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A NOTE ON LINE COLORINGS OF CUBIC GRAPHS

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All concepts used in this note may be found in [3] if not explicitely stated otherwise. The graphs considered throughout this note are connected, plane, cubic graphs which are assumed to have a 1-factorization C (i.e., a partition of the line set into three classes such that adjacent lines belong to different classes). The classes of C are called linear factors.

Let $C = \{L_1, L_2, L_3\}$ be a 1-factorization of G, and let $L = L_i \in C$ be a fixed linear factor. For each $e = [a, b] \in L$ there are four lines f_1, f_2, f_3, f_4 adjacent to e such that without loss of generality f_1, f_3 belong to L_2 (and f_2, f_4 belong to L_3). We say e is of type 1 in C if f_1 and f_3 belong to the same boundary of a face of G; otherwise e is said to be of type 2. For fixed C and fixed $L \in C$, we denote by $N(L)$ the number of lines e in L which are of type 2.

Theorem 1. *For arbitrary C of G and arbitrary L in C, $N(L)$ is even.*

In the proof of the theorem we shall use a concept called Q-extension: A line e of G is replaced with a quadrangle Q and the lines adjacent to e have (exactly) one of their endpoints in Q, such that the new (connected and cubic) graph is still plane. (For an exact definition of the Q-extension see [2]).

Proof of Theorem 1. Let C be a 1-factorization of G and L_i a fixed linear factor of G in C. For each line of L_i we apply the Q-extension and denote the graph obtained in this way by G^+. By [1, Theorem 2], G^+ is bipartite.

Now let $L_j, j \neq i$, be another linear factor of G in C. $T = L_i \cup L_j$ is a 2-factor of G, and to each cycle K of T corresponds K^+ of G^+ which is constructed as follows: If $e \in K$ belongs to L_j, then e belongs to K^+. If $e \in K$ belongs to L_i, then there are two lines f_1, f_2 of L_j adjacent to e in K; if e is in G of type 1, then there is in G^+ a path P joining f_1 and f_2 in Q (the quadrangle corresponding to e) and containing all points of Q, and we let P belong to K; but, if e is in G of type 2, then there are paths P_1 and P_2 in $Q \subset G^+$ joining f_1 and f_2 and each containing exactly three points of Q. We choose arbitrarily exactly one of P_1 and P_2 as belonging to K^+. By this construction, K^+ obviously is a cycle. Let T^+ denote the set of all these K^+. Since T is a set of disjoint cycles, therefore, T^+ is also a set of disjoint cycles, but T^+ is not a 2-factor of G^+ if L_i contains a line of type 2. In fact, to each $e \in L_i$, which is of type 2, corresponds exactly one $v(e) \in V(G^+)$ such that $v(e)$ does not belong to an element of T^+, and viceversa.

Obviously,

$$V(G^+) = (\bigcup_{K^+ \in T^+} V(K^+)) \cup \{v(e) \mid e \text{ is of type } 2\},$$

and

$$\bigcup_{K^+ \in T^+} V(K^+) \cap \{v(e) \mid e \text{ is of type } 2\} = \Phi.$$

Therefore,

$$\mid V(G^+) \mid - \mid \bigcup_{K^+ \in T^+} V(K^+) \mid = N(L_i),$$

and since T^+ is a set of disjoint cycles,

$$N(L_i) = \mid V(G^+) \mid - \sum_{K^+ \in T^+} \mid V(K^+) \mid.$$

Any cubic graph has an even number of points, and for any $K^+ \in T^+$, $\mid V(K^+) \mid$ is even because G^+ is bipartite. I.e., $N(L_i)$ is an even number. This proves the theorem.

In fact, it is possible to characterize the bipartite graphs in terms of $N(L_i)$. This is expressed by the following theorem.

Theorem 2. *G is bipartite if and only if G has a 1-factorization* $C = \{L_1, L_2, L_3\}$ *with* $N(L_i) = 0$ *for* $i = 1, 2, 3$.

Proof. 1. Assume G to be bipartite. Then G has a face-coloring with three colors 1, 2, 3 such that faces of the same color class have disjoint boundaries. We define $e \in E(G)$ as belonging to L_i if and only if e is boundary line for a face with color j and a face of color k such that $\{i, j, k\} = \{1, 2, 3\}$. One sees immediately that $C = \{L_1, L_2, L_3\}$ is a 1-factorization of G for which $N(L_1) = N(L_2) = N(L_3) = 0$.

2. If for some C of G and any $L_i \in C$ follows $N(L_i) = 0$, then we consider a face F and its boundary B. Assuming a line e of B belonging to L_i, it follows that the lines f_1, f_2 adjacent to e in B must belong to the same L_k, $k \neq i$, since $N(L_i) = 0$. Analogously, the lines adjacent to f_1, f_2 in B and different from e, belong to L_i since $N(L_k) = 0$, a.s.o. Thus we find that in B alternate lines of L_i and L_k, i.e., B is an even cycle. Since B is boundary of an arbitrarily chosen face of G we conclude by [1, Theorem 1] that G is bipartite. This finishes the proof of the theorem.

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