \in \dot{\sum\limits_{\iota \in \Gamma}} Z_\iota$, $x^\alpha = (\ldots, b_\iota^\alpha, \ldots)$ $(b_\iota^\alpha \in Z_\iota)$. Since A_ι are congruences on Ω-group, there holds $(\bigvee_{P\, \iota \in \Gamma} A_\iota)(O) = (\bigvee_{\mathscr{X}\, \iota \in \Gamma} A_\iota)(O) = \sum\limits_{\iota \in \Gamma} A_\iota(O)$. Hence $x^\alpha \bigvee\limits_{\iota \in \Gamma} A_\iota x^\beta$ $(\alpha, \beta \in \Gamma)$. By our supposition (see 4.4) there exists $x = (\ldots, b_\iota, \ldots) \in G$ such that $x^\alpha A_\alpha x$ $(\alpha \in \Gamma)$. Thus for every $\iota, \alpha \in \Gamma$, $\iota \neq \alpha$, we have got (4.12,1) $b_\iota = b_\iota^\alpha$.

Choose three distinct indices $\iota, \alpha, \beta \in \Gamma$ and pick x^α, x^β such that $b_\alpha^\iota \neq b_\iota^\beta$. But by (4.12,1) it follows $b_\iota^\alpha = b_\iota = b_\iota^\beta$, contrary to the choice of x^α, x^β.

4.13 Example 4.12 shows that, in general, the following theorem does not hold: If $A = \{A_\iota : \iota \in \Gamma\}$ is an associable system of congruences in an Ω-group G, B a congruence in G permutable with all A_ι, $\iota \in \Gamma$, then the system $\{B\} \cup A$ is associable.

The Theorem does not hold even in the case of congruences "on".

Proof. The system $\{A_1, A_2, A_3\}$ from example 4.1 (for card $\Gamma = 3$) which is not associable arises from the associable system $\{A_1, A_2\}$ being extended by a congruence A_3 (which commutes with A_1 and A_2 as well).

4.14 *Let A, B, C be partitions in a set G. Then $A(B \cap C) \subseteqq AB \cap AC$. If $A \leqq C$, then $A(B \cap C) = AB \cap AC = AB \cap C$. Analogous relations hold for binary relations $(B \cap C) A$, $BA \cap CA$, $BA \cap C$.*

Proof. $x[A(B \cap C)] z \Rightarrow y \in G$ exists such that $xAy(B \cap C) z \Rightarrow xAyBz, xAyCz \Rightarrow$ $\Rightarrow xACz, xABz \Rightarrow x(AB \cap AC) z$. $A(B \cap C) \subseteqq AB \cap AC$ is proved.

Let $A \leqq C$. Then $AC \subseteqq CC = C$, consequently $AB \cap AC \subseteqq AB \cap C$. Now, if $x(AB \cap C) z$, then xCz and $y \in G$ exists such that $xAyBz$; then with regard to $A \leqq C$ there will be xCy and with regard to xCz we get yCz. Hence $xAy(B \cap C) z$, $x[A(B \cap C)] z$ and therefore $AB \cap C \subseteqq A(B \cap C)$. Finally, $AB \cap C \leqq A(B \cap C) \subseteqq$ $\subseteqq AB \cap AC \subseteqq AB \cap C$.

4.14.1 *Let B, C be partitions in a set G. Then $B(B \cap C) = B \cap BC$, $(B \cap C) B =$ $= B \cap CB$.*

The proof follows directly from 4.14 if we put by turns B, C, B instead of A, B, C.

4.14.2 *Let A, B, C be partitions in a set G, $A \leqq C$. If A, B commute, then A, $B \wedge C$ commute and $A \wedge B$, $B \wedge C$ commute.*

Proof. By [12] 3.1, $AB \wedge C$ is a partition, by 4.14, $A(B \wedge C)$ is a partition so that by **[12]** 3.1 again A, $B \wedge C$ commute. If we apply the very proved assertion on the triple $B \wedge C$, A, B, we get the second assertion of the Theorem.

4.14.3 *If partitions B, C in a set G commute, then B, $B \wedge C$ commute.*

The proof follows from 4.14.2 if we put B, C, B instead of A, B, C.

4.15 1. *Let A, $B_\iota(\iota \in \Gamma)$ be partitions in a set G. If A commutes with every B_ι, $\iota \in \Gamma$, then A commutes with $\bigvee_{\iota \in \Gamma} B_\iota$.*

2. *Let A, $B_\iota(\iota \in \Gamma)$ be congruences in an Ω-group G. A, $\bigwedge_{\iota \in \Gamma} B_\iota$ commute if and only if $\bigcap_{\iota \in \Gamma} \cup B_\iota \supseteqq A(O)$, $\cup A \supseteqq \bigcap_{\iota \in \Gamma} B_\iota(O)$.*

Proof. The first proposition is Theorem 2.2 [6], the second one follows directly from [12] 3.9,

Remark. The second assertion holds if A, $B_\iota(\iota \in \Gamma)$ commute or more generally if $\cup B_\iota \supseteqq A(O)$ for all $\iota \in \Gamma$, $\cup A \not\supseteq B_\iota(O)$ for some $\iota \in \Gamma$. From 4.15 we get easily the following consequences a) to d):

For congruences A, B, C, D in an Ω-group G there holds:

a) *A, $A \wedge B$ commute if and only if $\cup B \supseteqq A(O)$.*

b) *Let A, B commute. A, $B \wedge C$ commute if and only if $\cup C \supseteqq A(O)$.* (See [6] Corollary 2.8.)

c) *Let A, B commute. Then $A \wedge D$, $B \wedge C$ commute if and only if $\cup C \supseteqq A(O) \cap$ $\cap D(O)$, $\cup D \supseteqq B(O) \cap C(O)$.* (See [6] Corollary 2.9).

d) *If A, B commute, $A \leqq C \leqq B$, then A, C commute and B, C commute.*

Infact, A, C commute by b), B, C by a).

The assertion d) is a special case of the following general assertion e) which follows directly from [12] 3.9.

e) *Let $\{A_\iota : \iota \in \Gamma\}$ be a system of congruences in an Ω-group G. Then*

$$A_\alpha, A_\beta \text{ commute } (\alpha, \beta \in \Gamma) \Leftrightarrow \bigcup_{\iota \in \Gamma} A_\iota(O) \subseteqq \bigcup_{\iota \in \Gamma} \cup A_\iota .$$

4.16 *Let A, $B_\iota(\iota \in \Gamma)$ be congruences in an algebra, let A commute with the $\mathcal{K}$-supremum of every finite subset of the system $B_\iota(\iota \in \Gamma)$. Then A commutes with $\bigvee_{\mathcal{K}\, \iota \in \Gamma} B_\iota$.*

The proof follows from 4.15(1) and from the fact that $\bigvee_{\mathcal{K}} B_\iota = \bigvee_P C_\varkappa$, where $C_\varkappa$ runs through $\mathcal{K}$-suprema of all finite subsets of the system $B_\iota(\iota \in \Gamma)$ (see [11] 1.2).

4.17 *Let $\boldsymbol{A} = \{A_\iota : \iota \in \Gamma\}$ be an associable system of partitions in a set G, $G_0 = \bigcap_{\iota \in \Gamma} \cup A_\iota$. Let $\boldsymbol{B} = \{B_\iota : \iota \in \Gamma\}$ be a system of partitions in G, $H_0 = \bigcap_{\iota \in \Gamma} \cup B_\iota$ and let there holds $A_\alpha \leqq B_\alpha \leqq \bigvee_{\iota \in \Gamma} A_\iota$, $\alpha \in \Gamma$. Then the system $\boldsymbol{B}$ is associable if and only if for $\alpha, \beta \in \Gamma$, $A_\alpha^1 \in A_\alpha$:*

(4.17,1) *$A_\alpha^1 \cap (H_0 \setminus G_0) \neq \emptyset \Rightarrow \boldsymbol{B} \sqcap A_\alpha^1$ is an associable system of partitions (on A_α^1);*

(4.17,2) *$A_\alpha^1 \cap (\bigcup B_\beta \setminus \bigcup A_\beta) \cap (G \setminus H_0) \neq \emptyset \Rightarrow A_\alpha^1 \subseteqq B_\beta^1$* for some $B_\beta^1 \in B_\beta$.

Remark. Condition (4.17,2) can be equivalently replaced by a formally stronger condition

(4.17,2′) $A_\alpha^1 \cap (G \setminus H_0) \neq \emptyset$, $A_\alpha^1 \cap \cup B_\beta \neq \emptyset \Rightarrow A_\alpha^1 \in B_\beta$.

Proof. Suppose $\boldsymbol{A}$ is associable and $A_\alpha \leqq B_\alpha \leqq \bigvee_{\iota \in \Gamma} A_\iota$, $\alpha \in \Gamma$.

I. $\alpha \in \Gamma$, $A_\alpha^1 \in A_\alpha$, $A_\alpha^1 \cap (G \setminus G_0) \neq \emptyset \Rightarrow A_\alpha^1 \in B_\alpha$.

Indeed, there exists $B_\alpha^1 \in B_\alpha$ with $A_\alpha^1 \subseteqq B_\alpha^1$. Let V be the block of the partition $\bigvee A_\iota = \bigvee B_\iota$ containing B_α^1. By 4.6.5 the block V is not of the 1st kind and by 4.6.6 $V = A_\alpha^1$. From the relations $B_\alpha^1 \subseteqq V = A_\alpha^1 \subseteqq B_\alpha^1$ there follows $A_\alpha^1 = B_\alpha^1$.

II. Let the conditions (4.17,1) and (4.17,2) be satisfied. We shall prove that the system $\boldsymbol{B}$ is associable. Let $V \in \bigvee_{\iota \in \Gamma} A_\iota = \bigvee_{\iota \in \Gamma} B_\iota$, $\mathfrak{A} = \{x^\iota : \iota \in \Gamma\} \subseteqq V$. If $V \cap G_0 \neq \emptyset$, then by (4.9,2) for $\boldsymbol{A}$ there holds $V \subseteqq G_0$. By 4.6.5 V is a block of the 1st kind for $\boldsymbol{A}$. Consequently $x \in G$ exists such that $x^\iota A_\iota x$, $\iota \in \Gamma$. From the relation $A_\iota \leqq B_\iota$ if follows then $x^\iota B_\iota x$, $\iota \in \Gamma$. Thus $\mathfrak{A}$ satisfies (4.1,1) for $\boldsymbol{B}$.

Let $V \subseteqq G \setminus G_0$. There exist $\alpha \in \Gamma$ and $A_\alpha^1 \in A_\alpha$ such that $V \supseteqq A_\alpha^1$. By (4.9,3) for $\boldsymbol{A}$ there is $V = A_\alpha^1$ and by I $V = B_\alpha^1$ for some block $B_\alpha^1 \in B_\alpha$. We can now distinguish two cases.

First, let $A_\alpha^1 \cap (H_0 \setminus G_0) \neq \emptyset$. V is a block of the partitions $\bigvee_{\iota \in \Gamma} (B_\iota \sqcap A_\alpha^1)$ because $B_\alpha \sqcap A_\alpha^1$ contains $B_\alpha^1 (= V)$ for its unique block and $\cup (B_\iota \sqcap A_\alpha^1) \subseteqq A_\alpha^1$ for all $\iota \in \Gamma$. By our assumption, $\boldsymbol{B} \sqcap A_\alpha^1$ is an associable system of partitions. It follows that every element of the system $\boldsymbol{B} \sqcap A_\alpha^1$ is a partition on A_α^1. Indeed, for the block A_α^1 of the partition $\{A_\alpha^1\} \in \boldsymbol{B} \sqcap A_\alpha^1$ there holds $A_\alpha^1 \cap \bigcup_{\iota \in \Gamma} \cup (B_\iota \sqcap A_\alpha^1) = A_\alpha^1 \cap H_0 \neq \emptyset$, then by (4.9,2) for $\boldsymbol{B} \sqcap A_\alpha^1$ there is $A_\alpha^1 \subseteqq \bigcap_{\iota \in \Gamma} \cup (B_\iota \sqcap A_\alpha^1) = H_0$, thus $A_\alpha^1 \supseteqq \cup (B_\iota \sqcap A_\alpha^1) \supseteqq \supseteqq A_\alpha^1, \cup (B_\iota \sqcap A_\alpha^1) = A_\alpha^1$ for every $\iota \in \Gamma$. By (4.17,1) and 4.4 $\mathfrak{A}$ satisfies (4.1,1) for $\boldsymbol{B}$.

Next, let $A_\alpha^1 \cap (G \setminus H_0) \neq 0$. Then $\mathfrak{A}$ verifies the condition (4.1,2) for $\boldsymbol{B}$. To show it, we recall the relations $\mathfrak{A} \subseteqq V = B_\alpha^1 = A_\alpha^1$ that was just proved. Now let $B_\alpha^1 \cap \cap \cup B_\beta \neq \emptyset$ for some $\beta \in \Gamma$. By our supposition A_α^1 is not incident with H_0, consequently $A_\alpha^1 \cap \cup B_\beta \cap (G \setminus H_0) \neq \emptyset$. If $A_\alpha^1 \cap (\cup B_\beta \setminus \cup A_\beta) \cap (G \setminus H_0) \neq \emptyset$, then using (4.17,2) $A_\alpha^1 \subseteqq B_\beta^1$ holds for some $B_\beta^1 \in B_\beta$. From the relations $V = B_\alpha^1 = A_\alpha^1 \subseteqq \subseteqq B_\beta^1 \subseteqq V$ it follows that $B_\alpha^1 = B_\beta^1$. If $A_\alpha^1 \cap \cup A_\beta \cap (G \setminus H_0) \neq \emptyset$, then from the associability of $\boldsymbol{A}$ it follows that $A_\alpha^1 = A_\beta^1$ for some $A_\beta^1 \in A_\beta$. Denote by B_β^1 the block of the partition B_β for which $B_\beta^1 \supseteqq A_\beta^1$. We have got $A_\beta^1 = A_\alpha^1 = B_\alpha^1 = V \supseteqq B_\beta^1 \supseteqq A_\beta^1$, then $B_\alpha^1 = B_\beta^1$. Thus the sufficiency of conditions (4.17,1) and (4.17,2) is proved.

III. Let the system $\boldsymbol{B}$ be associable. We shall prove (4.17,1). Let $A_\alpha^1 \cap (H_0 \setminus G_0) \neq \emptyset$ be for some $\alpha \in \Gamma$ and $A_\alpha^1 \in A_\alpha$. Then by (4.9,2) for $\boldsymbol{A}$ there is $A_\alpha^1 \subseteqq G \setminus G_0$ and by (4.9,3) for $\boldsymbol{A}$ $A_\alpha^1 \in \bigvee_{\iota \in \Gamma} A_\iota = \bigvee_{\iota \in \Gamma} B_\iota$. By (4.9,2) for $\boldsymbol{B}$ $A_\alpha^1 \subseteqq H_0$ is true (see also 4.8,1(a)) so that $B_\iota \sqcap A_\alpha^1$ is a partition on $A_\alpha^1 (\iota \in \Gamma)$. If we denote by B_α^1 the block of the partition B_α for which $B_\alpha^1 \supseteqq A_\alpha^1$, then according to I, there is $A_\alpha^1 = B_\alpha^1$. $A_\alpha^1 (= B_\alpha^1)$ is the unique block of the partition $\bigvee_{\iota \in \Gamma} (B_\iota \sqcap A_\alpha^1)$. To prove it, it suffices to consider that $\cup (B_\iota \sqcap A_\alpha^1) \subseteqq A_\alpha^1 \ (\iota \in \Gamma)$ and that $B_\alpha \sqcap A_\alpha^1$ has the unique block $A_\alpha^1 = B_\alpha^1$. Then it follows that $\boldsymbol{B} \sqcap A_\alpha^1$ is an associable system of partitions on A_α^1. Indeed, let $x^\lambda \bigvee_{\iota \in \Gamma} (B_\iota \sqcap A_\alpha^1) x^\mu (\lambda, \mu \in \Gamma)$ hold for $\mathfrak{A} = \{x^\iota : \iota \in \Gamma\}$; then $\mathfrak{A} \subseteqq A_\alpha^1$. Since $A_\alpha^1 \in \bigvee_{\iota \in \Gamma} B_\iota$, from the associability of $\boldsymbol{B}$ it follows either (4.1,1) or (4.1,2). In the first case there exists $x \in G$ such that $x^\iota B x \ (\iota \in \Gamma)$, thus x belongs to the same block of the partition $\bigvee_{\iota \in \Gamma} B_\iota$ as all $x^\iota (\iota \in \Gamma)$, i.e. $x \in A_\alpha^1$. Hence $x^\iota (B_\iota \sqcap A_\alpha^1)\, x \ (\iota \in \Gamma)$. Then $\mathfrak{A}$ satisfies (4.1,1) for $\boldsymbol{B} \sqcap A_\alpha^1$. In the second case we shall prove that $\mathfrak{A}$ satisfies (4.1,2) for $\boldsymbol{B} \sqcap A_\alpha^1$. There exist $\beta \in \Gamma$ and $B_\beta^1 \in B_\beta$ such that $\mathfrak{A} \subseteqq B_\beta^1$, then $\mathfrak{A} \subseteqq B_\beta^1 \cap A_\alpha^1 (\in B_\beta \sqcap A_\alpha^1)$. If $(B_\beta^1 \cap A_\alpha^1 \cap \cup (B_\gamma \sqcap A_\alpha^1) \neq \emptyset$ for some $\gamma \in \Gamma$, then $B_\beta^1 \cap \cup B_\gamma \neq \emptyset$, thus $B_\beta^1 \in B_\gamma$ and hence $B_\beta^1 \cap A_\alpha^1 \in B_\gamma \sqcap A_\alpha^1$. The associability of the system $\boldsymbol{B} \sqcap A_\alpha^1$ (i.e. the validity of the condition (4.17,1)) is proved in this way.

We shall prove (4.17,2'). Let $A_\alpha^1 \cap (G \setminus H_0) \neq \emptyset$, $A_\alpha^1 \cap \cup B_\beta \neq \emptyset$ for some $\beta \in \Gamma$. By (4.9,2) for $\boldsymbol{A}$ there is $A_\alpha^1 \subseteqq G \setminus G_0$. If $A_\alpha^1 \cap (H_0 \setminus G_0) \neq \emptyset$ held, then by the condition (4.17,1) the validity of which was just proved, it were $A_\alpha^1 \subseteqq \bigcap_{\iota \in \Gamma} \cup (B_\iota \sqcap A_\alpha^1) = = H_0 \cap A_\alpha^1, \subseteqq H_0$ a contradiction. Hence $A_\alpha^1 \subseteqq G \setminus H_0$ so that $A_\alpha^1 \cap \cup B_\beta \cap \cap (G \setminus H_0) \neq \emptyset$. By I there is $A_\alpha^1 = B_\alpha^1$ for some $B_\alpha^1 \in B_\alpha$. By (4.9,3) for $\boldsymbol{B}$ there will be $A_\alpha^1 = B_\alpha^1 \in B_\beta$. This completes the proof.

4.17.1 *Let the notation from* 4.17 *hold, the system* $\boldsymbol{A}$ *be associable and* $A_\alpha \leqq B_\alpha \leqq \leqq \bigvee_{\iota \in \Gamma} A_\iota$, $\alpha \in \Gamma$. *Then it holds*

a) *If* $G_0 = H_0$ *and if* (4.17,2) *is true, the assertion of* Theorem 4.17 *holds.*

b) *The condition* $G_0 = H_0$ *does not guarantee that* (4.17,2) *is fulfilled.*

c) *If* $\cup A_\iota = \cup B_\iota$, $\iota \in \Gamma$ *holds* (which is the condition of Th. 2.3 [6]), *in particula if* A_ι, $\iota \in \Gamma$, *are partitions on G, then the conditions* (4.17,1) *and* (4.17,2) *are satisfied.*

d) *The equality* $\cup A_\alpha = \cup B_\alpha$ *for some* $\alpha \in \Gamma$ *is satisfied if each block of the partition* B_α *contains a block of the partition* A_α.

Proof. a) is evident.

b) Example. Let $G = \{1, 2, 3, 4\}$, $A_1 = \{\{1\}\}$, $B_1 = \{\{1\}, \{3\}\}$, $A_2 = \{\{2\}\} = B_2$, $A_3 = \{\{3, 4\}\} = B_3$. Then $G_0 = \emptyset = H_0$, $A_\iota \leqq B_\iota$, $\iota \in \Gamma (= \{1, 2, 3\})$, $\cdot\bigvee_{\iota \in \Gamma} A_\iota = \bigvee_{i \in \Gamma} B_\iota$, $\{A_1, A_2, A_3\}$ is an associable system. (4.17,2) is not satisfied since for $A_3^1 = \{3, 4\}$ there holds $A_3^1 \cap (\cup B_1 \setminus \cup A_1) \cap (G \setminus H_0) \neq \emptyset$ while A_3^1 is not a subset of any block of the partition B_1.

c) Supposition $\cup A_\iota = \cup B_\iota$, $\iota \in \Gamma$, implies $G_0 = H_0$, then also (4.17,1). Let the supposition of condition (4.17,2) be satisfied. Then $A_\alpha^1 \cap (G \setminus H_0) \neq \emptyset$, $A_\alpha^1 \cap B_\beta^1 \neq \emptyset$ for some $B_\beta^1 \in B_\beta$. According to our supposition there exists $A_\beta^1 \in A_\beta$ such that $A_\beta^1 \subseteqq B_\beta^1$, $A_\alpha^1 \cap A_\beta^1 \neq \emptyset$. By (4.9,3) there is $A_\alpha^1 = A_\beta^1$ so $A_\alpha^1 \subseteqq B_\beta^1$.

d) From 4.8.1(a) it follows that $\cup A_\alpha$ does not respect the partition B_α. By supposition, any $B_\alpha^1 \in B_\alpha$ contains some $A_\alpha^1 \in A_\alpha$, thus $B_\alpha^1 \cap \cup A_\alpha \neq \emptyset$. Hence $\cup B_\alpha \subseteqq \cup A_\alpha$. The reverse inclusion being evident, thus $\cup A_\alpha = \cup B_\alpha$.

4.17.2 Corollary. *Let A, B, C be partitions in a set G, let A, B commute,* $A \leqq C \leqq A \vee B$. *Then B, C commute if and only if* $\cup C$ *respects the partition B.*

Proof. The condition is necessary by 4.8.1.

Sufficiency. Denoting $A_1 = A$, $A_2 = B = B_2$, $B_1 = C$, we get $G_0 = \cup A \cap \cup B$, $H_0 = \cup B \cap \cup C$. We shall prove that the conditions (4.17,1) and (4.17,2) will be satisfied. Let $A_\alpha^1 = A^1 \in A$, $x \in A^1 \cap H_0 \cap (G \setminus G_0) \neq \emptyset$. Because of $\cup A \subseteqq \cup C$, $x \in \cup A \sqcap \cup B \cap (G \setminus \cup A \cap \cup B) = \emptyset$ holds, which is a contradiction.

Let $A_\alpha^1 = B^1 \in B$, $B^1 \cap H_0 \cap (G \setminus G_0) \neq \emptyset$. $B \sqcap A_\alpha^1 = \{C \sqcap B^1, B \sqcap B^1\}$ holds. By assumption $C \sqcap B^1$ is a partition on B^1. Further $B \sqcap B^1 = \{B^1\}$. The partitions $C \sqcap B^1$, $B \sqcap B^1$ commute as they are comparable and "on" (on A_α^1), thus $B \sqcap A_\alpha^1$ is an associable system of partitions on A_α^1. Thus (4.17,1) is verified. We shall check (4.17,2').

Let $A_\alpha^1 = A^1 \in A$, $B_\beta^1 = C^1 \in C$, $A^1 \cap (G \setminus H_0) \neq \emptyset$, $A^1 \cap C^1 \neq \emptyset$. As A, B commute and $A \leqq C$, there holds $A^1 \cap \cup B = \emptyset$, thus $A^1 \in A \vee B$, consequently from the relations $A \leqq C \leqq A \vee B$ and from the supposition $A^1 \cap C^1 \neq \emptyset$ it follows that $A^1 = C^1$.

Let $A_\alpha^1 = A^1 \in A$, $B_\beta^1 = B^1 \in B$, $A^1 \cap (G \setminus H_0) \neq \emptyset$, $A^1 \cap B^1 = \emptyset$. Then from the commutativity of A, B and from the relation $A \leqq C$ it follows that $A^1 \cap \cup B = \emptyset$, which is a contradiction with $A^1 \cap B^1 \neq \emptyset$.

Let $A_\alpha^1 = B^1 \in B$, $B_\beta^1 = C^1 \in C$, $B^1 \cap (G \setminus H_0) \neq \emptyset$, $B^1 \cap C^1 \neq \emptyset$. Since $\cap C$

respects the partition B, there is $B^1 \cap \cup C = \emptyset$ – a contradiction with $B^1 \cap C^1 \neq \emptyset$.

Let $A_\alpha^1 = B^1 \in B$, $B_\beta^1 = B^2 \in B$, $B^1 \cap (G \setminus H_0) \neq \emptyset$, $B^1 \cap B^2 \neq \emptyset$. Then evidently $B^1 = B^2$. The proof is complete.

4.17.3 Corollary ([6] Cor. 2.4). *Let A, B, C be partitions in a set G, let A, B commute, $A \leqq C \leqq A \vee B$. If $\cup A = \cup C$ holds, then B, C commute.*

Indeed, since A, B commute, by 4.8.1(a) $\cup A$ ($= \cup C$) respects the partition B. The assertion follows then from 4.17.2.

4.18 Corollary. *Let $\{A_\iota : \iota \in \Gamma\}$ be an associable system of partitions in a set G, let $\{\Gamma_\varkappa : \varkappa \in K\}$ be a nonempty system of nonempty subsets of the set Γ. Then the system $\{\bigvee_{\iota \in \Gamma_\varkappa} A_\iota : \varkappa \in K\}$ is associable.*

Proof. Denote $\Gamma_0 = \bigcup_{\varkappa \in K} \Gamma_\varkappa$. By 4.5 the system $\{A_\iota : \iota \in \Gamma_0\}$ is associable. For $\alpha \in \Gamma_0$ let us denote by B_α any partition $\bigvee_{\iota \in \Gamma_\varkappa} A_\iota$ for which $\alpha \in \Gamma_\varkappa$. Further, denote $G_0 = \bigcap_{\iota \in \Gamma_0} \cup A_\iota$, $H_0 = \bigcap_{\iota \in \Gamma_0} \cup B_\iota$.

I. First we shall prove the following:

$A_\alpha^1 \cap \cup B_\beta \neq \emptyset$ for some $\alpha, \beta \in \Gamma_0$, $A_\alpha^1 \cap (G \setminus G_0) \neq \emptyset \Rightarrow A_\alpha^1 \in B_\beta$.

Let $B_\beta = \bigvee_{\iota \in \Gamma_\varkappa} A_\iota$ and fix $B_\beta^1 \in B_\beta$, $x \in A_\alpha^1 \cap B_\beta^1$. For every $y \in B_\beta^1$ there exist indices $\iota_0, \iota_1, \ldots, \iota_n \in \Gamma_\varkappa$ and elements $x_1, x_2, \ldots, x_n \in G$ such that $x A_{\iota_0} x_1 A_{\iota_1} x_2 \ldots x_n A_{\iota_n} y$. Thus for certain blocks $A_{\iota_0}^1 \in A_{\iota_0}, \ldots, A_{\iota_n}^1 \in A_{\iota_n}$ there holds $x, x_1 \in A_{\iota_0}^1$, $x_1, x_2 \in A_{\iota_1}^1, \ldots, x_n, y \in A_\iota^1$. By (4.9,3) we get successively $A_\alpha^1 = A_{\iota_0}^1$, $A_{\iota_0}^1 = A_{\iota_1}^1, \ldots, A_{\iota_{n-1}}^1 = A_{\iota_n}^1 \ni y$. Hence $A_\alpha^1 \supseteq B_\beta^1$. By the definition of B_β there is $B_\beta \geqq A_\beta$ and there exist $\gamma \in \Gamma_\varkappa$ and $A_\gamma^1 \in A_\gamma$ such that $B_\beta^1 \supseteq A_\gamma^1$ Hence $A_\alpha^1 \supseteq A_\gamma^1$ so that by (4.9,3) $A_\alpha^1 = A_\gamma^1$. Finally, $A_\alpha^1 \supseteq B_\beta^1 \supseteq A_\gamma^1 = A_\alpha^1$ and so $A_\alpha^1 = B_\beta^1$.

II. We shall show that (4.17.1) is true. Let $A_\alpha^1 \cap (H_0 \setminus G_0) \neq \emptyset$. If $A_\alpha^1 \cap \cup B_\beta = \emptyset$ for some $\beta \in \Gamma_0$, then $A_\alpha^1 \cap H_0 = \emptyset$ – a contradiction. Thus $A_\alpha^1 \cap \cup B_\beta \neq \emptyset$ for all $B \in \Gamma_0$. By I, $A_\alpha^1 \in B_\beta$ for all $\beta \in \Gamma_0$. Hence for all $\beta \in \Gamma_0$, $B_\beta \sqcap A_\alpha^1$ has the unique block A_α^1. Then $B \sqcap A_\alpha^1$ is an associable system of partitions (on A_α^1). Hence (4.17,1).

Clerarly the condition (4.17,2') is satisfied by I.

4.19 *Let $\{A_i : \iota \in \Gamma\}$ be an associable system of partitions in a set G, $\{\Gamma_\varkappa : \varkappa \in K\}$ a nonempty system of nonempty subsets of the set Γ, $\Gamma_0 = \bigcup_{\varkappa \in K} \Gamma_\varkappa$. Let $C = \{C_\varkappa : \varkappa \in K\}$ be a system of partitions in G, $\bigvee_{\varkappa \in K} C_\varkappa \leqq \bigvee_{\iota \in \Gamma_0} A_i$. Denote $G_0 = \bigcap_{\iota \in \Gamma_0} \cup A_i$, $H_0 = \bigcap_{\varkappa \in K} (\cup C_\varkappa \cup \bigcap_{\alpha \in \Gamma_\varkappa} \cup A_\alpha)$. Then the system $\{C_\varkappa \vee A_\alpha : \varkappa \in K, \alpha \in \Gamma_\varkappa\}$ is associable if and only if for $\varkappa, \lambda \in K$, $\alpha \in \Gamma_\varkappa$, $\beta \in \Gamma_\lambda$, $A_\alpha^1 \in A_\alpha$ there holds:*

(4.19,1) $A_\alpha^1 \cap (H_0 \setminus G_0) \neq \emptyset$ = $C(A_\alpha^1) = \{(C_\lambda \vee A_\beta) \sqcap A_\alpha^1 : \lambda \in K, \beta \in \Gamma_\lambda\}$ *is an associable system of partitions (on A_α^1).*

(4.19,2) $A_\alpha^1 \cap (\cup C_\lambda \setminus \cup A_\beta) \cap (G \setminus H_0) \neq \emptyset \Rightarrow A_\alpha^1 \in C_\lambda$ or $A_\alpha^1 \in A_\beta$.

Remark. The system $C(A^1_\alpha)$ in the condition (4.19,1) consists of the one-block partition $\{A^1_\alpha\}$ and of all partitions $C_\lambda \sqcap A^1_\alpha (\lambda \in K)$ for which $\beta \in \Gamma_\lambda$ exists such that $A^1_\alpha \cap \cup A_\beta = \emptyset$.

In the condition (4.19,2) the statement on the right hand side of the implication can be replaced by $A^1_\alpha \in C_\lambda \vee A_\beta$.

Proof. For $\varkappa \in K$, $\alpha \in \Gamma_\varkappa$ let us denote $A_{\varkappa,\alpha} = A_\alpha$, $B_{\varkappa,\alpha} = C_\varkappa \vee A_\alpha$. The system $\{A_\iota : \iota \in \Gamma_0\}$ is associable, consequently so is the system $A = \{A_{\varkappa,\alpha} : \varkappa \in K, \alpha \in \Gamma_\varkappa\}$. For $\varkappa \in K$, $\alpha \in \Gamma_\varkappa$ there holds
$A_{\varkappa,\alpha} = A_\alpha \leqq C_\varkappa \vee A_\alpha = B_{\varkappa,\alpha} \leqq \bigvee_{\varkappa \in K} C_\varkappa \vee \bigvee A_i = \bigvee_{\iota \in \Gamma_0} A_\iota = \bigvee_{\iota \in \Gamma_0} \{A_{\varkappa,\alpha} : \varkappa \in K, \alpha \in \Gamma_\varkappa\}$.

By 4.17, the system $B = \{B_{\varkappa,\alpha} : \varkappa \in K, \alpha \in \Gamma_\alpha\} = \{C_\varkappa \vee A_\alpha : \varkappa \in K, \alpha \in \Gamma_\varkappa\}$ is associable if and only if the conditions (4.17,1) and (4.17,2) are satisfied, where $G_0 = \bigcap_{\iota \in \Gamma_0} \cup A_\iota$, $H_0 = \bigcap_{\varkappa \in K} [\cup C_\varkappa \cup \bigcap_{\alpha \in \Gamma_\varkappa} \cup A_\alpha]$.

Let $\varkappa, \lambda \in K$, $\alpha \in \Gamma_\varkappa$, $\beta \in \Gamma_\lambda$, $A^1_{\varkappa,\alpha} \in A_{\varkappa,\alpha}$, $B^1_{\lambda,\beta} \in B_{\lambda,\beta}$.
We reformulate (4.17,1) for the present situation:

$A^1_{\varkappa,\alpha} \cap (H_0 \setminus G_0) \neq \emptyset \Rightarrow B \sqcap A^1_{\varkappa,\alpha}$ is an associable system of partitions (on $A^1_{\varkappa,\alpha}$).

Consider a partition $B_{\lambda,\beta} \sqcap A^1_{\varkappa,\alpha}$ belonging to the system $B \sqcap A^1_{\varkappa,\alpha}$: $B_{\lambda,\beta} \sqcap A^1_{\varkappa,\alpha} = (C_\lambda \vee A_\beta) \sqcap A^1_\alpha$. By (4.9,3) for the associable system $\{A_\iota : \iota \in \Gamma_0\}$ there holds

$$A^1_\alpha \in \bigvee_{\iota \in \Gamma_0} A_\iota \geqq C_\lambda \vee A_\beta \geqq C_\lambda.$$

Hence we obtain: if $A^1_\alpha \cap \cup A_\beta = \emptyset$, then $(C_\lambda \vee A_\beta) \sqcap A^1_\alpha = C_\lambda \sqcap A^1_\alpha$. If $A^1_\alpha \cap \cup A_\beta \neq \emptyset$, then $A^1_\alpha \in A_\beta$, consequently $(C_\lambda \vee A_\beta) \sqcap A^1_\alpha$ has the unique block A^1_α. Such a case necessarily occurs, namely for $\lambda = \varkappa$, $\beta = \alpha$. Then the condition (4.17,1) in our case reads:

$A^1_\alpha \cap (H_0 \setminus G_0) \neq \emptyset \Rightarrow C(A^1_\alpha)$ is an associable system of partitions (on A^1_α).

Analogously we reformulate the condition (4.17,2):

$$A^1_{\varkappa,\alpha} \cap (\cup B_{\lambda,\beta} \setminus \cup A_{\lambda,\beta}) \cap (G \setminus H_0) \neq \emptyset \Rightarrow A^1_{\varkappa,\alpha} \in B_{\lambda,\beta}.$$

Using the original notation, this condition reads:

(1) $$A^1_\alpha \cap (\cup C_\lambda \setminus \cup A_\beta) \cap (G \setminus H_0) \neq \emptyset \Rightarrow A^1_\alpha \in C_\lambda \vee A_\beta.$$

Let us consider that $A^1_\alpha \in \bigvee_{\iota \in \Gamma_0} A_\iota$. If A^1_α does not contain any block of the partition A_β, then it will be $A^1_\alpha \in C_\lambda$, if it contains, it will be $A^1_\alpha \in A_\beta$ (by 4.9,3)). Conversely, let $A^1_\alpha = C^1_\lambda \in C_\lambda$ or $A^1_\alpha = A^1_\beta \in A_\beta$ (and, of course, let the statement on the left of (1) be true). Evidently there holds $A^1_\alpha \in \bigvee_{\iota \in \Gamma_0} A_\iota$. If $A_\alpha \cap \cup A_\beta = \emptyset$, then $A^1_\alpha = C^1_\lambda \in C_\lambda \vee A_\beta$; if $A^1_\alpha \cap \cup A_\beta \neq \emptyset$, then $A^1_\alpha = A^1_\beta$ for some $A^1_\beta \in A_\beta$ (by 4.9,3)) and with regard to the relations $A_\beta^1 = A_\alpha^1 \in \bigvee_{\iota \in \Gamma_0} A_\iota \geqq \bigvee_{\varkappa \in K} C_\varkappa \geqq C_\lambda$ it will be $A_\alpha^1 = A_\beta^1 \in C_\lambda \vee A_\beta$. Thus it is established that the statement on the reight of (1) can be equivalently written in the form $A_\alpha^1 \in C_\lambda$ or $A_\alpha^1 \in A_\beta$.

4.19.1 The conditions (4.19,1) and (4.19,2) are satisfied trivially if there holds

(4.19,3) $$\cup C_\varkappa \subseteqq \cup A_\alpha, \quad \varkappa \in K, \quad \alpha \in \Gamma_\varkappa.$$

Theorem 4.17 will be obtained as a consequence of Theorem 4.19 if we put there $K = \Gamma$, $\Gamma_\iota = \{\iota\}$, $C_\iota = B_\iota$ for all $\iota \in K (= \Gamma)$.

Theorems 2.4 and 2.4a [6] are corollaries to Theorem 4.19, as well. Indeed, by introducing a suitable notation the condition (4.19,3) is evidently satisfled. (E.g. if we put in 4.19 $K = \Gamma$, $\Gamma_\iota = \{\iota\}$ for all $\iota \in K$, $C_\iota = B_2$ for $\iota \in \Gamma_1$, $C_\iota = B_1$ for $\iota \in \Gamma_2$ we obtain Th. 2.4a)

4.20 *Let A, B, A', B' be partitions in a set G, $A' \vee B' \leqq A \vee B$, let A, B commute. Then $A \vee A'$, $B \vee B'$ commute if and only if*

(4.20,1) $$\cup B' \setminus \cup B \text{ respects the partition } A,$$
$$\cup A' \setminus \cup A \text{ respects the partition } B.$$

Proof. In Theorem 4.19 let us put $K = \{1, 2\}$, $\Gamma_\iota = \{\iota\}$ for $\iota \in K$, $A_1 = A$, $A_2 = B$, $C_1 = A'$, $C_2 = B'$. Condition (4.19,2) is satisfied trivially. Indeed, $H_0 = (\cup A \cup \cup A') \cap (\cup B \cup \cup B') \supseteqq (\cup A \cap \cup B') \cup (\cup B \cap \cup A')$, then $A^1 \cap (\cup B' \setminus \cup B) \cap (G \setminus H_0) \subseteqq \cup A \cap \cup B' \cap (G \setminus H_0) = \emptyset$, $B^1 \cap (\cup A' \setminus \cup A) \cap (G \setminus H_0) \subseteqq \cup B \cap \cup A' \cap (G \setminus H_0) = \emptyset$ and evidently $A^1 \cap (\cup A' \setminus \cup A) \subseteqq \cup A \cap (\cup A' \setminus \cup A) = \emptyset$, $B^1 \cap (\cup B' \setminus \cup B) = \emptyset$. Condition (4.19,1) reads (for $\alpha = 1$):

(1) $$A^1 \cap (H_0 \setminus G_0) \neq \emptyset \Rightarrow \{A^1\},\ B' \sqcap A^1 \text{ are commuting partitions.}$$

Partitions $\{A^1\}$, $B' \sqcap A^1$ commute if and only if $B' \sqcap A^1$ is a partition on A^1 (or empty but this is evidently eliminated in (1)) and this is equivalent to the relation $\cup B' \supseteqq A'$ and to the relation $\cup B' \setminus \cup B \supseteqq A^1$ as well since by (4.9,2) $A^1 \cap \cup B \neq \emptyset \Rightarrow A^1 \cap G_0 \neq \emptyset \Rightarrow A^1 \subseteqq G_0$ – a contradiction. The first statement in (1) is equivalent to the statement $A^1 \cap (\cup B' \setminus \cup B) \neq \emptyset$. Indeed, $A^1 \cap (H_0 \setminus G_0) = (A^1 \cap H_0) \setminus G_0 = A^1 \cap (\cup A \cup \cup A') \cap (\cup B \cup \cup B') \setminus G_0 = (A^1 \cap \cup B) \cup (A^1 \cap \cup B') \setminus (\cup A \cap \cup B) = (A^1 \cap \cup B') \setminus (\cup A \cap \cup B) = (A^1 \cap \cup B') \setminus (A^1 \cap \cup B) = A^1 \cap (\cup B' \setminus \cup B)$. Thus the first condition in (4.20,1) is verified. The second is obtained similarly.

4.20.1 Condition (4.20,1) is evidently satisfied if there holds

(4.20,2) $$\cup A' \subseteqq \cup A, \cup B' \subseteqq \cup B.$$

4.20.2 Cor. 2.6 and Cor. 2.7a in [6] are consequences of Theorem 4.20. In the first case it suffices to put $A = A'$, $B' = C$. In both cases condition (4.20,2) is satisfied.

4.20.3 We specialize Theorems 4.20 by putting requirements on A and A' or B and B', respectively.

Let A, A', B, B', C, D be partitions in a set G, A, B commute, $A' \vee B' \leqq A \vee B$, $C \leqq A \vee B$. Then the following assertions 1 *to* 5 *hold*:

(1) *If $A \leqq A'$, $B \leqq B'$, then the following conditions are equivalent*:

a) *A', B' commute*

b) (4.20,1) *is true*

c) $(\cup A' \cap \cup B') \setminus (\cup A \cap \cup B)$ *respects the partitions A and B*

(2) *A, $B \vee C$ commute if and only if $\cup C \setminus \cup B$ respects the partition A. If we suppose moreover that $A \wedge B \leqq C$, then A, $B \vee C$ commute if and only if $\cup C$ respects the partition A.* (See Cor. 2.6[6].)

(3) *$A \vee C$, $B \vee C$ commute if and only if $\cup C \setminus \cup B$ respects the partition A and $\cup C \setminus \cup A$ respects the partition B.*

(4) *If $A \leqq C$, then B, C commute if and only if $\cup C$ respects the partition B* (See 4.17.2).

(5) *If $\cup B \cap \cup C \subseteqq \cup A$, $\cup A \cap \cup D \subseteqq \cup B$, then the partitions $A \vee (B \wedge C)$, $B \vee (A \wedge D)$ commute.* (Cf. Cor. 2.12 [6] and 4.25.).

Proof: (1) We apply Theorem 4.19 in the same manner as in the proof to Theorem 4.20. Condition (4.19,2) is satisfied trivially (see proof to 4,20). Condition (4.19,1) can be formulated as follows (for $\alpha = 1$, $A_1 = A$, $A^1 \in A$):

(i) $A^1 \cap (H_0 \setminus G_0) \neq \emptyset \Rightarrow \{A^1\}$, $B' \sqcap A^1$ are commuting partitions on A^1.

The first statement in (i) is equivalent to the following

$$A^1 \cap [(\cup A' \cap \cup B') \setminus (\cup A \cap \cup B) \neq \emptyset.$$

The second statement in (i) is equivalent to the fact that $B' \sqcap A^1$ is a partition on A^1 and this is equivalent to the relation $\cup B' \supseteqq A^1$ and also to the relation $\cup B' \setminus \setminus (\cup A \cap \cup B) \supseteqq A^1$ (since $A^1 \cap (H_0 \setminus G_0) \neq \emptyset \Rightarrow A^1 \subseteqq G \setminus G_0$ – by (4.9,2)). Because of $A^1 \subseteqq \cup A'$, the last relation can be equivalently formulated in the form $(\cup A' \cap \cup B') \setminus (\cup A \cap \cup B) \supseteqq A^1$. It follows that the condition (i) is equivalent to the statement: $(\cup A' \cap \cup B') \setminus (\cup A \cap \cup B)$ respects the partition A. That it also respects the partition B can be proved by a similar argument (formulate (4.19,1) for $\alpha = 2$). So we have got the equivalence between a) and c). The equivalence between a) and b) follows directly from 4.20.

(2) The first assertion follows from 4.20 by putting $A = A'$, $B' = C$. The second assertion: If A, $B \vee C$ commute, then $\cup C \setminus \cup B$ respects the partition A. Hence if $A^1 \cap (\cup C \setminus \cup B) \neq \emptyset$, then $A^1 \subseteqq \cup C$. If $A^1 \cap \cup B \cap \cup C \neq \emptyset$, then from the

commutativity of A, B and from the supposition $A \wedge B \leqq C$ it follows that $A^1 \subseteqq \subseteqq \cup B \cap \cup A \subseteqq \cup C$. If now, conversely, $\cup C$ respects A and $A^1 \cap (\cup C \setminus \cup B) \neq \emptyset$, then using commutativity of A, B it follows that $A^1 \subseteqq \cup C$ and $A^1 \subseteqq G \setminus \cup B$, then $A^1 \subseteqq \cup C \setminus \cup B$.

(3) can be obtained from 4.20 for $A' = B' = C$.

(4) 4.20 for $A \leqq A' = C$, $B' = B$.

(5) If we put $A' = B \wedge C$, $B' = A \wedge D$, the condition (4.20,2) will be satisfied.

4.21 *Let A, B, A', B' be congruences in an Ω-group. Then $A \wedge A'$, $B \wedge B'$ commute if and only if $[A'(O) \cap A(O)] \cup [B'(O) \cap B(O)] \subseteqq \cup A \cap \cup B \cap \cup A' \cap \cup B'$.*

The assertion is obtained immediately from [12] 3.9.

4.21.1 Corollary. *Let A, B, A', B' be congruences in an Ω-group, A, B commute. Then $A \wedge A'$, $B \wedge B'$ commute if one of the following conditions is satisfied:*

a) $A'(O) \cap A(O) \subseteqq \cup B'$, $B'(O) \cap B(O) \subseteqq \cup A'$ (*the condition is also necessary*)

b) $A \leqq B'$, $B \leqq A'$

c) A', B' *commute.*

Proof. a) follows immediately from 4.21, b) and c) follow from a).

4.21.2 Remark. Assertion 4.21.1 b) agrees to Cor. 2.9 [6] applied to congruences in Ω-group.

4.22 *Let $\boldsymbol{A} = \{A_\iota : \iota \in \Gamma\}$ be an associable system of partitions in a set G, B a partition in G, $G_0 = \bigcap_{\iota \in \Gamma} \cup A_\iota$, $H = \cup B \cap G_0$, $\Gamma_1 \subseteqq \Gamma$, $\Gamma_2 = \Gamma \setminus \Gamma_1$. The system $\boldsymbol{B} = \{A_\alpha : \alpha \in \Gamma_1\} \cup \{B \wedge A_\mu : \mu \in \Gamma_2\}$ is associable if and only if the following conditions 1, 2, 3 and 4 hold provided that $\Gamma_2 \geqq 2$, $\Gamma_1 \neq \emptyset$, conditions 1′, 2, 3 and 4 provided that* card $\Gamma_2 = 1$, *and condition 1′ provided that $\Gamma_1 = \emptyset$:*

(4.22,1) $B \sqcap H \geqq A_\alpha \sqcap H$, $\alpha \in \Gamma_1$

(4.22,1′) $\boldsymbol{B} \sqcap H$ *is an associable system of partitions* (*on* H)

(4.22,2) H *respects the partition* A_α, $\alpha \in \Gamma_1$

(4.22,3) $\alpha, \beta \in \Gamma_1 \Rightarrow A_\alpha \sqcap (G_0 \setminus H) = A_\beta \sqcap (G_0 \setminus H)$

(4.22,4) $\alpha \in \Gamma_1, \mu \in \Gamma_2, A_\alpha^1 \cap A_\mu^1 \cap B^1 \cap (G \setminus G_0) \neq \emptyset \Rightarrow A_\alpha^1 \subseteqq B^1$.

Remark. For the cases of card $\Gamma_2 = 1$ or $\Gamma_1 = \emptyset$ the problem is solved unsatisfactorily in the sense that the Theorem gives a trivial result for partitions "on".

Proof. Denote $B_\alpha = A_\alpha$ for $\alpha \in \Gamma_1$, $B_\mu = B \wedge A_\mu$ for $\mu \in \Gamma_2$.

I. Suppose that the systems $\boldsymbol{A}$, $\boldsymbol{B}$ are associable.

1. By (4.9,1) for $\boldsymbol{B}$, $\boldsymbol{B} \sqcap H$ is an associable system of partitions on H. This is a stronger variant of the condition 1′. Now let card $\Gamma_2 \geqq 2$, let $a \bigvee (\boldsymbol{B} \sqcap H)\, z$ hold for $a, z \in H$. Choose $\mu, \nu \in \Gamma_2$, $\mu \neq \nu$. Let $x^\varkappa \bigvee (\boldsymbol{B} \sqcap H)\, x^\lambda$, $\varkappa, \lambda \in \Gamma$, hold for a system $\{x^\iota : \iota \in \Gamma\}$ where $x^\mu = a$, $x^\nu = z$. Then there exists $x \in H$ such that $x^\varkappa (B_\varkappa \sqcap H)\, x$, $\varkappa \in \Gamma$, then $x^\varkappa (B \sqcap H)\, x$, $\varkappa \in \Gamma_2$, hence $a = x^\mu (B \sqcap H)\, x^\nu = z$. The

element a can run through the entire block of the partition $\bigvee(\boldsymbol{B} \sqcap H)$ containing the element z. Therefore $\boldsymbol{B} \sqcap H \geqq \bigvee(\boldsymbol{B} \sqcap H) \geqq A_\alpha \sqcap H, \alpha \in \Gamma_1$, which is the condition 1.

2. From (4.9,2) for $\boldsymbol{B}$ we get 2.

3. By (4.9,3) for $\boldsymbol{B}$ one gets:

$$\alpha, \quad \beta \in \Gamma_1, \quad A_\alpha^1 \cap A_\beta^1 \cap (G_0 \setminus H) \neq \emptyset \Rightarrow A_\alpha^1 = A_\beta^1.$$

From (4.9,2) for $\boldsymbol{A}$ it follows that $A_\alpha \sqcap (G_0 \setminus H)$ and $A_\beta \sqcap (G_0 \setminus H)$ are partitions on $G_0 \setminus H$, thus $A_\alpha \sqcap (G_0 \setminus H) = A_\beta \sqcap (G_0 \setminus H)$, which is the condition 3.

4) From the condition $\alpha \in \Gamma_1$, $\mu \in \Gamma_2$, $A_\alpha^1 \cap A_\mu^1 \cap B^1 \cap (G \setminus G_0) \neq \emptyset$ it follows by (4.9,3) for $\boldsymbol{B}$ that $A_\alpha^1 = A_\mu^1 \cap B^1 \subseteqq B^1$.

II. Sufficiency. 1. Condition (4.9,1) for $\boldsymbol{B}$ is identical with the condition 1′ of our Theorem. Then if card $\Gamma_2 = 1$ or $\Gamma_1 = \emptyset$, (4.9,1) is satisfied for $\boldsymbol{B}$. Let $\Gamma_1 \neq \emptyset$, card $\Gamma_2 \geqq 2$. If we prove that the system $\boldsymbol{A} \sqcap H$ is associable, the suppositions of Theorem 2.5 [6] for the systems $\boldsymbol{A} \sqcap H$, $\boldsymbol{B} \sqcap H$ will be satisfied as it follows from the condition 1 of our Theorem, hence the system $\boldsymbol{B} \sqcap H$ will be associable, i.e. the condition (4.9,1) will be satisfied for $\boldsymbol{B}$. Then let $x^\varkappa \bigvee(\boldsymbol{A} \sqcap H)\, x^\lambda$, $\varkappa$, $\lambda \in \Gamma$, hold for $\{x^\iota : \iota \in \Gamma\}$. Then $x^\varkappa \bigvee(\boldsymbol{A} \sqcap G_0)\, x^\lambda$, $\varkappa$, $\lambda \in \Gamma$. Since, by (4.9,1) for $\boldsymbol{A}$ $\boldsymbol{A} \sqcap G_0$ is an associable system of partitions on G_0, $x \in G_0$ will exist by 4.4 such that $x^\iota(A_\iota \sqcap G_0)\, x$, $\iota \in \Gamma$. For $\alpha \in \Gamma_1$ and for suitable $A_\alpha^1 \in A_\alpha$ there is $x^\alpha \in A_\alpha^1 \cap H$, then by condition 2 of our Theorem there si $A_\alpha^1 \subseteqq H$ and consequently $x \in H$ since $x \in A_\alpha^1$. Hence $x^\iota(A_\iota \sqcap H)\, x$, $\iota \in \Gamma$, i.e. the system $\boldsymbol{A} \sqcap H$ is associable.

2. Condition (4.9,2) for $\boldsymbol{B}$ is satisfied provided that $\alpha \in \Gamma_1$ as it follows from the condition 2 of our Theorem. Let $\mu \in \Gamma_2$, $B_\mu^1 \cap H = \emptyset$, $B_\mu^1 = A_\mu^1 \cap B^1$. Then $A_\mu^1 \cap G_0 \neq \neq \emptyset$, thus $A_\mu^1 \subseteqq G_0$ and hence $B_\mu^1 = A_\mu^1 \cap B^1 \subseteqq G_0 \cap \cup \boldsymbol{B} = H$; so condition (4.9,2) applied to $\boldsymbol{B}$ holds for $\mu \in \Gamma_2$ as well.

3. First case. Let α, $\beta \in \Gamma_1$, $A_\alpha^1 \cap A_\beta^1 \cap (G \setminus H) \neq \emptyset$. If $A_\alpha^1 \cap A_\beta^1 \cap (G \setminus G_0) \neq \emptyset$, then $A_\alpha^1 = A_\beta^1$ by (4.9,3) for $\boldsymbol{A}$. If $A_\alpha^1 \cap A_\beta^1 \cap (G_0 \setminus H) \neq \emptyset$, then A_α^1, $A_\beta^1 \subseteqq G_0$ holds by (4.9,2) for $\boldsymbol{A}$ and by condition 2 of our Theorem there will be A_α^1, $A_\beta^1 \subseteqq G \setminus H$, thus A_α^1, $A_\beta^1 \subseteqq G_0 \setminus H$. From the condition 3 of our Theorem it follows that $A_\alpha^1 = A_\beta^1$. Thereofre (4.9,3) for $\boldsymbol{B}$ is verified for the first case.

Second case. Let now $\alpha \in \Gamma_1$, $\mu \in \Gamma_2$, $A_\alpha^1 \cap B_\mu^1 \cap (G \setminus H) \neq \emptyset$, $B_\mu^1 = A_\mu^1 \cap B^1$. Since $A_\alpha^1 \cap A_\mu^1 \cap B^1 \cap (G \setminus H) \subseteqq \cup \boldsymbol{B}$, there holds $A_\alpha^1 \cap A_\mu^1 \cap B^1 \cap (G \setminus G_0) \neq \emptyset$, then by (4.9,3) for $\boldsymbol{A}$ there is $A_\alpha^1 = A_\mu^1$ and by the condition 4 of our Theorem one gets $A_\alpha^1 \subseteqq B^1$, consequently $A_\alpha^1 = A_\alpha^1 \cap B^1 = A_\mu^1 \cap B^1 = B_\mu^1$, i.e. (4.9,3) holds for $\boldsymbol{B}$ again.

Third case. Let μ, $\nu \in \Gamma_2$, $(A_\mu \wedge B)^1 \cap (A_\nu \wedge B)^1 \cap (G \setminus H) \neq \emptyset$. Evidently $(A_\mu \wedge B)^1 = A_\mu^1 \cap B^1$, $(A_\nu \wedge B)^1 = A_\nu^1 \cap B^1$ where A_μ^1, A_ν^1, B^1 are blocks of the partitions A_μ, A_ν, B, respectively, containing the element x. Then we have $A_\mu^1 \cap \cap A_\nu^1 \cap B^1 \cap (G \setminus H) \neq \emptyset$. As in the previous case there will be $A_\mu^1 \cap A_\nu^1 \cap B^1 \cap \cap (G \setminus G_0) \neq \emptyset$, consequently by (4.9,3) for $\boldsymbol{A}$ we have $A_\mu^1 = A_\nu^1$. Hence $(A_\mu \wedge \wedge B)^1 = A_\mu^1 \cap B^1 = A_\nu^1 \cap B^1 = (A_\nu \wedge B)^1$. This completes the proof.

4.22.1 *If* $\Gamma_1 \neq \emptyset$, $B \geqq A_\alpha$, $\alpha \in \Gamma_1$, *then the conditions of* Theorem 4.22 *are satisfied.*

Remark. Th. 2.5 [6] is thus a consequence of Theorem 4.22.

Proof. From the condition $B \geqq A_\alpha$, $\alpha \in \Gamma_1 \neq \emptyset$, it follows that $G_0 = H$, thus condition 2 follows from the associability of the system $\boldsymbol{A}$ and conditions 1, 3, 4 are satisfied trivially. Condition 1′ is also satisfied since, by Theorem 2.5 [6] system $\boldsymbol{B}$ is associable and by (4.9,1) for $\boldsymbol{B}$, the system $\boldsymbol{B} \sqcap H$ is associable as well.

4.22.2 Corollary. *Let* $\boldsymbol{A} = \{A_\iota : \iota \in \Gamma\}$ *be an associable system of partitions on a set* G, B *a partition on* G, $\emptyset \neq \Gamma_1 \subseteqq \Gamma$, card $(\Gamma \setminus \Gamma_1) \geqq 2$. *Then the system* $\boldsymbol{B} = \{A_\alpha : \alpha \in \Gamma_1\} \cup \{B \wedge A_\mu : \mu \in \Gamma \setminus \Gamma_1\}$ *is associable if and only if* $B \geqq A_\alpha$, $\alpha \in \Gamma_1$.

Proof follows directly from 4.22.

4.22.3 Remark. Theorem 4.22.2 does not hold when card $(\Gamma \setminus \Gamma_1) = 1$. In greater detail: Let $\emptyset \neq \Gamma_1$, card $(\Gamma \setminus \Gamma_1) = 1$, let the system $\boldsymbol{A}$ be associable. Then it holds: $B \geqq A_\alpha$, $\alpha \in \Gamma_1 \Rightarrow$ the system $\boldsymbol{B}$ is associable (by 4.22 and 4.22.1). The implication cannot be inverted (not even in the case of partitions "on") as the following example shows: the system $\boldsymbol{B} = \{A_1, B \wedge A_2\}$ is associable, the system $\boldsymbol{A} = \{A_1, A_2\}$ is associable as well but $B < A_1$, where $G = \{1, 2, 3, 4, 5, 6\}$, $A_1 = \{\{1, 2, 3, 4\}, \{5, 6\}\}$, $A_2 = \{\{1, 3, 5\}, \{2, 4, 6\}\}$, $B = \{\{1, 2\}, \{3, 4\}, \{5, 6\}\}$.

4.22.4 Definition. Let A be a partition in a set G, $F \subseteqq G$. Under $A \sqsupset F$ we understand the set of all blocks of the partition A that are incident with the set F. The partition $A \sqsupset F$ is called the *closure* of the set F in the partition A ([3,4] 2.3).

4.22.5 Corollary. *Let* $\boldsymbol{A} = \{A_\iota : \iota \in \Gamma\}$ *be an associable system of partitions in a set* G, B *a partition in* G, $\cup B \supseteqq G_0 = \bigcap_{\iota \in \Gamma} \cup A_\iota$, $\Gamma_1 \subseteqq \Gamma$, $\Gamma_2 = \Gamma \setminus \Gamma_1$. *The system* $\boldsymbol{B} = \{A_\alpha : \alpha \in \Gamma_1\} \cup \{B \wedge A_\mu : \mu \in \Gamma_2\}$ *is associable if and only if the following condition* 5 *holds provided that* card $\Gamma_2 \geqq 2, \Gamma_1 \neq \emptyset$, *conditions* 5′ *and* 6 *provided that* card $\Gamma_2 = 1$ *and condition* 5′ *provided that* $\Gamma_1 = \emptyset$:

(4.22,5) $B \geqq A_\alpha \sqsupset (\cup A_\mu \cap \cup B)$, $\alpha \in \Gamma_1$, $\mu \in \Gamma_2$,
(4.22,5′) $\boldsymbol{B} \sqcap G_0$ *is an associable system of partitions* (*on* G_0)
(4.22,6) $B \geqq A_\alpha \sqsupset [\cup A_\mu \cap \cup B \cap (G \setminus G_0)]$, $\alpha \in \Gamma_1$, $\mu \in \Gamma_2$.

Proof. From the supposition $\cup B \supseteqq G_0$ it follows that $H = G_0 \cap \cup B = G_0$. Condition (4.22,1′) is then identical with condition (4.22,5′) (so the case $\Gamma_1 = \emptyset$ is established), condition (4.22,2) follows from the associability of the system $\boldsymbol{A}$ (see (4.9,2)) and condition (4.22,3) is satisfied trivially. From the condition (4.22,4) it follows that for $\alpha \in \Gamma_1$, $\mu \in \Gamma_2$ and for $F_\mu = \cup A_\mu \cap \cup B \cap (G \setminus G_0)$ it holds $B \geqq A_\alpha \sqsupset F_\mu$. Conversely, from this condition we obtain (4.22,4) for $x \in A_\alpha^1 \cap A_\mu^1 \cap B^1 \cap (G \setminus G_0)$ implies $x \in A_\alpha^1 \cap F_\mu$, thus $B^1 \supseteqq A_\alpha^1$. Hence the case card $\Gamma_2 = 1$ is established.

Now let card $\Gamma_2 \geqq 2$, $\Gamma_1 \neq \emptyset$. From the condition (4.22,1) it follows $B \geqq B \sqcap G_0 \geqq A_\alpha \sqcap G_0 = A_\alpha \sqsupset G_0$ for $\alpha \in \Gamma_1$, which together with (4.22,6) gives $B \geqq A_\alpha \sqsupset (\cup A_\mu \cap \cup B)$ for $\alpha \in \Gamma_1$, $\mu \in \Gamma_2$, which is condition (4.22,5). Conversely, let (4.22,5) hold. Then on the one hand $B \geqq A_\alpha \sqsupset F_\mu$ for $F_\mu \subseteqq \cup A_\mu \cap \cup B$ – then (4.22,4) holds, and on the other hand $B \geqq A_\alpha \sqsupset G_0$ for $G_0 \subseteqq \cup A_\mu \cap \cup B$ and hence $B \geqq A_\alpha \sqsupset G_0 = A_\alpha \sqcap G_0$, then $B \sqcap G_0 \geqq A_\alpha \sqcap G_0 (\alpha \in \Gamma_1)$ – thus condition (4.22,1) is verified as required. The Theorem is proved.

4.22.6 Corollary. *Let* $\{A_\iota : \iota \in \Gamma\}$ *be an associable system of congruences in an* Ω*-group* G, B *a congruence in* G, $G_0 = \bigcap_{\iota \in \Gamma} \cup A_\iota$, $H = G_0 \cap \cup B$, $\Gamma_1 \subseteqq \Gamma$, $\Gamma_2 = \Gamma \setminus \Gamma_1$. *The system* $\boldsymbol{B} = \{A_\alpha : \alpha \in \Gamma_1\} \cup \{B \wedge A_\mu : \mu \in \Gamma_2\}$ *is associable if and only if the following conditions* 7, 8, 9 *and* 10 *hold provided that* card $\Gamma_2 \geqq 2$, $\Gamma_1 \neq \emptyset$, *conditions* 7′, 8, 9 *and* 10 *provided that* card $\Gamma_2 = 1$ *and condition* 7′ *provided that* $\Gamma_1 = \emptyset$;

(4.22,7) $G_0 \cap B(O) \supseteqq A_\alpha(O) \cap \cup B$, $\alpha \in \Gamma_1$
(4.22,7′) $\boldsymbol{B} \sqcap H$ *is an associable system of congruences (on* H*)*
(4.22,8) $A_\alpha(O) \subseteqq \cup B$, $\alpha \in \Gamma_1$
(4.22,9) $\cup B \nsupseteq G_0 \Rightarrow A_\alpha(O) = A_\beta(O)$ *for all* $\alpha, \beta \in \Gamma_1$
(4.22,10) $\alpha \in \Gamma_1$, $\mu \in \Gamma_2$, $\cup A_\alpha \cap \cup A_\mu \cap \cup B \cap (G \setminus G_0) \neq \emptyset \Rightarrow A_\alpha(O) \subseteqq B(O)$.

The Theorem is an immediate consequence of Theorem 4.22.

4.22.7 *Let* $\{A_\iota : \iota \in \Gamma\}$ *be an associable system of congruences in an* Ω*-group* G, B *a congruence in* G, $\cup B \supseteqq G_0 = \bigcap_{\iota \in \Gamma} \cup A_\iota$, $\Gamma_1 \subseteqq \Gamma$, $\Gamma_2 = \Gamma \setminus \Gamma_1$. *Then the system* $\boldsymbol{B} = \{A_\alpha : \alpha \in \Gamma_1\} \cup \{B \wedge A_\mu : \mu \in \Gamma_2\}$ *is associable if and only if condition* 11 *is satisfied provided that* card $\Gamma_2 \geqq 2$, $\Gamma_1 \neq \emptyset$, *conditions* 11′, 12 *provided that* card $\Gamma_2 = 1$ *and condition* 11′ *provided that* $\Gamma_2 = \emptyset$:

(4.22,11) $B(O) \supseteqq A(O)$, $\alpha \in \Gamma_1$
(4.22,11′) $\boldsymbol{B} \sqcap G_0$ *is an associable system of congruences (on* G_0*)*
(4.22,12) $\alpha \in \Gamma_1$, $\mu \in \Gamma_2$, $\cup A_\alpha \cap \cup A_\mu \cap \cup B \cap (G \setminus G_0) \neq \emptyset \Rightarrow A_\alpha(O) \subseteqq B(O)$.

The Theorem is a consequence of 4.22.5.

4.22.8 In particular, it holds: *Let congruences* A, B *in an* Ω*-group* G *commute, let* $\cup C \supseteqq \cup A \cap \cup B$ *hold for a congruence* C *in* G. *Then the congruences* A, $B \wedge C$ *commute.*

Proof. We shall use Theorem 4.22.7 for card $\Gamma_2 = 1$. Condition (4.22,12) is satisfied trivially, condition (4.22,11′) requires the permutability of congruences $A \sqcap (\cup A \cap \cup B)$, $(C \wedge B) \sqcap (\cup A \cap \cup B)$. Regarding $\cup C \supseteqq \cup A \cap \cup B$, these congruences are on $\cup A \cap \cup B$, and so commute.

Remark. In Theorem 4.22.8, which has been just proved, the following fact can be demonstrated: From the associability of the system $\{A_\iota : \iota \in \Gamma\}$ and the validity of

(4.22,11′) and (4.22,12) (see Theorem 4.22.7) may follow the associability of the system $\boldsymbol{B}$, even if the relation $\cup\, \boldsymbol{B} \supseteqq G_0$ or – using the same notation as in 4.22.8 – the relation $\cup\, C \supseteqq \cup\, A \cap \cup\, B$ fails.

Indeed, the assertion of Theorem 4.22.8 holds if the requirement $\cup\, C \supseteqq \cup\, A \cap \cup\, B$ is replaced by the condition $\cup\, C \supseteqq A(O)$ (this can be easily deduced from [12] 3.9). Commuting congruences A, B in Ω-group can be easily found as well, for which $\cup\, A \cap \cup\, B \neq A(O)$. If we choose a congruence C such that $\cup\, C = A(O)$, it will be $\cup\, A \cap \cup\, B \supsetneqq \cup\, C$.

4.22.9 (I) *Let A, B, C be partitions in a set G, A, B commute, $H = \cup\, A \cap \cup\, B \cap \cup\, C$. Then A, $B \wedge C$ commute if and only if the partitions $A \sqcap H$, $(B \wedge C) \sqcap H$ commute and H respects the partition A.*

(II) *Let A, B, C be congruences in an Ω-group G, A, B commute. Then A, $B \wedge C$ commute if and only if $A(O) \subseteqq \cup\, C$.*

Proof. (I) It suffices to formulate the conditions 1′, 2, 3 and 4 of Theorem 4.22 for the case that the partitions A_1, A_2, B are denoted by A, B, C:

(1′) $A \sqcap H$, $(B \wedge C) \sqcap H$ commute, (2) H respects the partition A. Condition (3) is satisfied evidently, condition (4) trivially.

(II) If A, B, C are congruences, then $A \sqcap H$, $(B \wedge C) \sqcap H$ as congruences on the Ω-group H commute, therefore (1′) holds. Condition (2) is equivalent to the condition $A(O) \subseteqq \cup\, C$. It is evidently implied by condition (2) and it implies it as well for $A(O) \subseteqq \cup\, A \cap \cup\, B$.

Remark. 1. Assertion (II) can be very easily deduced from 4.22.6 or from 4.21.1(a).

2. From (I) there follows Cor. 2.8 [6]. Namely, if $A \leqq C$, then $\cup\, C$ and consequently even H respect A. $A \sqcap H$, $(B \wedge C) \sqcap H$ commute as well, since for $x, y \in H$ there holds:

$xA(B \wedge C)\, y \Rightarrow xABy, xACy$ (see, e.g. 4.14) $\Rightarrow xBAy, xCAy$ (since the comparable partitions $A \sqcap H \leqq C \sqcap H$ on the set H commute) $\Rightarrow x(BA \wedge CA)\, y \Rightarrow x(B \wedge C)\, Ay$ (see 4.14). The reverse implications hold as well (by an analogous argument). This completes the proof.

4.23 *Let $\boldsymbol{A} = \{A_\iota : \iota \in \Gamma\}$ be a system of partitions in a set G, B a partition in G, $G_0 = \bigcup_{\iota \in \Gamma} \cup\, A_\iota$, $H = G_0 \cap \cup\, B$, $\Gamma_1 \subseteqq \Gamma$, $\Gamma_2 = \Gamma \setminus \Gamma_1$. Let the system $\boldsymbol{B} = \{A_\alpha : \alpha \in \Gamma_1\} \cup \{B \wedge A_\mu : \mu \in \Gamma_2\}$ be associable, $\bigvee \boldsymbol{A} = \bigvee \boldsymbol{B}$. Then the system $\boldsymbol{A}$ is associable if and only if there holds:*

a) *$\alpha \in \Gamma_1$, $A^1_\alpha \cap (G_0 \setminus H) \neq \emptyset \Rightarrow \boldsymbol{A} \sqcap A^1_\alpha$ is an associable system of partitions (on A^1_α),*

b) $\alpha \in \Gamma_1$, $\mu \in \Gamma_2$, $A^1_\alpha \cap (\cup\, A_\mu \setminus \cup\, B) \cup (G \setminus G_0) \neq \emptyset \Rightarrow A^1_\alpha \in A_\mu$.

Proof. Denoting $B_\iota = A_\iota (\iota \in \Gamma_1)$, resp. $= B \wedge A_\iota (\iota \in \Gamma_2)$ it will be $B_\iota \leqq A_\iota \leqq$

$\leqq \bigvee_{\iota \in \Gamma} B_\iota$. By 4.17 the system A is associable if and only if (4.17,1) and (4.17,2) hold. The first condition will be reduced to a) in the present case, the second to b).

4.23.1 Corollary. *Let $A = \{A_\iota : \iota \in \Gamma\}$ be a system of partitions on a set G, B a partition on G, $\Gamma_1 \subseteqq \Gamma$, $\Gamma_2 = \Gamma \setminus \Gamma_1$. Let the system $\boldsymbol{B} = \{A_\alpha : \alpha \in \Gamma_1\} \cup \cup \{B \wedge A_\mu : \mu \in \Gamma_2\}$ be associable, $\bigvee A = \bigvee B$. Then the system A is associable.*

4.23.2 Remark. The associability of both the systems A and $\boldsymbol{B}$ does not imply the equation $\bigvee A = \bigvee \boldsymbol{B}$ (not even in the case of partitions "on") as it is shown by the following example: $G = \{1, 2, 3, 4\}$, $A_1 = \{\{1, 2\}, \{3, 4\}\}$, $A_2 = \{\{1, 3\}, \{2, 4\}\}$, $B = A_1$. The partitions A_1, A_2 commute, the partitions A_1, $B \wedge A_2$, too. There holds $A_1 \vee A_2 = G_{\max}$, $A_1 \vee (B \wedge A_2) = A_1 \neq G_{\max}$.

4.24 *Let $A = \{A_\iota : \iota \in \Gamma\}$ be an associable system of partitions in a set G, B, C partitions in G, $\Gamma_1 \subseteqq \Gamma$, $\Gamma_2 = \Gamma \setminus \Gamma_1$,* card $\Gamma_1 \geqq 2$, card $\Gamma_2 \geqq 2$, $\cup B \supseteqq \bigcap_{\iota \in \Gamma} \cup A_\iota$ $(= G_0)$, $\cup C \supseteqq G_0 \cap \cup B$. *The system $\boldsymbol{C} = \{C \wedge A_\alpha : \alpha \in \Gamma_1\} \cup \{B \wedge A_\mu : \mu \in \Gamma_2\}$ is associable if there holds*

(4.24,1) $B \geqq A \sqsupset (\cup A_\mu \cap \cup B)$, $\alpha \in \Gamma_1$, $\mu \in \Gamma_2$
(4.24,2) $C \geqq A_\mu \sqsupset (\cup A_\alpha \cap \cup C)$, $\alpha \in \Gamma_1$, $\mu \in \Gamma_2$.

Proof. From 4.22.5 it follows that condition (4.24,1) implies the associability of the system $\boldsymbol{B} = \{A_\alpha : \alpha \in \Gamma_1\} \cup \{B \wedge A_\mu : \mu \in \Gamma_2\}$. From condition (4.24,2) it follows immediately that $C \geqq (B \wedge A_\mu) \sqsupset (\cup A_\alpha \cap \cup C)$, $\alpha \in \Gamma_1$, $\mu \in \Gamma_2$. Since we assume $\cup C \supseteqq G_0 \cap \cup B$, the system $\boldsymbol{C}$ is associable by 4.22.5.

4.25 The following two theorems represent modifications of Cor. 2.12 [6] for congruences in Ω-group.

(1) *Let A, B, C, D be congruences in an Ω-group G, A, B commute, $A \leqq C$, $B \leqq D$. Then the congruences*

$A_1 = A(B \wedge C) = C \wedge AB$ *and* $B_1 = B(A \wedge D) = D \wedge AB$ *commute. The partitions*

$A_2 = A \vee_P (B \wedge C) = C \wedge (A \vee_P B)$ *and* $B_2 = B \vee_P (A \wedge D) = D \vee (B \vee_P D)$ *commute as well.*

(2) *Let A, B, C, D be congruences in an Ω-group G. Let every two successive partitions in the sequence A, B, C, D, A commute. Then the partitions*

$$A_2 = A \vee_P (B \wedge C),\ B_2 = B \vee_P (A \wedge D)$$

commute.

Remark. Further commuting pairs will be obtained by cyclic permutations of the ordered set A, B, C, D.

Proof. (1) Two expressions of A_1 and B_1 follow from 4.14. A_1 is a congruence for $C \wedge AB$ is a congruence by [12] 3.3. Similarly for B_1.

Two expressions of A_2 follow from the commutativity of A, B; by [12] 3.9.2 there holds namely $(B, A)\, M^*$ in $P(G)$. Similarly for B_2.

The commutativity of A_1, B_1:

By [12] 3.5.5 there holds (with regard to the relations

$$A(O) \cup B(O) \subseteqq \cup A \cap \cup B, \cup A \subseteqq \cup C, \cup B \cup \cup D)$$

$$\cup A_1 = A(O) + \cup A \cap \cup B \cap \cup C = \cup A \cap \cup B, \cup B_1 = \cup A \cap \cup B$$

$$A_1(O) = A(O) + \cup A \cap B(O) \cap C(O) \subseteqq \cup A \cap \cup B = \cup B_1, B_1(O) \subseteqq \cup A_1 .$$

Then the congruences A_1, B_1 commute by [12] (3.9).

In proving commutativity of the partitions A_2, B_2 we need prove at first that each of the partitions A and $B \wedge C$ commutes with each of B and $A \wedge D$.

A, $B \wedge C$ commute by 4.14.2 and since $B \wedge C \leqq D$, $B \wedge C$, $A \wedge D$ commute as well. The congruences A, $A \wedge D$ commute ([12] 3.9) since

$$A(O) \subseteqq \cup B \subseteqq \cup D \Rightarrow A(O) \subseteqq \cup A \cap \cup D = \cup (A \wedge D),$$

$$(A \wedge D)(O) = A(O) \cap D(O) \subseteqq \cup A.$$

Analogously, B, $B \wedge C$ commute. By 4.15

$A_2 = A \vee_P (B \wedge C)$ commutes with B and with $A \wedge D$,

consequently, by 4.15 again, A_2 commutes with $B \vee_P (A \wedge D) = B_2$.

(2) Similarly as in (1), to prove the commutativity of A_2, B_2, it suffices to prove that each of the partitions A and $B \wedge C$ commutes with each of B and $A \wedge D$. By 4.13.3 A, $A \wedge D$ commute and B, $B \wedge C$ as well. The congruences $B \wedge C$, $A \wedge D$ also commute since $(B \wedge C)(O) = B(O) \cap C(O) \subseteqq \cup A \cap \cup D = \cup (A \wedge D)$, $(A \wedge D)(O) = A(O) \cap D(O) \subseteqq \cup B \cap \cup C = \cup (B \wedge C)$.

4.26 *Let A, B, C be congruences in an Ω-group, A, B commute, $C = (A \vee_P C) \wedge$ $\wedge (B \vee_P C)$ and $C(O) \subseteqq \cup A$. Then A, C commute. If, moreover, $C \leqq A \vee_P B$ holds, then B, C also commute.*

Remark. 4.26 is a generalization of Th. 2.8 [6] referred to congruences.

Proof. Condition $C = (A \vee_P C) \wedge (B \vee_P C)$ implies $\cup C = (\cup A \cup \cup C) \cap$ $\cap (\cup B \cup \cup C) \supseteqq \cup A \cap \cup B$. The condition of commutativity of A, B leads to $A(O) \subseteqq \cup A \cap \cup B \subseteqq \cup C$. This together with the condition $C(O) \subseteqq \cup A$ gives the commutativity of A, C.

If, moreover, there holds $C \leqq A \vee_P B$, we have $A \leqq A \vee_P C \leqq A \vee_P B$. By 4.17.2 $A \vee_P C$, B will commute if $\cup A \cap \cup C$ respects B. This holds, however, for

$$\cup A \supseteqq B(O), \qquad \cup C \supseteqq \cup A \cap \cup B \supseteqq B(O).$$

Now we apply Theorem 4.22.9 (I) to the triple of partitions B, $A \vee_P C$, C. Then B, $(A \vee_P C) \wedge C = C$ commute if and only if $B \sqcap H$, $C \sqcap H$ (where $H = \cup B \cap \cup C$) commute (which is satisfied for there are considered congruences on the Ω-group H)

and if $B^1 \cap \cup B \cap \cup C \neq \emptyset$ implies $B^1 \subseteqq \cup C$. This later condition means that the set $\cup C$ respects the partition B, i.e. $B(O) \subseteqq \cup C$. This, however, holds as well as we have shown above.

4.27 *Let A, B, C be congruences in an Ω-*group, *A, B commute, C is between A and B (i.e. $(A \wedge C) \vee_P (B \wedge C) = C = (A \vee_P C) \wedge (B \vee_P C)$). Then A, C commute and B, C commute.*

Proof. By 3.5.7 there holds

$$C(O) = [A(O) \cap C(O) + \cup A \cap \cup C \cap B(O) \cap C(O)] \cup [\cup B \cap \cup C \cap A(O) \cap \\ \cap C(O) + B(O) \cap C(O)].$$

With regard to the relation $A(O) \cup B(O) \subseteqq \cup A \cap \cup B$ ([12] 3.9) one gets

$$C(O) = A(O) \cap C(O) + B(O) \cap C(O) \subseteqq \cup A \cap \cup B.$$

So it is proved that the conditions of Theorem 4.26 are satisfied. Then A, C commute and B, C commute.

4.27.1 The previous Theorem represents a generalization of Cor. 2.13 [6] related to the congruences in an Ω-group.

4.28 (See Th. 2.10' [6]). *Let A, B be congruences in an Ω-group G. The following conditions are equivalent.*

(1) *Every partition $C \in P(G)$, $A \wedge B \leqq C \leqq A$, commutes with B.*
(2) *Every partition $C \in P(G)$, $A \wedge B \leqq C \leqq B$, commutes with A.*
(3) *A, B commute and $A(O)$, $B(O)$ are comparable sets.*
(4) *$A(O) \subseteqq B(O) \subseteqq \cup A$ or $B(O) \subseteqq A(O) \subseteqq \cup B$.*

Proof. We shall use Theorem 2.10 [6] by which (1) is equivalent to the following condition:

(5) A, B commute and any block V of the partition $A \vee_P B$ either does not contain any block of the partition A or contains a block $A^1 \in A$ such that any block $A^2 \in A$, $A^2 \neq A^1$, $A^2 \subseteqq V$, is contained in a block of the partition B.

In proving the equivalence (3) $\Leftrightarrow$ (5), we shall prove (1) $\Leftrightarrow$ (3). Since condition (3) is symmetric with regard to A, B, we have immediately (2) $\Leftrightarrow$ (3). The equivalence (4) $\Leftrightarrow$ (3) follows from 3.9.

Let the condition (5) be satisfied. The block $V \in A \vee_P B$ containing $O \in G$ contains $A(O)$ and $B(O)$. From the commutativity of A, B it follows by 4.8.1 that $\cup A \cap \cup B$ respects the partition $A \vee_P B$, then $V \subseteqq \cup A \cap \cup B$. If $A(O)$ is the unique block of A contained in V, then V is on the one hand equal to $A(O)$ and on the other hand is union of these blocks of B which are incident with $A(O)$, i.e. $A(O) = V = A(O) + B(O)$. Hence $B(O) \subseteqq A(O)$. Let V contain a block of A different from $A(O)$, e.g. $v + A(O)$. Then from (5) it follows that $A(O)$ or $v + A(O)$ is a subset of a block of B. The first case gives $A(O) \subseteqq B(O)$, the other $v + A(O) \subseteqq v + B(O)$, hence again $A(O) \subseteqq B(O)$. Finally: $A(O)$ and $B(O)$ are comparable sets.

Let (3) hold. From 4.8.1 it follows that $\cup A \cap \cup B$ respects the partition $A \vee_P B$. Then any block $V \in A \vee_P B$ is a subset of some of the sets $\cup A \setminus \cup B$, $\cup A \cap \cup B$, $\cup B \setminus \cup A$. Let V contain two distinct blocks of A. Then $V \subseteqq \cup A \cap \cup B$. The partitions $\bar{A} = A \sqcap (\cup A \cap \cup B)$ and $\bar{B} = B \sqcap (\cup A \cap \cup B)$ are congruences on the Ω-group $\cup A \cap \cup B$ and $V \in \bar{A} \vee_P \bar{B}$. If $A(O) \subseteqq B(O)$, then $\bar{A} \leqq \bar{B}$, consequently each block of $\bar{A}$ which is a subset of V is a subset of a block of B. If $B(O) \subseteqq A(O)$, then $\bar{B} \leqq \bar{A}$, thus V contains the unique block of $\bar{A}$ and consequently of A as well. This completes the proof.

References

[1] Birkgoff, G.: *Teorija struktur.* Moskva 1952.

[2] Borůvka, O.: *Teorie rozkladů v množině.* Publ. Fac. Sci. Univ. Brno, No. 278 (1946), 1—37.

[3] Borůvka, O.: *Grundlagen der Gruppoid-und Gruppentheorie.* Berlin 1960 (English version 1974).

[4] Borůvka, O.: *Základy teorie grupoidů a grup.* ČSAV Praha 1962.

[5] Draškovičová, H.: *The lattice of partitions in a set.* Acta F. R. N. Univ. Comen. — Math. 24 (1970), 37—65.

[6] Draškovičová, H.: *On a generalization of permutable equivalence relations.* Mat. čas. SAV 22 (1972), 297—308.

[7] Draškovičová, H.: *Permutability, distributivity of equivalence relations and direct products.* Matemat. čas. SAV 23 (1973), 69—87.

[8] Kolibiar, M.: *Über direkte Produkte von Relativen.* Acta Fac. Nat. Univ. Comen. 10, III (1965), 1—8.

[9] Ore, O.: *Theory of equivalence relations.* Duke Math. J. 9 (1942), 573—627.

[10] Šik, F.: *Über Charakterisierung kommutativer Zerlegungen.* Publ. Fac. Sci. Univ. Brno, No. 354 (1954), 1—6.

[11] Tran Duc Mai, *Partitions and congruences in algebras. I. Fundamental properties.* Archivum math. (Brno), 10 (1974) (to appear).

[12] Tran Duc Mai, *Partitions and congruences in algebras. III. Commutativity of congruences.* Archivum math. (Brno), 10 (1974) (to appear).

[13] Trevisan, G.: *Construzione di quasigruppi con relazioni di congruenza non permutabili.* Rendiconti del Seminario matematico della università di Padova, XXII (1953), 11—22.

[14] Wenzel, G. H.: *Note on a subdirect representation of universal algebras.* Acta Math. Acad. Sci. Hung. 18 (1967), 329—333.

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