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Josef Zapletal

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ON THE CHARACTERIZATION OF SEMILATTICES SATISFYING THE DESCENDING CHAIN CONDITION AND SOME REMARKS ON DISTINGUISHING SUBSETS

JOSEF ZAPLETAL, BRNO

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1. INTRODUCTION

In this paper one internal characterization of semilattices satisfying the descending chain condition is given. Further the distinguishing subsets of semigroups are constructed. The results concerning the distinguishing subsets of semilattices satisfying the maximum condition are given in [4]. In the introductory part we give some known definitions and results.

1.1. Definition. Let G be an ordered set, $E \subseteqq G$. The set E is called *an initial segment of* G if, for all elements $x \in E$ and $y \in G$, the condition $y \leqq x$ implies $y \in E$.

1.2. Lemma. *Let G be a lower semilattice' $J \subseteqq G$. Then J is an ideal in G iff J is an initial segment.*

Proof. Let $J \subseteqq G$ be an ideal. Let $x \in J$, $y \in G$, $y \geqq x$. Then $x = x \wedge y \in J$. Hence J is an initial segment in G.

Let J be an initial segment in G. Let $x \in J$, $y \in G$ be arbitrary. Then we have $x \wedge y \leqq x$ and hence $x \wedge y \in J$ and J is an ideal in G.

1.3. Definition. Let G be an ordered set and let H be a well-ordered set. A one-one isotone mapping φ of G into H is called *a good extension of* G.

Theorem (V. NovÁk [1], Theorem 2.3) – Let G be an ordered set. Then G has a good extension if and only if G satisfies the descending chain condition.

1.4. Definition. Let G be a semigroup, $L \subseteqq G$ its subset. For $x, y \in G$ we put $(x, y) \in \Xi_{(G,L)}$ if, for any $u, v \in G$, the condition $uxv \in L$ is equivalent to $uyv \in L$.

1.5. Remark. It is easy to prove that the relation $\Xi_{(G,L)}$ is a congruence-relation on G (See [2], [3]).

1.6. Definition. Let G a semigroup, $L \subseteqq G$ a set, $u, v \in G$. We say that the elements $x, y \in G$, $x \neq y$, are *distinguished by u and v with respet to L* if the condition $uxv \in L$, $uyv \in L$, are not equivalent. We say that L *distinguishes* G if, for any $x, y \in G$, $x \neq y$, there are $u, v \in G$ such that x, y are distinguished by u and v with respet to L.

2. A CHARACTERIZATION OF SEMILATTICES SATISFYING THE DESCENDING CHAIN CONDITION

2.1. Theorem. *Let G be a semigroup satisfying the folowing conditions:*

1. *There exists a (transfinite or finite) sequence $(a_\iota)_\iota < \vartheta$ (where ϑ is an ordinal) in which each element of G appears precisely once such that for every $\alpha < \vartheta$ the set $\{a_\iota; \iota < \alpha\}$ is an ideal in G.*
2. *For every $x \in G$ there exists a natural number $n(x) \geqq 2$ such that $x^{n(x)} = x$.*

Then G is a lower semilattice satisfying the descending chain condition.

Proof. a) First we prove that all elements of G are idempotent.

Let $x \in G$. Let $\lambda < \vartheta$ be the least ordinal number such that $x \in J_\lambda = \{a_\iota; \iota < \lambda\}$. We show that λ is an isolated ordinal number. Let us admit that λ is a limit ordinal number. Then according to the condition 1. $J_\lambda = \bigcup_{\mu < \lambda} J_\mu$. Hence $x \in J_\mu$ for $\mu < \lambda$ and it is a contradiction with the minimality of λ. Therefore λ is isolated.

Simultaneously $x \in J_\lambda$, $x \notin J_{\lambda-1}$. For $n(x) = 2$ it follows $x^2 = x$ and x is an idempotent element in G. Let $n(x) > 2$. From the definition of the ideal we have $x^{n(x)-2} . x = x^{n(x)-1} \in J_\lambda$. According to the condition 1. it is $J_\lambda - J_{\lambda-1} = \{x\}$. Hence $x^{n(x)-1} \in J_{\lambda-1}$ or $x^{n(x)-1} \in \{x\}$. Let $x^{n(x)-1} \in J_{\lambda-1}$. Then $x^{n(x)-1} . x = x \in J_{\lambda-1}$ which is a contradiction with the assumption that $x \notin J_{\lambda-1}$. Hence $x^{n(x)-1} = x$. Analogously it can be proved that $x^{n(x)-k} = x$ for every k for which $n(x) - k \geqq 2$.

b) We prove that under the conditions 1. and 2. the semigroup operation is commutative.

Let us suppose that there exist $x, y \in G$ such that $xy \neq yx$. Let $\lambda < \vartheta$ be the least ordinal number with the property $xy \in J_\lambda$. By a), λ is isolated. We can suppose, without loss of generality, that $yx \notin J_\lambda$. If we had $xy \in J_\lambda$, there would exist an isolated ordinal number $\alpha < \lambda$ such that $yx \in J_\alpha$ and $xy \notin J_\alpha$ and the whole following part of proof would be done analogously. Thus, we suppose $xy \in J_\lambda$ $yx \notin J_\lambda$. The condition $xy \in J_\lambda$ implies $y(xy)\, yx \in J_\lambda$. In the part a) of the proof we have proved that every element $x \in G$ is idempotent. Hence $yxyyx = yxyx = yx$. We have $yx \in J_\lambda$ which is a contradiction to the assumption that $yx \notin J_\lambda$. Thus we have proved that there are no elements $x, y \in G$ for which $xy \neq yx$ holds. Such semigroup – operation is commutative.

From the parts a) and b) this proof it follows that G is an idempotent and commutative semigroup and hence a semilattice. Let the binary semigroup operation be written in the form of infimum ($\wedge$). Then G is a lower semilattice. We prove that the semilattice-ordering of G has a good extension.

c) If $\lambda \neq \mu$, then $a_\lambda \neq a_\mu$. We prove that $a_\lambda \wedge a_\mu = a_\mu$ implies $\mu \leqq \lambda$. Let us admit $\lambda < \mu$. Then $J_{\lambda+1}$ contains a_λ and does not contain a_μ. But $J_{\lambda+1}$ contains also $a_\lambda \wedge a_\mu = a_\mu$ and it is a contradiction. Hence there exists a good extension of the primary partial ordering.

Using the part c) of this proof and NovÁk's Theorem (see the introduction) we have that G satisfies the descending chain condition.

2.2. Theorem. *Let G be a semigroup. Then the following statements are equivalent:*

A. *G is a lower semilattice satisfying the descending chain condition.*

B. a) *There exists a (transfinite or finite) sequence $(a_\iota)_{\iota<\vartheta}$ in which each element of G appears precisely once such that for every $\alpha < \vartheta$ the set $\{a_\iota; \iota < \alpha\}$ is an ideal in G.*

b) *For every $x \in G$ there exists a natural number $n(x) \geqq 2$ such that $x^{n(x)} = x$.*

Proof. The statement B implies A according to Theorem 2.1. A implies B follows from [1] Theorem 2.3.

2.3. Theorem. *Let G be a semigroup, $I \subseteqq G$ an ideal in G. Let L distinguish I. Let the following conditions be valid for G:*

1. *There exists a (transfinite or finite) sequence $(a_\iota)_{\iota<\vartheta}$ in which each element of $G - I$ appears precisely once such that $J_\alpha = I \cup \{a_\iota; \iota < \alpha\}$ is an ideal in G for every $\alpha < \vartheta$.*

2. *$x^3 = x$ for every $x \in G - I$.*

Then there exists $L \subseteqq G$ distinguishing G.

Proof. First, we carry out some preparatory considerations.

a) An arbitrary union of ideals of a semigroup is an ideal in this semigroup. Let us have an element $x \in \bigcup_{\varkappa \in K} J_\varkappa$. For arbitrary $a, b \in G$, $ax, xb \in \bigcup_{\varkappa \in K} J_\varkappa$. Indeed, let $\varkappa_0$ be an index from K for which $x \in J_{\varkappa_0}$. For arbitrary $a, b \in G$ $ax, xb \in J_{\varkappa_0}$ and hence $ax, xb \in \bigcup_{\varkappa \in K} J_\varkappa$.

b) In this part of the proof we construct distinguishing subsets for the given sequence of ideals satisfying the conditions of this Theorem.

The ideal I_0 has a distinguishing subset L_0 according to the assumption of the Theorem.

Let $0 < \alpha < \vartheta$ be an ordinal. Suppose that, for each $\lambda < \alpha$, we have defined a distinguishing subset L_λ of J_λ satisfying the following condition: *If* $\lambda < \mu < \alpha$ then $L_\lambda = J_\lambda \cap L_\mu$. Then we define L_α in the following way:

Let α be isolated. Then $J_{\alpha-1}$ is an ideal in G such that $L_{\alpha-1}$ distinguishes $J_{\alpha-1}$. Then the sets $L_{\alpha-1}$ or $L_{\alpha-1} \cup \{a_{\alpha-1}\}$ distinguish $J_\alpha = J_{\alpha-1} \cup \{a_{\alpha-1}\}$.

Indeed, if there exist for every $x \in J_{\alpha-1}$ elements u_x, $v_x \in L_{\alpha-1}$ such that $L_{\alpha-1}$ contains precisely one of the elements $u_x x v_x$, $u_x a_{\alpha-1} v_x$ then both sets $L_{\alpha-1}$, $L_{\alpha-1} \cup \cup \{a_{\alpha-1}\}$ distinguish the ideal J_α. We define L_α to be one of these sets.

If there exists $x \in J_{\alpha-1}$ such that for every u, $v \in J_{\alpha-1}$ the set $L_{\alpha-1}$ contains either both elements uxv, $ua_{\alpha-1}v$ or none, then there exists precisely one such element x. We have $a_{\alpha-1}xa_{\alpha-1} \in J_{\alpha-1}$. If $a_{\alpha-1}xa_{\alpha-1} \in L_{\alpha-1}$ we define the distinguishing subset L_α of the ideal J_α to be $L_{\alpha-1}$. This is possible for $a_{\alpha-1}a_{\alpha-1}a_{\alpha-1} = a_{\alpha-1}$ according to assumption 2) of this theorem and according to $a_{\alpha-1} \notin L_{\alpha-1}$. Therefore the set $L_{\alpha-1}$ distinquishes J_α. If $a_{\alpha-1}xa_{\alpha-1} \notin L_{\alpha-1}$ then we define the distinguishing set L_α of J_α to be $L_{\alpha-1} \cup \{a_{\alpha-1}\}$. This is possible because $a_{\alpha-1}xa_{\alpha-1} \in J_{\alpha-1}$ and hence $a_{\alpha-1}xa_{\alpha-1} \neq a_{\alpha-1}a_{\alpha-1}a_{\alpha-1} = a_{\alpha-1}$ and simultaneously $a_{\alpha-1} \in L_\alpha = L_{\alpha-1} \cup \{a_{\alpha-1}\}$.

If α is a limit number then $J_\alpha = \bigcup_{\nu<\alpha} J_\nu$ and we define the distinguishing subset L_α to be the union $\bigcup_{\nu<\alpha} L_\nu$.

If $\lambda < \alpha < \vartheta$ then $L_\lambda = J_\lambda \cap L_\alpha$. We prove it by induction. For each ordinal α we denote by $V(\alpha)$ the following assertion: If $\lambda < \alpha$ then $L_\lambda = J_\lambda \cap L_\alpha$.

Then $V(0)$ holds trivially as there is no ordinal $\lambda < 0$.

Let us have $0 < \alpha < \vartheta$ and suppose that $V(\beta)$ holds for each β, $0 \leqq \beta < \alpha$. If α is isolated and $\lambda = \alpha - 1$ then we have $L_\alpha = L_{\alpha-1}$ or $L_\alpha = L_{\alpha-1} \cup \{a_{\alpha-1}\}$, where $a_{\alpha-1} \in J_\alpha$. Then $L_{\alpha-1} = J_{\alpha-1} \cap L_\alpha$.

If α is isolated and $\lambda < \alpha - 1$ then $L_\lambda = J_\lambda \cap L_{\alpha-1} = J_\lambda \cap (J_{\alpha-1} \cap L_\alpha) = (J_\lambda \cap J_{\alpha-1}) \cap L_\alpha = J_\lambda \cap L_\alpha$.

If α is a limit ordinal then $J_\lambda \cap L_\alpha = J_\lambda \cap (\bigcup_{\iota<\alpha} L_\iota) = \bigcup_{\iota<\alpha} (J_\lambda \cap L_\iota) = \bigcup_{\lambda<\iota<\alpha} (J_\lambda \cap L\iota) \cup \cup \bigcup_{\iota\leqq\lambda} (J_\lambda \cap L_\iota) = L_\lambda$, Thus, $V(\alpha)$ holds.

Let α be a limit number. Then $J_\alpha = \bigcup_{\nu<\alpha} J_\nu$ is an ideal by (a). Let us admit that $L_\alpha = \bigcup_{\nu<\alpha} L_\nu$ does not distinguish J_α. It means there exists at least one pair $x, y \in J_\alpha$ such that it holds for all elements u, $v \in J_\alpha$, simultaneously either uxv, $uyv \in L_\alpha$ or uxv, $uyv \notin L_\alpha$. Let us consider the first case. The second is analogous.

It holds x, $y \in J_\alpha = \bigcup_{\nu<\alpha} J_\nu$, therefore there exists $\nu_0 < \alpha$ such that x, $y \in J_{\nu_0}$. Since J_{ν_0} has a distinguishing subset L_{ν_0}, there exist u_0, $v_0 \in J_{\nu_0}$ such that either $u_0xv_0 \notin L_{\nu_0}$, $u_0yv_0 \in L_{\nu_0}$ or $u_0xv_0 \in L_{\nu_0}$, $u_0yv_0 \notin L_{\nu_0}$. We consider the first case again, the second being analogous. Since x, $y \in J_{\nu_0}$ then also $u_0xv_0 \in J_{\nu_0}$ and $u_0yv_0 \in J_{\nu_0}$.

Simultaneously $u_0xv_0 \notin L_{\nu_0}$, $u_0yv_0 \in L_{\nu_0}$ and $u_0yv_0 \in L_\alpha = \bigcup_{\nu<\alpha} L_\nu$. From the construction of distinguishing subsets in the part b) and from the condition $u_0xv_0 \notin L_{\nu_0}$ it follows that u_0xv_0 is an element of no distinguishing subset $L_\varkappa$ for $\varkappa \geqq \nu_0$ and hence u_0xv_0 does not lie in $L_\alpha = \bigcup_{\nu<\alpha} L_\nu$ and L_α distinguishes J_α.

2.4 Example. *Let S be a lower semillattice with the operation $\wedge$ satisfying the descending chain condition with the least element e. Further let G be the free idempotent monoid with the operation $\circ$ and with the unit e and generators a, b. Let $G \cap S = \{e\}$. Let us put $\mathcal{G} = G \cup S$ and let us define*

$$x \,.\, y = \begin{cases} e & if \;\; x = e = y \\ x & if \;\; y = e \neq x \\ y & if \;\; x = e \neq y \\ x \wedge y & if \;\; x \neq e \neq y;\, x, y \in S \\ x \circ y & if \;\; x \neq e \neq y;\, x, y \in G \\ x & if \;\; x \neq e \neq y;\, x \in G,\, y \in S \\ y & if \;\; x \neq e \neq y;\, x \in S,\, y \in G \end{cases}$$

Then the following statements hold:

(i) *$.$ is a monoidal operation on $\mathcal{G}$.*
(ii) *G is an ideal in $\mathcal{G}$.*
(iii) *The set $L_0 = \{a \,.\, b \,.\, a, b \,.\, a \,.\, b\}$ distinguishes G.*
(iv) *There exists $L \subseteqq \mathcal{G}$ such that L distinguishes $\mathcal{G}$.*

Proof. (i) Evidently, e is the unit in $\mathcal{G}$. We must prove that the operation $\bullet$ is associative. The following evidently holds from the definition of monoidal operation: Let $x \in G$, $y \in S$ then $x \,.\, y = x = y \,.\, x$. If the elements $u, v, w \in \mathcal{G}$ are all from G or from S then $(u \,.\, v) \,.\, w = u \,.\, (v \,.\, w)$. If one of elements a, b, c – let us denote it by z – is in G and the remaining two are in S then certainly $(u \,.\, v) \,.\, w = z = u \,.\, (v \,.\, w)$. If two of the elements u, v, w are in G then let us denote their product by z. We have again $(u \,.\, v) \,.\, w = z = u \,.\, (v \,.\, w)$.

(ii) The second statement follows from the definition of the monoidal operation on $\mathcal{G}$.

(iii) We find, for every two elements $x, y \in G$, $x \neq y$, the elements u, v such that either $uxv \in L_0$, $uyv \notin L_0$ or $uxv \notin L_0$, $uyv \in L_0$. We choose all the possible unordered pairs $x, y \in G$ and for every pair the respective elements $u, v \in G$.

x, y	u, v	$u \,.\, x \,.\, v$	$u \,.\, y \,.\, v$
e, a	b, b	$b \notin L_0$	$b \,.\, a \,.\, b \in L_0$
e, b	a, a	$a \notin L_0$	$a \,.\, b \,.\, a \in L_0$
$e, a \,.\, b$	e, a	$a \notin L_0$	$a \,.\, b \,.\, a \in L_0$

$e, b . a$	e, b	$b \notin L_0$	$b . a . b \in L_0$
$e, a . b . a$	e, e	$e \notin L_0$	$a . b . a \in L_0$
$e, b . a . b$	e, e	$e \notin L_0$	$b . a . b \in L_0$
a, b	a, a	$a \notin L_0$	$a . b . a \in L_0$
$a, a . b$	e, a	$a \notin L_0$	$a . b . a \in L_0$
$a, b . a$	e, b	$a . b \notin L_0$	$b . a . b \in L_0$
$a, a . b . a$	e, e	$a \notin L_0$	$a . b . a \in L_0$
$a, b . a . b$	e, e	$a \notin L_0$	$b . a . b \in L_0$
$b, a . b$	e, a	$b . a \notin L_0$	$a . b . a \in L_0$
$b, b . a$	e, b	$b \notin L_0$	$b . a . b \in L_0$
$b, a . b . a$	e, e	$b \notin L_0$	$a . b . a \in L_0$
$b, b . a . b$	e, e	$b \notin L_0$	$b . a . b \in L_0$
$a . b, b . a$	e, b	$a . b \notin L_0$	$b . a . b \in L_0$
$a . b, a . b . a$	e, e	$a . b \notin L_0$	$a . b . a \in L_0$
$a . b, b . a . b$	e, e	$a . b \notin L_0$	$b . a . b \in L_0$
$b . a, a . b . a$	e, e	$b . a \notin L_0$	$a . b . a \in L_0$
$b . a, b . a . b$	e, e	$b . a \notin L_0$	$b . a . b \in L_0$
$a . b . a, b . a . b$	b, e	$b . a \notin L_0$	$b . a . b \in L_0$

We have chosen for every pair x, y from G Some elements u, v such that $uxv \notin L_0$, $uyv \in L_0$. Hence L_0 distinguishes G. (iv) The monoid $\mathcal{G}$ satisfies the assumptions of the Theorem 2.3. Therefore there exists a distinguishing subset L in $\mathcal{G}$.

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J. Zapletal
602 00 Brno, Hilleho 6
Czechoslovakia