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ON PLAIN ABSOLUTE EQUILIBRIUM POINTS IN GENERAL NON-ORDERED GAMES WITH PERFECT INFORMATION [I]

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An essential generalization of Berge's variant of the Zermelo—von Neumann theorem (and of the original result of Zermelo) is proved. Our theorem concerns quite general non-ordered games with perfect information and with chain-valued pay-off functions, and it admits infinite plays. Important particular cases are considered. A further generalization (poset-valued pay-offs) is shown.

§ 0. INTRODUCTION

We shall consider quite general non-ordered games with perfect information, but without chance moves. ("Non-ordered games" means "non-initial games on oriented graphs". The considered games may have infinitely many positions or players, infinite plays are admitted, and the pay-off functions are chain-valued.)

Under our conception, at any moment of every play (of a game with perfect information) the moving player knows the preceding course (including the momentary position) of the play. In a contradistinction to the usually investigated games on (finite rooted) trees, in the considered games position need not "involve" the preceding course of play; consequently, at defining the general (pure) strategies it is necessary to introduce certain auxiliary notions ("segments" etc., see § 5). Nevertheless, the case if players use only the knowledge of momentary position is to be considered as the most important; the strategies corresponding to this case are called plain (§§ 2.6, 5.6.3, 5.7.6).

The notion of (pure) equilibrium point in a position (of a game with perfect information) can be introduced in the usual way; a system of strategies which is an equilibrium point in each position of a game is called an absolute equilibrium point of the game. An equilibrium point is called plain if it consists only of plain strategies. (Cf. §§ 2 and 5.)

The basic known result – the so-called Zermelo – von Neumann theorem – concerns only the games (with perfect information; chance moves are admitted) on finite rooted trees. This theorem was obtained by H. W. Kuhn ([7], § 4) as a corollary of his considerations on the decomposition of certain games; of course, it can be proved directly, by induction (cf., e.g., McKinsey's book [10], ch. VI, § 2, Th. 6.1). Nevertheless, the original result of Zermelo (in [14]) concerns (chess and similar) games which admit also infinite plays.

C. Berge has introduced and investigated (cf. [1], Ch. 1 of [2], and Ch. 6 and Appendix 1 of [3]) games (with perfect information) which are more natural and more general than the usually considered games on finite rooted trees, namely non--ordered games with perfect information. (Our conception of the latter games is somewhat more general than that of Berge. Cf. § 2.) The Berge equilibrium point theorem (it involves – after the elimination of chance moves – the above mentioned variant of the Zermelo – von Neumann theorem as a special case see [2], Ch. I, § 7 – the fundamental theorem, and [3], Ch. 6 – the Zermelo – von Neumann theorem) says that if a Bergean game with perfect information is locally finite (i.e., it has no infinite play) and each of its evaluation functions is finite-valued, then the game has (speaking in the terminology of the present paper) a plain absolute equilibrium point (but cf. §§ 3.6, 3.5, 2.8 – 9!). (Berge considers very special pay-off functions, namely those corresponding to "active" or "passive" players. Cf. § 2.4.0.) The proof can be performed in a natural way, but by means of transfinite induction (starting from the end-positions).

The original result of Zermelo (see [14]) concerns very special antagonistic (see § 2.11) games with perfect information. This result can be generalized in several ways; I proved several such theorems even for a somewhat more general kind of non-ordered antagonistic games (namely antagonistic complete games; the results of this character are based mainly on theorems 6.25/1 – 3 and 3.11 of [4]; they will be published in some of the following parts of [4], their preliminary variant is presented in [5]).

The main purpose of this paper is to generalize Berge's variant of the Zermelo – von Neumann theorem. Our main theorem gives an essential generalization in four ways: the notion of plain absolute equilibrium point is "stronger" and more natural than Berge's notion of absolute equilibrium point (cf. §§ 3.6, 2.8 – 9); the class of considered game structures (i.e. games without respecting pay-off functions) is somewhat richer (cf. §2); the class of pay-off functions (those satisfying the sufficient condition of the theorem) is much richer, also after restriction to the locally finite case; there exist games having infinite plays and satisfying the sufficient condition of the theorem. The latter way of generalization is to be considered as the most principal, since the usual proof idea is quite inapplicable if infinite plays are possible. (Cf. § 5.9.) Naturally, that sufficient condition is sizably strong (e.g., for each player of a game, all the infinite plays give the same pay-off) – the existence of a plain absolute equilibrium point is a "very strong property" (cf. § 5.8.1-2).

A certain part of the proof method (for the main theorem) is taken from the proof of the equilibrium point theorem in [6] (§ 3.13; this theorem concerns a class of finite complete (two-player) games), but the fact that infinite games are admitted in the present paper has led to essential complications (connected, among others, with the necessity of the use of transfinite induction), while the simpler structure of games with perfect information (in comparison with complete games) made possible to use some simplifications (cf. § 5.11); there are some other disparities.

Of course, we wish to obtain a theorem which sufficiently utilizes the new proof idea. This aim has led, especially, to the introduction of two auxiliary "technical" conditions ((A) and (B)) in the main theorem. There is a number of various particular cases (having self-contained meanings) in which the satisfaction of the whole sufficient condition is to be seen immediately; among others, the Berge theorem and the original result of Zermelo belong to these particular cases. (Cf. § 3.)

In this paper we consider chain-valued pay-off functions. Nevertheless, it is possible to introduce poset-valued pay-off functions, and to obtain some generalization of our results (§ 3) in a simple way; cf. §§ 5.0-5.5.

§ 1. ORIENTED GRAPHS. PSEUDOLENGTHS. SPECIAL PAY-OFF FUNCTIONS

1.0. Preliminaries. We shall use the accepted logical and set theoretical denotations and notions ($\neg$, $\wedge$, $\vee$, $\Rightarrow$; "iff" is to be read "if and only if"; $\emptyset$ means the empty set, X denotes the general cartesian product, etc.). For a set A, card A denotes the cardinal number of A, and we write $\exp A = \{B;\ B \subseteq A\}$ (the *Boolean* of A). Under a *binary relation* we mean a set of ordered pairs. *Mappings* are considered as special binary relations: $f = \{(f(x), x);\ x \in \operatorname{dom} f\}$ for any mapping f (where "dom" is the domain), and we write $f = (f(x);\ x \in \operatorname{dom} f)$, while $\{f(x);\ x \in \operatorname{dom} f\} = \operatorname{im} f$. There is exactly one mapping with empty domain (the *empty mapping*), namely $\emptyset$. For a mapping f and a set $A \subseteq \operatorname{dom} f$, the *restriction* of f to A is denoted by $f \mid A$ $(= (f(x);\ x \in A) = (\operatorname{im} f \times A) \cap f)$; of course, if $g \subseteq f$, then $g = f \mid \operatorname{dom} g$. At mappings denoted by Greek letters, sometimes we do not write parentheses.

A *partially ordered set* (*poset*) is a pair $\mathscr{V} = (V, \leqq)$, where V is a set and $\leqq\ \subseteq V \times V$ is a binary relation which is reflexive (on V), antisymmetric, and transitive; $\mathscr{V}$ is said to be a *chain* (or a *totally* ordered set, or a *linearly* ordered set) iff, moreover, $\leqq$ is full on V. (We say that $\rho \subseteq V \times V$ is *full* on V iff $V \times V = \rho^{-1} \cup \rho$.) We shall use the accepted elementary notions, denotations, and conventions for posets; especially, if we consider several posets or a system of posets, we often use the symbols $\leqq$, $>$, sup, min etc. *without index* (whenever no misunderstanding can arise by it).

1.1. Chains. Let $\mathscr{V} = (V, \leqq)$ be a chain. $\mathscr{V}$ is said to be *complete* iff sup A and inf A exist for any $A \subseteq V$ (i.e., iff the chain $\mathscr{V}$ is a complete lattice). $\mathscr{V}$ is said to be *well-ordered* [*inversely well-ordered*] iff min A [max A] exists for any nonempty $A \subseteq V$. It is easy to prove that *$\mathscr{V}$ is well-ordered and* (*at the same time*) *inversely well-ordered iff V is finite.*

1.2. Preference relations. Under a *preference relation* on a set X we mean a binary relation $\overset{\circ}{\leqq}\ \subseteq X \times X$ which is reflexive (on X), full (on X), and transitive.

Let X be a set, $\mathring{\mathscr{V}} = (V, \overset{\circ}{\leqq})$ be a set with a preference relation, let $f : X \to V$. Then the relation $\overset{\circ}{\leqq}_{(f, \mathring{\mathscr{V}})} = \{(x_1, x_2);\ x_1, x_2 \in X,\ f(x_1) \overset{\circ}{\leqq} f(x_2)\}$ is a preference relation on X.

On the other hand, if X is a set and $\overset{\circ}{\leqq}$ is a preference relation on X, then there exist a chain $\mathscr{V} = (V, \leqq)$ and a mapping $f : X \to V$ such that $\overset{\circ}{\leqq} = \overset{\circ}{\leqq}_{(f, \mathring{\mathscr{V}})}$. [It is sufficient to choose $V = X/(\overset{\circ}{\leqq}^{-1} \cap \overset{\circ}{\leqq}) = \{\{y;\ y \in X,\ y \overset{\circ}{\leqq} x,\ x \overset{\circ}{\leqq} y\};\ x \in X\}$ (the decomposition corresponding to the equivalence relation $\overset{\circ}{\leqq}^{-1} \cap \overset{\circ}{\leqq}$), $f = (\{y;\ y \in X, y \overset{\circ}{\leqq} x, y \overset{\circ}{\leqq} y\}; x \in X)$ (the natural surjection of X onto V), and $\leqq\ = \{(f(x_1), f(x_2));\ x_1, x_2 \in X, x_1 \overset{\circ}{\leqq} x_2\}$. We shall denote $\mathscr{V}_{\overset{\circ}{\leqq}} = \mathscr{V}$ and $f_{\overset{\circ}{\leqq}} = f$ for those $\mathscr{V}$ and f.]

1.3. Quasiorderings. Under a *quasiordering* on a set X we mean a binary relation (in X) which is reflexive (on X) and transitive. It is easy to see that it is admissible to re-formulate § 1.2 in the following way: "quasiordering" is to be written instead of "preference relation", "poset" is to be written instead of "chain", and "full (on X), " is to be omitted.

1.4. Let X be a set, let $\mathscr{V}_k = (V_k, \leqq_k)$ be chains and $f_k : X \to V_k$ for $k = 1, 2$. We say that f_1 with $\mathscr{V}_1$ and f_2 with $\mathscr{V}_2$ *express the same preference* iff $\overset{\circ}{\leqq}_{(f_1, \mathscr{V}_1)} = \overset{\circ}{\leqq}_{(f_2, \mathscr{V}_2)}$. We say that f_1 with $\mathscr{V}_1$ and f_2 with $\mathscr{V}_2$ *express antagonistic preferences* iff $\overset{\circ}{\leqq}_{(f_1, \mathscr{V}_1)} = (\overset{\circ}{\leqq}_{(f_2, \mathscr{V}_2)})^{-1}$.

1.5. Definition, remarks. Let $\overset{\circ}{\leqq}$ be a preference relation on a set X. We introduce binary relations $\overset{\circ}{\leqq}^{\varepsilon}$ ($\varepsilon = +, -$) on $(\exp X) \setminus \{\emptyset\}$ in such a way: for $\emptyset \neq A, B \subseteq X$

$A \overset{\circ}{\leqq}^{+} B \Leftrightarrow$ for each $a \in A$ there exists $b \in B$ such that $a \overset{\circ}{\leqq} b$;

$A \overset{\circ}{\leqq}^{-} B \Leftrightarrow$ for each $b \in B$ there exists $a \in A$ such that $a \overset{\circ}{\leqq} b$.

These two relations are preference relations on $(\exp X) \setminus \{\emptyset\}$. [Evidently, they are reflexive and transitive. If $\emptyset \neq A, B \subseteq X$, $\neg B \overset{\circ}{\leqq}{}^{+} A$, then there exists $b_0 \in B$ such that $\neg b_0 \overset{\circ}{\leqq} a$ for each $a \in A$, but then $a \overset{\circ}{\leqq} b_0$ for each $a \in A$, and, therefore, $A \overset{\circ}{\leqq}{}^{+} B$. The fullness of $\overset{\circ}{\leqq}^{-}$ can be proved analogously.]

In particular, if $\overset{\circ}{\leqq} = \overset{\circ}{\leqq}_{(f, \mathscr{V})}$ for some complete chain $\mathscr{V} = (V, \leqq)$ and a suitable mapping $f : X \to V$, then (for $\varepsilon = +, -$) there holds: $\overset{\circ}{\leqq}^{\varepsilon} = \overset{\circ}{\leqq}_{(f^{\varepsilon}, \mathscr{V})}$, where the mapping $f^{\varepsilon} : (\exp X) \setminus \{\emptyset\} \to V$ is defined in such a way:

$$f^{+}(A) = \sup \{f(a);\ a \in A\}, \quad f^{-}(A) = \inf \{f(a);\ a \in A\},$$

where $\emptyset \neq A \subseteq V$, and sup and inf are taken in $\mathscr{V}$. (The proof is simple.)

The motivation of the introduction of $\overset{\circ}{\leqq}^{+}$ and $\overset{\circ}{\leqq}^{-}$ is connected with preference relations and pay-off functions in Bergean games with perfect information. (See § 2.4.1.)

1.6. Graphs. Under an *oriented graph* (or only: a *graph*; we shall consider only oriented graphs) we undersand a mapping Γ such that

$$\operatorname{im} \Gamma \subseteq \exp \operatorname{dom} \Gamma$$

(this definition conforms to Berge's conception of oriented graphs; we do not denote a graph by (Γ, X) with $\Gamma : X \to \exp X$, as $X = \operatorname{dom} \Gamma$ is given by Γ).

1.7. Convention. (Positions.) If a fixed graph Γ is considered, we use symbols P, P_0, Z in the following sense:

$$P = \operatorname{dom} \Gamma, \quad P_0 = \{x; x \in P, \Gamma x = \emptyset\}, \quad Z = P \setminus P_0;$$

the elements of P (i.e., the vertices of the graph Γ) $[P_0; Z]$ will be called *positions* [*final positions*, or *terminal positions; nonfinal positions*, or *nonterminal positions*] (respectively).

1.8 Our **interpretation of graphs** corresponds to that of Berge: if Γ is a graph, $x \in P$, then $y \in \Gamma x$ occurs iff "Γ contains an *edge* which goes from x to y"; therefore, a *vertex* (position) is terminal iff there does not exist an edge going from it. In our considerations (in the following §§), Γ will usually be the *graph of a game*, and then Γx means the set of (all) positions which can follow immediately after x; any *play* (cf. § 1.13) of such a game is performed in the following way: at a nonfinal position x, the moving player chooses an element $y \in \Gamma x$ as the next (following) position, while at any final position the play terminates.

1.9. Transformations. Let Γ be a graph. The set

$$\mathrm{T}(\Gamma) = \bigcup_{Y \subseteq Z} \bigtimes_{z \in Y} \Gamma z$$

is said to be the set of *plain Γ-transformations*; we shall say "*Γ-transformation*", too (as the general Γ-transformations are not considered in the main parts of this paper; cf. § 5.6.1). E.g., $\emptyset$ is a Γ-transformation (the *empty Γ-transformation*). $\sigma \in \mathrm{T}(\Gamma)$ is said to be *full* iff $\operatorname{dom} \sigma = Z$. $\mathrm{T}_F(\Gamma)$ will denote the set of full Γ-transformations. Under a *conservative Γ-transformation* we mean $\sigma \in \mathrm{T}(\Gamma)$ such that $\operatorname{im} \sigma \subseteq P_0 \cup \cup \operatorname{dom} \sigma$; of course, the empty Γ-transformation and also all full Γ-transformations are conservative. Clearly, a subset of a Γ-transformation is a Γ-transformation, too.

1.10. Denotation. In the whole paper, we denote

$$W = \{0, 1, 2, \ldots\} \cup \{\infty\}$$

and, for any $l \in W$,

$$W_l = \{k; k \in W, k < 1 + l\} \left(= \begin{Bmatrix} \{0, 1, \ldots, l\} \\ W \setminus \{\infty\} \end{Bmatrix} \quad \text{if} \quad l \begin{Bmatrix} < \\ = \end{Bmatrix} \infty \right),$$

$(1 + \infty = \infty)$. $\mathscr{W}$ will denote W with the natural ordering.

1.11. Definition. Let $l \in W$, let $\mathbf{y} = (y_k; k \in W_l)$ (be a mapping of W_l), and let x be an element. Then we put $x \oplus \mathbf{y} = (x_k; k \in W_{1+l})$, where $x_0 = x$ and $x_{k+1} = y_k$ for each $k \in W_l$.

1.12. Definition. Let $l \in W$, let $\boldsymbol{x} = (x_k;\ k \in W_l)$ (be a mapping of W_l), let $m \in W_l$. Then we put $\boldsymbol{x}^{[m]} = (x_{k+m};\ k \in W_{l-m})$ ($\infty - m = \infty$); $\boldsymbol{x}^{[m]}$ is called the mth *remainder* of $\boldsymbol{x}$.

1.13. Definitions, remarks. (Plays.) Let Γ be a graph. We say that $\boldsymbol{x}$ is a *Γ-play* (or only "a *play*", if Γ is fixed) iff $\boldsymbol{x} = (x_k;\ k \in W_l)$ for some $l \in W$ and some elements $x_k \in P$ such that $x_{k+1} \in \Gamma x_k$ for each $k < l$, and $x_k \in P_0$ if $l < \infty$ (cf. § 1.10). $\boldsymbol{X}_\Gamma$ (or only $\boldsymbol{X}$) denotes *the set of all Γ-plays.* We say that $\boldsymbol{x} = (x_k;\ k \in W_l) \in \boldsymbol{X}$ *starts from* x iff $x_0 = x$ $\boldsymbol{\Gamma} x$ will denote *the set of all plays which start from* $x \in P$, and $\boldsymbol{\Gamma}$ is considered as the corresponding mapping, i.e. $\boldsymbol{\Gamma} = (\boldsymbol{\Gamma} x;\ x \in P)$. Clearly, $\boldsymbol{X} = \bigcup_{x \in P} \boldsymbol{\Gamma} x$, and $\boldsymbol{\Gamma} x \cap \boldsymbol{\Gamma} y = \emptyset$ if $x, y \in P$, $x \neq y$; it is easy to see that $\boldsymbol{\Gamma} x \neq \emptyset$ for each $x \in P$. If $x \in P_0$, then $\boldsymbol{\Gamma} x$ contains exactly one element, namely $\boldsymbol{\Gamma} x = \{(x_k;\ k = 0)\}$, where $x_0 = x$; we shall denote this Γ-play by (x). We say that $\boldsymbol{x} = (x_k;\ k \in W_l) \in \boldsymbol{X}$ *passes in* $Y \subseteq P$ iff $\{x_k;\ k \in W_l\} \subseteq Y$.

For $\boldsymbol{x} = (x_k;\ k \in W_l) \in \boldsymbol{X}$ we denote $\mathrm{L}(\boldsymbol{x}) = l$ (the *length* of $\boldsymbol{x}$; $\boldsymbol{x}$ is said to be *infinite* iff $\mathrm{L}(\boldsymbol{x}) = \infty$). L (or L_Γ) itself is considered as the corresponding mapping ($\mathrm{L}_\Gamma = \mathrm{L} = (\mathrm{L}(\boldsymbol{x})\ \ \boldsymbol{x} \in \boldsymbol{X}) : \boldsymbol{X} \to W$), and it is called ***the (natural) length on*** Γ.

Γ is said to be *locally finite* (or *progressively finite*; cf. [2], ch. I, § 7, or [3], ch. 3) iff $\mathrm{L}(\boldsymbol{x}) < \infty$ for each $\boldsymbol{x} \in \boldsymbol{X}$ (i.e. iff Γ has no infinite play). Of course, it may happen that Γ is finite (i.e. P is finite) and is not locally finite, or conversely.

Evidently, if $\boldsymbol{x} \in \boldsymbol{X}$ and $m \in W_{\mathrm{L}(\boldsymbol{x})}$, then $\boldsymbol{x}^{[m]} \in \boldsymbol{X}$. If $x, y \in P$, $\boldsymbol{y} \in \boldsymbol{\Gamma} y$, then $x \oplus \boldsymbol{y} \in \boldsymbol{X}$ iff $y \in \Gamma x$.

Supposition. In the remainder of § 1, let Γ be a fixed graph, $\boldsymbol{X} = \boldsymbol{X}_\Gamma$.

1.14. Transformations and plays. Let $\sigma \in \mathrm{T}(\Gamma)$, $\boldsymbol{x} = (x_k;\ k \in W_l) \in \boldsymbol{X}$. We say that $\boldsymbol{x}$ *complies with* σ iff

$$k < l,\ x_k \in \operatorname{dom} \sigma \Rightarrow x_{k+1} = \sigma x_k .$$

It is easy to see that if σ is a *conservative* Γ-transformation and $x \in P_0 \cup \operatorname{dom} \sigma$, then there exists *exactly one* $\boldsymbol{x} \in \boldsymbol{\Gamma} x$ which complies with σ; this Γ-play will be denoted by $\boldsymbol{p}(x, \sigma)$.

Clearly, if $x \in P$, $y \in \Gamma x$, $\boldsymbol{y} \in \boldsymbol{\Gamma} y$, $\sigma \in \mathrm{T}(\Gamma)$, and if $\boldsymbol{y}$ complies with σ, then $x \oplus \boldsymbol{y}$ complies with σ iff $y = \sigma x$.

1.15. Plain sets of plays. $\boldsymbol{Y} \subseteq \boldsymbol{X}$ is said to be *plain* iff there holds: if $\boldsymbol{x} = (x_k;\ k \in W_r) \in \boldsymbol{Y}$, $\boldsymbol{y} = (y_k;\ k \in W_s) \in \boldsymbol{Y}$, $m < r$, $n < s$, $x_m = y_x$, then $x_{m+1} = y_{n+1}$.

It is easy to see that $\boldsymbol{Y} \subseteq \boldsymbol{X}$ is plain iff there exists $\sigma \in \mathrm{T_F}(\Gamma)$ such that any $\boldsymbol{x} \in \boldsymbol{Y}$ complies with σ.

1.16. Pay-off functions. A (general) *pay-off function* on Γ is given by a chain $\mathscr{V} = (V, \leqq)$ and a mapping $\boldsymbol{f} : \boldsymbol{X} \to V$ (but usually $\boldsymbol{f}$ is called "pay-off function", while $\mathscr{V}$ is to be given separately); the pay-off function is said to be *real-valued* iff the chain of real numbers may be taken as that $\mathscr{V}$.

1.17. Pseudolengths. A ***pseudolength*** on Γ is given by a chain $\mathscr{W}^* = (W^*, \leqq^*)$ having the greatest element (the latter will be denoted by ∞^*) and by a mapping $\mathrm{L}^* : \boldsymbol{X} \to W^*$ such that the following conditions are satisfied (for any $\boldsymbol{x} \in \boldsymbol{X}$, $x \in P$):

D(1) $\mathrm{L}(\boldsymbol{x}) = \infty \Rightarrow \mathrm{L}^*(\boldsymbol{x}) = \infty^*$

D(2) $y \in \Gamma x,\ \boldsymbol{y}_1, \boldsymbol{y}_2 \in \boldsymbol{\Gamma} y,\ \mathrm{L}^*(\boldsymbol{y}_1) \leqq^* \mathrm{L}^*(\boldsymbol{y}_2) \Rightarrow \mathrm{L}^*(x \oplus \boldsymbol{y}_1) \leqq^* \mathrm{L}^*(x \oplus \boldsymbol{y}_2)$

D(3) $y \in \Gamma x,\ \boldsymbol{y} \in \boldsymbol{\Gamma} y,\ \mathrm{L}^*(x \oplus \boldsymbol{y}) <^* \infty^* \Rightarrow \mathrm{L}^*(x \oplus \boldsymbol{y}) >^* \mathrm{L}^*(\boldsymbol{y})$;

of course, in such a case there holds

D′(2) $y \in \Gamma x,\ \boldsymbol{y}_1, \boldsymbol{y}_2 \in \boldsymbol{\Gamma} y,\ \mathrm{L}^*(\boldsymbol{y}_1) = \mathrm{L}^*(\boldsymbol{y}_2) \Rightarrow \mathrm{L}^*(x \oplus \boldsymbol{y}_1) = \mathrm{L}^*(x \oplus \boldsymbol{y}_2)$

D′(3) $y \in \Gamma x,\ \boldsymbol{y} \in \boldsymbol{\Gamma} y \Rightarrow \mathrm{L}^*(x \oplus \boldsymbol{y}) \geqq^* \mathrm{L}^*(\boldsymbol{y})$.

E.g., L (with $\mathscr{W}$) is a pseudolength. Further, the constant mapping of $\boldsymbol{X}$ into $\{\infty\}$ (with the one-element chain containing ∞) is a pseudolength; it will be called the *trivial pseudolength.* (We have not said "...onto $\{\infty\}$", as it may happen $\Gamma = \emptyset$, then $\boldsymbol{X} = \emptyset$, and the trivial pseudolength is the empty mapping.)

1.18. Qualitative pay-off functions. Let L^* with a chain $\mathscr{W}^*$ be a pseudolength on Γ. Under an L^*-***qualitative*** (or, more exactly speaking, $(\mathrm{L}^*, \mathscr{W}^*)$-qualitative) ***pay-off function*** (on Γ) we mean a real-valued pay-off function $\boldsymbol{f}^*$ on Γ such that the following conditions are satisfied (for any $\boldsymbol{x} \in \boldsymbol{X}$):

D(o) $$\boldsymbol{f}^*(\boldsymbol{x}) \in \{-1, 0, +1\}$$

D(i) $$\mathrm{L}(\boldsymbol{x}) = \infty \Rightarrow \boldsymbol{f}^*(\boldsymbol{x}) = 0$$

D(ii) $$\boldsymbol{f}^*(\boldsymbol{x}) = 0 \Rightarrow \mathrm{L}^*(\boldsymbol{x}) = \infty^*$$

D(iii) $$0 < \mathrm{L}(\boldsymbol{x}) \Rightarrow \boldsymbol{f}^*(\boldsymbol{x}) = \boldsymbol{f}^*(\boldsymbol{x}^{[1]});$$

of course, in such a case there holds (for $\boldsymbol{x} = (x_k;\, k \in W_l)$)

D′(iii) $$\mathrm{L}(\boldsymbol{x}) < \infty \Rightarrow \boldsymbol{f}^*(\boldsymbol{x}) = \boldsymbol{f}^*((x_{\mathrm{L}(\boldsymbol{x})}))\ (= \boldsymbol{f}^*(\boldsymbol{x}^{[\mathrm{L}(\boldsymbol{x})]})),$$

i.e., the pay-off for a finite play is determined by the terminal position of the play, while (cf. D(i)) any infinite play gives 0 as the pay-off.

E.g., the constant mapping of $\boldsymbol{X}$ into $\{0\}$ is an L^*-qualitative pay-off function.

1.19. Quasiqualitative pay-off functions. Let L^* with a chain $\mathscr{W}^*$ be a pseudolength on Γ, let $\boldsymbol{f}^*$ be an L^*-qualitative pay-off function. Let $\boldsymbol{f}$ with a chain $\mathscr{V}$ be a pay-off function on Γ. We say that $\boldsymbol{f}$ is an L^*-***quasiqualitative pay-off function complying***

with $\boldsymbol{f^*}$ iff the following conditions are satisfied (for any $\mathbf{x}, \mathbf{y}, \mathbf{y}_1, \mathbf{y}_2 \in \mathbf{X}$, $x \in P$; cf. § 1.0):

$$\mathrm{D}(\bar{1}) \qquad y \in \Gamma x,\ \mathbf{y}_1, \mathbf{y}_2 \in \boldsymbol{\Gamma} y, f(\mathbf{y}_1) \leqq f(\mathbf{y}_2) \Rightarrow f(x \oplus \mathbf{y}_1) \leqq f(x \oplus \mathbf{y}_2)$$

$$\mathrm{D}(\bar{2}) \qquad f^*(\mathbf{x}) < f^*(\mathbf{y}) \Rightarrow f(\mathbf{x}) \leqq f(\mathbf{y})$$

$$\mathrm{D}(\bar{3}) \qquad f^*(\mathbf{x}) = f^*(\mathbf{y}) = \begin{Bmatrix} +1 \\ -1 \end{Bmatrix}, \qquad \mathrm{L}^*(\mathbf{x}) < \mathrm{L}^*(\mathbf{y}) \Rightarrow f(\mathbf{x}) \begin{Bmatrix} \geqq \\ \leqq \end{Bmatrix} f(\mathbf{y}),$$

$$\mathrm{D}(\bar{4}) \qquad f^*(\mathbf{x}) = f^*(\mathbf{y}) \geqq 0,\ \ \mathrm{L}^*(\mathbf{x}) = \infty^* = \mathrm{L}^*(\mathbf{y}) \Rightarrow f(\mathbf{x}) = f(\mathbf{y});$$

of course, $\mathrm{D}(\bar{4})$ can be expressed equivalently in such a way:

$$\mathrm{D}(\overline{4.1}) \qquad f^*(\mathbf{x}) = f^*(\mathbf{y}) = 1,\ \ \mathrm{L}^*(\mathbf{x}) = \infty^* = \mathrm{L}^*(\mathbf{y}) \Rightarrow f(\mathbf{x}) = f(\mathbf{y}),$$

$$\mathrm{D}(\overline{4.2}) \qquad f^*(\mathbf{x}) = f^*(\mathbf{y}) = 0 \Rightarrow f(\mathbf{x}) = f(\mathbf{y})$$

(cf. D(o), D(ii)).

E.g., the function $\boldsymbol{f^*}$ itself (with the chain of real numbers) is an L*-quasiqualitative pay-off function complying with $\boldsymbol{f^*}$ (namely, then D(1) follows from D(iii)).

1.20. The interpretation (cf. §§ 1.17 – 19). Let L* with a chain $\mathscr{W}^*$ be a pseudolength, let $\boldsymbol{f^*}$ be an L*-qualitative pay-off function, let $\boldsymbol{f}$ be an L*-quasiqualitative pay-off function complying with $\boldsymbol{f^*}$.

L* can be interpreted as a certain *criterion of the continuance of plays*; ∞^* means the "*very long*" continuance. The conditions D(1) – D(3) are natural and their meanings are clear.

+1 [0; −1] means (as a value of $\boldsymbol{f^*}$) "*win*" ["*draw*"; "*loss*"]; therefore, *any infinite play is drawn* (under $\boldsymbol{f^*}$, see D(ii)), and the "*qualitative result*" *of any finite play* $\mathbf{x}$ *is equal to the natural evaluation* $\boldsymbol{f^*}((x))$ *of its terminal position* x (cf. Di(iii)). The condition D(ii) has a special character; *only "very long" plays may be drawn.*

Note that the important condition (of a certain *monotony* at one-move extensions of plays) expressed by D(2) for L* and by $\mathrm{D}(\bar{1})$ for f is not so strong as could be expected; cf. § 1.25.1.

The meanings of the other conditions are clear: $\mathrm{D}(\bar{2})$ expresses the *compliance of* $\boldsymbol{f}$ *with* $\boldsymbol{f^*}$; $\mathrm{D}(\bar{3})$ says that "*more rapidly*" (in the sense given by L*) *won* [*lost*] Γ-*plays are (nonstrongly) better* [*worse*]: $\mathrm{D}(\overline{4.1})$ says that *all "very long" won plays have the same value* (under $\boldsymbol{f}$); similarly, $\mathrm{D}(\overline{4.2})$ says that *all drawn plays* (they are "very long", see D(ii)) *have the same value.*

1.21. Lemma. *Let* Γ *be locally finite. Let* f *with a chain* $\mathscr{V}$ *be a pay-off function on* Γ. *Then the following statements are equivalent*:

(A) *There exists a pseudolength* L* (*on* Γ) *with a chain* $\mathscr{W}^*$ *and an* L*-*qualitative pay-off function* $\boldsymbol{f^*}$ (*on* Γ) *such that* $\boldsymbol{f}$ *is an* L*-*quasiqualitative pay-off function complying with* $\boldsymbol{f^*}$.

(B) *f is an* L^0*-quasiqualitative pay-off function complying with* f^0 *where* L^0 *is the trivial pseudolength on* Γ, *and* (*the* L^0*-qualitative pay-off function*) f^0 *is the mapping of* $\boldsymbol{X}$ *into* $\{-1\}$.

(C) *f satisfies the condition* $D(\bar{1})$.

Proof. Evidently, f^0 is an L^0-qualitative pay-off function. Of course, (B) $\Rightarrow$ (A), and (A) $\Rightarrow$ (C). If (C) is satisfied, then D(o)$-$(iii) and $D(\bar{2})-(\bar{4})$ with $L^* = L^0$, $f^* = f^0$ are satisfied in a trivial way, while $D(\bar{1})$ is satisfied by the supposition; hence, (B) holds. Therefore, (C) $\Rightarrow$ (B).

1.22. Lemma. *Let f be a real-valued pay-off function on* Γ. *Let the following conditions be satisfied* (*for any* $\boldsymbol{x} \in \boldsymbol{X}$, $x \in P$);

$$(\alpha) \qquad L(\boldsymbol{x}) = \infty \Rightarrow f(\boldsymbol{x}) = 0$$

$$(\beta) \qquad y \in \Gamma x,\ \boldsymbol{y}_1, \boldsymbol{y}_2 \in \boldsymbol{\Gamma} y,\ |f(\boldsymbol{y}_1)| \leqq |f(\boldsymbol{y}_2)| \Rightarrow |f(x \oplus \boldsymbol{y}_1)| \leqq |f(x \oplus \boldsymbol{y}_2)|$$

$$(\gamma) \qquad L(\boldsymbol{x}) > 0, \qquad f(\boldsymbol{x}^{[1]}) \begin{Bmatrix} \geqq \\ \leqq \end{Bmatrix} 0 \Rightarrow f(\boldsymbol{x}^{[1]}) \begin{Bmatrix} \geqq \\ \leqq \end{Bmatrix} f(\boldsymbol{x}) \begin{Bmatrix} \geqq \\ \leqq \end{Bmatrix} 0.$$

Let $\mathscr{W}^* = (W^*, \leqq^*)$ *be the chain with* $W^* = [0, \infty] \times [0, \infty]$ *and with the lexicographic ordering as* $\leqq^*$, *let* $\infty^* = (\infty, \infty)$ (*the greatest element of* $\mathscr{W}^*$).

Let mappings f^* *and* L^* *of* $\boldsymbol{X}$ *be defined in the following way:*

$$\boldsymbol{f}^*(\boldsymbol{x}) = \begin{Bmatrix} \operatorname{sgn} \boldsymbol{f}(\boldsymbol{x}^{[L(\boldsymbol{x})]}) \\ 0 \end{Bmatrix} \quad \text{if} \quad L(\boldsymbol{x}) \begin{Bmatrix} < \\ = \end{Bmatrix} \infty,$$

$$L^*(\boldsymbol{x}) = \begin{Bmatrix} \left(\dfrac{1}{|\boldsymbol{f}(\boldsymbol{x})|}, L(\boldsymbol{x}) \right) \\ \infty^* \end{Bmatrix} \quad \text{if} \quad f(\boldsymbol{x}) \begin{Bmatrix} \neq \\ = \end{Bmatrix} 0,$$

(*for any* $\boldsymbol{x} \in \boldsymbol{X}$), *where* $\operatorname{sgn} a = 1\ [0\colon -1]$ *if* $a > 0$ $[a = 0\colon a < 0]$.

Then L^* *with* $\mathscr{W}^*$ *is a pseudolength* (*on* Γ), f^* *is an* L^**-qualitative pay-off function, and f is an* L^**-quasiqualitative pay-off function complying with* f^*.

Proof. There holds;

$$(\beta') \quad [y \in \Gamma x,\ \boldsymbol{y}_1, \boldsymbol{y}_2 \in \boldsymbol{\Gamma} y, f(\boldsymbol{y}_1) \leqq f(\boldsymbol{y}_2) \Rightarrow f(x \oplus \boldsymbol{y}_1) \leqq f(x \oplus \boldsymbol{y}_2)]\ (\equiv D(\bar{1}))$$

$$(\gamma') \qquad L(\boldsymbol{x}) > 0 \Rightarrow |f(\boldsymbol{x}^{[1]})| \geqq |f(\boldsymbol{x})|$$

$$(\gamma'') \qquad |f(\boldsymbol{x})| \begin{Bmatrix} \leqq |f(\boldsymbol{x}^{[L(\boldsymbol{x})]})| \\ = 0 \end{Bmatrix} \quad \text{if} \quad L(\boldsymbol{x}) \begin{Bmatrix} < \\ = \end{Bmatrix} \infty,$$

$$(\gamma''') \qquad f^*(\boldsymbol{x}) = \operatorname{sgn} f(\boldsymbol{x})$$

[see: (β) and (γ); (γ); (γ') and (α); (γ) and (α) (respectively)].

The satisfaction of DD(o), (i), (iii), $(\bar{4})$ follows from the definitions given by the lemma. The satisfaction of DD(1), (2), (3), (ii), $(\bar{1})$, $(\bar{2})$, $(\bar{3})$ follows from: (α); (β); (γ'); (γ''); (β'); (γ'''); (γ'') (respectively). Thus, the lemma is proved.

1.23. Lemma. *Let* $0 \neq T \subseteq [0, \infty)$, *let* $\varphi : T \times \{0, 1, 2, \ldots\} \to [0, \infty)$ *be such that for any* $t, t_1, t_2 \in T$, $l_1, l_2 \in \{0, 1, 2, \ldots\}$ *there holds*;

(a.1) $$t_1 < t_2 \Rightarrow \varphi(t_1, l_1) < \varphi(t_2, l_2)$$

(a.2) $$l_1 < l_2 \Rightarrow \varphi(t, l_1) \geqq \varphi(t, l_2)$$

(b) $$\varphi(t, l_1) = \varphi(t, l_1 + 1) \Rightarrow \varphi(t, l_2) = \varphi(t, l_1) \quad \textit{for each } l_2 \geqq l_1.$$

Let h be a (real-valued) function on P_0 *such that*

$$\{|\, h(x)\,|;\, x \in P_0\} \subseteq T.$$

Let f be the real-valued pay-off function (on Γ*) such that for an arbitrary* $\mathbf{x} = (x_k;\, k \in W_l) \in \mathbf{X}$ *there holds*

$$f(\mathbf{x}) = \begin{Bmatrix} \varphi(|\, h(x_l)\,|, l) \,.\, \operatorname{sgn} h(x_l) \\ 0 \end{Bmatrix} \quad \textit{if} \quad l \begin{Bmatrix} < \\ = \end{Bmatrix} \infty.$$

Then the conditions (α), (β), (γ) (§ 1.22) *are satisfied (by this* f*).*

Proof. (α) follows immediately from the definition of f, (γ) is to be derived by means of (a.2). Let $x \in Z$, $\mathbf{x}, \mathbf{y} \in \Gamma x$, $\mathbf{x} = (x_k;\, k \in W_{l_1})$, $\mathbf{y} = (y_k;\, k \in W_{l_2})$, $|f(\mathbf{x}^{[1]})| \leqq |f(\mathbf{y}^{[1]})|$. If $l_1 = \infty$, then $|f(\mathbf{x})| = 0 \leqq |f(\mathbf{y})|$. If $l_2 = \infty$, then $\mathrm{L}(\mathbf{y}^{[1]}) = \infty$, $|f(\mathbf{x})| \leqq |f(\mathbf{x}^{[1]})| \leqq |f(\mathbf{y}^{[1]})| = 0$ (see (γ), (α)), i.e. $|f(\mathbf{x})| = 0 = |f(\mathbf{y})|$. Let $l_1 < \infty$, $l_2 < \infty$. We put $r_1 = |\, h(x_{l_1})\,|$, $r_2 = |\, h(y_{l_2})\,|$. Thus, $|f(\mathbf{x})| = \varphi(r_1, l_1)$, $|f(\mathbf{y})| = \varphi(r_2, l_2)$, $\varphi(r_1, l_1 - 1) = |f(\mathbf{x})^{[1]}| \leqq |f(\mathbf{y}^{[1]})| = \varphi(r_2, l_2 - 1)$. Hence, $r_1 \leqq r_2$ (see (a.1)). If $r_1 < r_2$, then (cf. above and again (a.1)) $|f(\mathbf{x})| < |f(\mathbf{y})|$. Let $r_1 = r_2 = r$. If $l_1 \geqq l_2$, then $|f(\mathbf{x})| = \varphi(r, l_1) \leqq \varphi(r, l_2) = |f(\mathbf{y})|$ (see (a.2)). Now, let $l_1 < l_2$. Then $l_1 - 1 < l_1 \leqq l_2 - 1$, and $\varphi(r, l_1 - 1) \geqq \varphi(r, l_1) \geqq \varphi(r, l_2 - 1) \geqq \varphi(r, l_1 - 1)$ (cf. above and (a.2)), hence $\varphi(r, l_1 - 1) = \varphi(r, l_1)$, thus (cf. (b)) $|f(\mathbf{x})| = \varphi(r, l_1) = \varphi(r, l_2) = |f(\mathbf{y})|$. We have proved that $|f(\mathbf{x})| \leqq |f(\mathbf{y})|$ in any case. Consequently, there holds even

(β') $$\mathbf{x}_1, \mathbf{x}_2 \in \Gamma x,\ |f(\mathbf{x}_1^{[1]})| \leqq |f(\mathbf{x}_1^{[1]})| \Rightarrow |f(\mathbf{x}_1)| \leqq |f(\mathbf{x}_2)|,$$

which, of course, implies (β).

1.24. Important particular cases of the situations considered in §§ 1.22 – 1.23.

1.24.1. (Cf. § 1.21.) Let Γ be *locally finite*. Let $\mathscr{V} = (V, \leqq)$ be a chain, let

$$F : \{(x, y, v);\, x \in P,\, y \in \Gamma x,\, v \in V\} \to V,$$

let (for any (x, y, v_1), $(x, y, v_2) \in \operatorname{dom} F$)

(*) $$v_1 \leqq v_2 \Rightarrow F(x, y, v_1) \leqq F(x, y, v_2).$$

Let

$$h_0 : P_0 \to V.$$

It is easy to prove (by induction) that there exists exactly one mapping $f: \mathbf{X} \to V$ (it, together with $\mathscr{V}$, will be considered as a pay-off function on Γ) such that

$$f((x)) = h_0(x) \quad \text{if} \quad x \in P_0,$$
$$f(x \oplus \mathbf{y}) = F(x, y, F(\mathbf{y})) \quad \text{if} \quad x \in P, y \in \Gamma x, \mathbf{y} \in \boldsymbol{\Gamma} y.$$

Evidently, *f satisfies the condition* $\mathrm{D}(\bar{1})$.

In particular, if $\circ$ is a binary operation on V such that $(V, \circ, \leqq)$ is a linearly ordered abelian semigroup (hence, if $v_1, v_2, v_3, v_4 \in V$, then $v_1 \leqq v_2 \wedge v_3 \leqq v_4$ implies $v_1 \circ v_3 \leqq v_2 \circ v_4$), and $h : P \to V$ (the so-called *evaluation function*), then, if we choose F and h_0 such that

$$F(x, y, v) = h(x) \circ v \text{ for any } x \in Z, y \in \Gamma x, v \in V,$$
$$h_0 = h \mid P_0,$$

we have a particular case of the above introduced situation, and

$$f(\mathbf{x}) = h(x_0) \circ h(x_1) \circ \ldots \circ h(x_l)$$

for any $\mathbf{x} = (x_k ; k \in W_l) \in \mathbf{X}$. E.g., it is possible to choose the set of real numbers as V, the natural ordering as $\leqq$, and $\circ = \max$ or $\circ = \min$ (where "max" and "min" are considered as binary operations) or $\circ = +$; clearly, the case $\circ = \max$ [$\circ = \min$] gives exactly the "*active*" ["*passive*"] *pay-off functions* (on the locally finite graph Γ) in the sense introduced by Berge (cf., e.g., [2], ch. I. § 2).

It is easy to generalize somewhat the case with ("*vertex-*") *evaluation*, e.g. by introducing some "*mixed*" *evaluation* (given by some *end-vertex evaluation* $h_0 : P_0 \to \to V$ and an "*edge evaluation*" $h_E : \{(x, y); x \in P, y \in \Gamma x\} \to V$).

1.24.2. (Cf. § 1.23.) Let g be a real-valued pay-off function on Γ, let $g(\mathbf{x}) = 0$ *if* $\mathrm{L}(\mathbf{x}) = \infty$, let *$g(\mathbf{x})$ depend only on the terminal position of $\mathbf{x}$ if* $\mathrm{L}(\mathbf{x}) < \infty$. Choosing $T = [0, \infty)$, and (see § 1.23) $\varphi(t, r) = t$ for each $t \in T$, $r = 0, 1, 2, \ldots$, $h(x) = g((x))$ for each $x \in P_0$, we see that $g = f$ where f is given by § 1.23 to the above described T, φ, h.

1.24.3. (Cf. § 1.23.) A real-valued pay-off function f^* is said to be ***qualitative*** iff it satisfies the conditions DD(o) (i), (iii); the interpretation is the same as that (of f^*) mentioned in § 1.20. If we wish to express such an interest under which (at a given qualitative pay-off function f^*) more rapidly won [lost] plays are considered as better [worse], we can choose, e.g., $T = \{0, 1\}$, $\varphi(t, r) = \dfrac{t}{1 + r}$ $(t \in T, r = 0, 1, 2, \ldots)$ $h(x) = f^*((x))$ (see § 1.23), and then the pay-off function f given by § 1.23 expresses the above mentioned interest.

1.25. D($\bar{1}$), D($\bar{4}$), and some stronger conditions. Let f with a chain $\mathscr{V} = (V, \leqq)$ be a pay-off function on Γ.

1.25.1. We introduce condition

D($\bar{1}^+$) $y_k \in \Gamma x$, $\mathbf{y}_k \in \Gamma y_k (k = 1, 2,), f(\mathbf{y}_1) \leqq f(\mathbf{y}_2) \Rightarrow f(x \oplus \mathbf{y}_1) \leqq f(x \oplus \mathbf{y}_2)$,

which is, evidently, stronger than D($\bar{1}$).

E.g., the pay-off functions f in the special case in § 1.24.1 (that in which f is given by h, at $(V, \circ, \leqq)$) and in §§ 1.24.2-3 satisfy even D($\bar{1}^+$).

1.25.2. Further, we introduce condition

D($\bar{4}^+$) $\qquad f^*(\mathbf{x}) = f^*(\mathbf{y}),\ \ \mathrm{L}^*(\mathbf{x}) = \infty^* = \mathrm{L}^*(\mathbf{y}) \Rightarrow f(\mathbf{x}) = f(\mathbf{y})$,

which is, evidenty stronger than D($\bar{4}$).

E.g., the pay-off function f considered in § 1.22 (which, of course, involves the cases considered in §§ 1.23, 1.24.2-3), with L* and f^* introduced there, satisfies D($\bar{4}^+$) (as, even, $\mathrm{L}^*(\mathbf{x}) = \infty^*$ implies $f(\mathbf{x}) = 0$, in § 1.22).

1.26. On the linear transformations at real-valued pay-off functions.

Let f be a real-valued pay-off function on Γ, let $\lambda \neq 0$ and c be real numbers, let $g = c + \lambda f (= (c + \lambda f(\mathbf{x}); \mathbf{x} \in \mathbf{X}))$. There is a number of trivial but very useful auxiliary propositions on transfering the properties of f to g; we shall need the following ones:

a) If $\lambda > 0$ [$\lambda < 0$], then f and g express the same preference [antagonistic preferences].

b) If f satisfies (α), (β), (γ) (§ 1.22) and $c = 0$, then g satisfies (α), (β), (γ). If f is given by some φ and h in the sense described in § 1.23, then g is given by φ and h . . sgn λ.

c) Let L* (with some $\mathscr{W}^*$) be a pseudolength (on Γ), let f^* be an L*-qualitative pay-off function. Let $g^* = f^*$. sgn λ. Then g^* is an L*-qualitative pay-off function. If f is an L*-quasiqualitative pay-off function complying with f^*, and if

either $\lambda > 0$,
or $\lambda < 0$ and f satisfies D($\bar{4}^+$),

then g is an L*-quasiqualitative pay-off function complying with g^*.

Note that linear transformations can be applied also in such a way: real numbers $\lambda_k \neq 0$, $c_k (k = 1, 2)$ are given, and pay-off functions $f_k = c_k + \lambda_k f$ are considered; of course, then $f_{3-k} = \left(c_{3-k} - \dfrac{c_k \lambda_{3-k}}{\lambda_k}\right) + \dfrac{\lambda_{3-k}}{\lambda_k} f_k$ for $k = 1, 2$.

§ 2. GAMES WITH PERFECT INFORMATION

2.1. Partitions. We say that $(P(j); j \in J)$ is a partition of a set P iff (J is a set, $P(j)$ are sets, and) $\bigcup_{j \in J} P(j) = P$.

Convention. If a partition $(P(j); j \in J)$ of a set P is given, we denote, for each $x \in P$, by $j(x)$ that element of J for which

$$x \in P(j(x))$$

(of course, there exists exactly one such $j(x)$).

2.2. Definition. Under a ***game with perfect information*** (we write only *g. p. i.*, too) we mean a 4-tuple

$$\mathcal{G} = (\Gamma, (P(j); j \in J), (\mathcal{V}_j = (V_j, \leqq_j); j \in J), (f_j; j \in J))$$

such that Γ is an oriented graph, $(P(j); j \in J)$ is a partition of $P = \operatorname{dom} \Gamma$, $\mathcal{V}_j$ is a chain, and $f_j : \mathbf{X}_\Gamma \to V_j$ (i.e., f_j with $\mathcal{V}_j$ is a pay-off function on Γ) for each $j \in J$. *$\mathcal{G}$ is said to have a property introduced for graphs* (cf. § 1.13) *iff Γ has this property.*

2.3. Definition; remarks on the interpretation. Let

$$\mathcal{G} = (\Gamma, (P(j); j \in J), (\mathcal{V}_j = (V_j, \leqq_j); j \in J), (f_j; j \in J))$$

be a g.p.i. Γ is said to be the *graph* of $\mathcal{G}$, $(\Gamma, (P(j); j \in J))$ is called the *game structure* given by $\mathcal{G}$, and

$$(\Gamma, (P(j); j \in J); (\mathring{\leqq}_{(f_j, \mathcal{V}_j)}; j \in J))$$

is called the *preference form* of $\mathcal{G}$.

The interpretation of Γ, P, P_0, Z, $\mathbf{X} = \mathbf{X}_\Gamma$ (cf. convention 1.7) *is given by* § 1.8. J is the *set of players* (if $J = 0$, then necessarily $P = \emptyset$). f_j with $\mathcal{V}_j (j \in J)$ is the pay-off function of player j. $P(j)$ is the *set of all the positions at which it is the player j's turn to move* (of course, actually j moves only at the positions from $Z \cap P(j)$). A *play* (starting from some position $x_0 \in P$) is performed in the natural way (cf. § 1.8); if x is a momentary nonfinal position of the play, then the *moving* (i.e. the choice of an element of Γx (as the next position) performing) *player* is $j(x)$.

The preference form of a g.p.i. is not so natural as the formally defined g.p.i. (§ 2.2) itself, but, on the other hand, *all the notions and properties introduced in this paper for games with perfect information could be formulated in terms of the preference forms* [see, in particular, § 2.10; of course, some notions (e.g., plain strategy) can be introduced in terms of game structures or even in terms of graphs (e.g. plays)].

Note that any triple

$$(\Gamma, (P(j); j \in J), (\mathring{\leqq}_j; j \in J))$$

where Γ is an oriented graph, $(P(j); j \in J)$ is a partition of $P = \operatorname{dom} \Gamma$, and $\overset{\circ}{\leqq}_j (j \in J)$ are preference relations on $\mathbf{X}_\Gamma$ – such an object will be called a *g.p.i. in preference form* – is the preference form of a suitable g.p.i. (e.g., of

$$(\Gamma, (P(j); j \in J), (\mathscr{V}_{\overset{\circ}{\leqq}_j}; j \in J), (f_{\overset{\circ}{\leqq}_j}; j \in J)),$$

cf. § 1.2). Therefore, for the aims of this work *the concept of g.p.i. in preference form could be given as the basic notion*, instead of the notion of g.p.i.

Note that also the notion of g.p.i. in preference form is somewhat redundant from the formal point of view (as the graph Γ can be omitted; namely, if $J = \emptyset$, then $P = \emptyset = \Gamma$, if $J \neq \emptyset$, then any $\overset{\circ}{\leqq}_j$ determines $\mathbf{X}_\Gamma$ and, hence, Γ uniquely).

2.4. Bergean games with perfect information.

2.4.0. Any *Bergean game with perfect information* (that introduced by his general definition in [2], Ch. I, § 2) can be defined formally as

$$(\Gamma, n, (P_1, \ldots, P_n), (\overset{\circ}{\leqq}_1, \ldots, \overset{\circ}{\leqq}_n), N^+, N^-)$$

where Γ is a graph, n is a positive integer, $N^+ \cup N^- = \{1, \ldots, n\}$ $(= N)$, $N^+ \cap N^- = \emptyset$, $(P_j; j \in J)$ is a partition of $P = \operatorname{dom} \Gamma$, and $\overset{\circ}{\leqq}_j (j \in J)$ are preference relations *on* P; moreover, Berge supposes that $P(j(x)) \cap \Gamma x = \emptyset$ for each $x \in P$. Here N is the set of *players* (or of the numbers of players), elements of $N^+[N^-]$ are called *active* [*passive*] players. For any $j \in N$, there exists exactly one $\varepsilon \in \{+, -\}$ such that $j \in N^\varepsilon$; we shall denote this ε by $\varepsilon(j)$.

2.4.1. It is easy to see that the adequate description of the interests of the players in a Bergean g.p.i. (in the sense of § 2.4.0) can be given, in terms of preference relations on $\mathbf{X}_\Gamma$, in the following way: if $j \in N$, then the player j's preference relation *on* $\mathbf{X} = \mathbf{X}_\Gamma$ equals $\overset{\circ}{\leqq}_{(g, \overset{\circ}{\mathscr{V}}_j)}$ where $g = (\{x_k;\ k \in W_l\};\ (x_k;\ k \in W_l) \in \mathbf{X})$ $(: \mathbf{X} \to \to (\exp P \setminus \{\emptyset\}))$, and $\overset{\circ}{\mathscr{V}}_j = (\mathbf{X}, \overset{\circ}{\leqq}{}_j^{\varepsilon(j)})$ (§ 1.5). In such a way we have transformed any Bergean g.p.i. to a g.p.i. in preference form, and the latter could be "naturally represented" by a g.p.i. (cf. § 2.3). In this sense and from our point of view (see § 2.3), *Bergean gs. p.i. may be considered as a particular case of gs. p.i. introduced in* § 2.2.

2.4.2. In the most usual case, *the preference relations* $\overset{\circ}{\leqq}_j$ *on* P of a Bergean g.p.i. (§ 2.4.0) *are given by real-valued "evaluation functions"* f_j on P, i.e. $\overset{\circ}{\leqq}_j = \overset{\circ}{\leqq}_{(f_j, \mathscr{R})}$ (cf. § 1.2) where $\mathscr{R}$ is the chain of real numbers. It is easy to see that in this case the *natural pay-off function* $f_j = f_j^{\varepsilon(j)} \,.\, g$ [where $f_j^{\varepsilon(j)}$ is defined by § 1.4 to $\mathscr{V} = \mathscr{R}^* = ([-\infty, +\infty], \leqq)$ ($\leqq$ is the natural ordering), $X = \mathbf{X}$, and $f = f_j$] with $\mathscr{R}^*$ gives exactly the player j's preference relation (on $\mathbf{X}$) described in § 2.4.1, i.e., $\overset{\circ}{\leqq}_{(g,j)} = \overset{\circ}{\leqq}_{(f_j, \mathscr{R}^*)}$.

2.5. Of course, also the usually considered *games* with perfect information, with real-valued pay-off functions, and without chance moves *on trees* are involved in our definition (§ 2.2); the most usual case (the tree is finite, and the pay-offs are determined by the terminal positions) can be considered as a special case of Bergean games with perfect information (each player may be considered as active or as passive, and then his evaluation of nonfinal positions is less than or greater than (respectively) any evaluation of final positions). (Here we consider trees as special oriented graphs, and then the root of any tree is determined by the tree itself; thus, the initial position of a game of the mentioned kind need not be presented in the formal definition of the game.)

Supposition. In the remainder of § 2, let

$$\mathscr{G} = (\Gamma, (P(j); j \in J), (\mathscr{V}_j = (V_j, \leqq_j); j \in J), (\boldsymbol{f}_j; j \in J))$$

be a game with perfect information. (Cf. §§ 1.0, 1.7, 2.1.)

2.6. Convention, definition. $\boldsymbol{p}$ will have the meaning introduced in § 1.14. We shall write

$$Z(j) = Z \cap P(j),$$

$$\mathring{S}(j) = \bigtimes_{z \in Z(j)} \Gamma z, \qquad \mathring{S} = \bigtimes_{j \in J} \mathring{S}(j).$$

Elements of $\mathring{S}(j)$ are called the *plain strategies of j* (in $\mathscr{G}$). Elements of $\mathring{S}$ are said to be the *plain strategic situations.*

2.7. Remark. There exists a *natural bijection of* $\mathring{S}$ *onto* $\mathrm{T}_{\mathrm{F}}(\Gamma)$, namely $(\sigma_j; j \in J) \mapsto \bigcup_{j \in J} \sigma_j$ (for each $(\sigma_j; j \in J) \in \mathring{S}$). (If $J = \emptyset$, then $P = \emptyset$, $\Gamma = \emptyset$, $\mathring{S} = \{\emptyset\} = \mathrm{T}_{\mathrm{F}}(\Gamma)$.)

2.8. In accordance to the well-known fundamental definitions (cf., e.g., [10], Ch. VI. 2), the definition of equilibrium point consisting (only) of plain strategies (or, as we say, of *plain equilibrium point*) *in some* $x^0 \in P$ (this position is considered as an initial position) is to be formulated in such a way: $\sigma = (\sigma_j; j \in J) \in \mathring{S}$ is a plain equilibrium point (of $\mathscr{G}$) in x° iff for each $j_0 \in J$ and each $\boldsymbol{x} \in \boldsymbol{\Gamma} x^\circ$ complying with σ_j for any $j \in J \setminus \{j_0\}$ there holds $\boldsymbol{f}_{j_0}(\boldsymbol{x}) \leqq_{j_0} \boldsymbol{f}_{j_0}(\boldsymbol{y})$ where $\boldsymbol{y}$ is that element of $\boldsymbol{\Gamma} x^0$ which complies with σ_j for each $j \in J$; note that *it may happen that that* $\boldsymbol{x}$ *does not comply with any plain strategy of the player j.*

Using the 1-1 correspondence among the elements of S and those of $\mathrm{T}_{\mathrm{F}}(\Gamma)$ (§ 2.7), we can give the direct definition of plain absolute equilibrium point (cf. § 0) in terms of full Γ-transformations:

2.9. Definition. Under a ***plain absolute equilibrium point*** of $\mathscr{G}$ we mean $\sigma \in \mathrm{T}_{\mathrm{F}}(\Gamma)$ such that for each $x \in P$, $j \in J$, and for any $\boldsymbol{x} \in \boldsymbol{\Gamma} x$ complying with $\sigma \mid (Z \setminus Z(j))$ there holds

$$\boldsymbol{f}_j(\boldsymbol{x}) \leqq_j \boldsymbol{f}_j(\boldsymbol{p}(x, \sigma)).$$

2.10. Remark. Of course, in the situation from § 2.9 $\boldsymbol{f}_j(\mathbf{x}) \leqq_j \boldsymbol{f}_j(\mathbf{p}(x, \sigma))$ if and only if $\mathbf{x} \mathrel{\overset{\circ}{\leqq}}_{(f_j, \mathscr{V}_j)} \mathbf{p}(x, \sigma)$; therefore, *the notion of plain absolute equilibrium point could be formulated in terms of preference forms of gs. p.i.*

2.11. Definition, remark. The game $\mathscr{G}$ is called ***antagonistic*** iff card $J = 2$ and, for $\{j_1, j_2\} = J$, the pay-off functions $\boldsymbol{f}_{j_1}, \boldsymbol{f}_{j_2}$ (with $\mathscr{V}_{j_1}, \mathscr{V}_{j_2}$) express antagonistic preferences.

E.g., if $J = \{j_1, j_2\}$, $j_1 \neq j_2$, and if there exist a real-valued pay-off function $\boldsymbol{f}$ (on Γ) and real numbers λ_k, $c_k (k = 1, 2)$ such that $\boldsymbol{f}_{j_k} = c_k + \lambda_k \boldsymbol{f} (k = 1, 2)$ and $\lambda_1 \lambda_2 < 0$, then $\mathscr{G}$ is antagonistic. (Cf. § 1.2.6.)

If $\mathscr{G}$ is antagonistic, then we also say "... *saddle point*" instead of "... equilibrium point".

§ 3. THE MAIN THEOREM. PARTICULAR CASES

(See the conventions in §§ 1.0, 1.7, 2.1, 2.6.)

3.0. The main theorem. *Let*

$$\mathscr{G} = (\Gamma, (P(j); j \in J), (\mathscr{V}_j = (V_j, \leqq_j); j \in J), (\boldsymbol{f}_j; j \in J))$$

be a game with perfect information.

Let there exist a pseudolength L* *(on* Γ*) with a chain* $\mathscr{W}^* = (W^*, \leqq^*)$, *and, for each* $j \in J$, *an* L*-*qualitative pay-off function* $\boldsymbol{f}_j^*$ *such that the following statements* (0), (*A*), (*B*) *are satisfied*:

(0) *For each* $j \in J$, $\boldsymbol{f}_j$ *(with* $\mathscr{V}_j$*) is an* L*-*quasiqualitative pay-off function complying with* $\boldsymbol{f}_j^*$.

(A):

(A.1) $\{L^*((x)); x \in P_0\}$ *is well-ordered (in* $\mathscr{W}^*$*).*

(A.2) *Let* $\nabla \subseteq Z$, $(y(z);\ z \in \nabla) \in \bigtimes_{z \in \nabla} (\Gamma z \setminus \nabla)$, $(\mathbf{y}(z);\ z \in \nabla) \in \bigtimes_{z \in \nabla} \Gamma y(z)$; *let set* $\{\mathbf{y}(z); z \in \nabla\}$ *be plain, and let each* $\mathbf{y}(z)$ $(z \in \nabla)$ *pass in* $P \setminus \nabla$. *Then* $\{L^*(z \oplus \mathbf{y}(z)); z \in \nabla\}$ *is well-ordered (in* $\mathscr{W}^*$*).*

(B) *Let* $z \in Z$, $(\mathbf{y}(z); y \in \Gamma z) \in \bigtimes_{y \in \Gamma z} \Gamma y$, *and let set* $\{\mathbf{y}(y); y \in \Gamma z\}$ *be plain. Then*

(B/1) $\{\boldsymbol{f}_{j(z)}(z \oplus \mathbf{y}(y)); y \in \Gamma z\}$ *is inversely well-ordered in* $\mathscr{V}_{j(z)}$;

(B/2) *if* $\{\boldsymbol{f}_{j(z)}(\mathbf{y}(y)); y \in \Gamma z\} \left\{\begin{smallmatrix} = \\ \neq \end{smallmatrix}\right\} \{-1\}$, *then set* $\{L^*(z \oplus \mathbf{y}(y)); y \in \Gamma z\}$ *is* $\left\{\begin{matrix} \textit{inversely well-ordered} \\ \textit{well-ordered} \end{matrix}\right\}$ *(in* $\mathscr{W}^*$*).*

Then the game $\mathscr{G}$ *has a plain absolute equilibrium point.*

3.1. Remark. The proof is presented in § 4. The idea of the construction of a plain absolute equilibrium point of the game $\mathscr{G}$ is based mainly on the condition (0);

each of the conditions DD (1) – (3), (o) – (iii), $(\bar{1})$ – $(\bar{4})$ *is used essentially in the proof.* The realization of this idea necessitates certain auxiliary "technical" conditions; we have chosen the conditions (A) and (B). This choice seems to be the most useful; our aim is not the full utilization of such conditions, we only wish to obtain immediately a number of important particular cases from a common result, and the theorem serves well for this aim, as we show in the following.

3.2. Theorem. *Let*

$$\mathscr{G} \doteq (\Gamma, (P(j);\ j \in J), (\mathscr{V}_j = (V_j, \leqq_j); j \in J), (f_j;\ j \in J))$$

be a game with perfect information.

Let L^* *be a pseudolength* (*on* Γ) *with a chain* $\mathscr{W}^* = (W^*, \leqq^*)$, *and, for each* $j \in J$, *let* f_j^* *be an* L^**-qualitative pay-off function such that* f_j (*with* $\mathscr{V}_j$) *is an* L^**-quasiqualitative pay-off function complying with* f_j^*.

Let there occur at least one of the following three cases:

(I) *P is finite.*
(II) *im* L^* *is well-ordered* (*in* $\mathscr{W}^*$), *and* Γx *is finite for each* $x \in P$.
(III) *im* L^* *is finite, im* f_j *is inversely well-ordered for each* $j \in J$.

Then the conditions (*A*) *and* (*B*) *from the main theorem are satisfied, and* (*therefore*) *the game* $\mathscr{G}$ *has a plain absolute equilibrium point.*

(The proof is trivial.)

3.3. Remark. Of course, at applying the theorem, further trivial or simple propositions could be used ($\mathbf{X}$ is finite $\Rightarrow$ P is finite; $\mathscr{W}^*$ is finite $\Rightarrow$ im L^* is finite, etc.; see § 1.1). The case (II) is involved, e.g., in the following *case* (II^+), and the latter *is sufficient* (*in the situation from* § 3.2) *for the satisfaction of* (*A*) *and* (*B*), too:

(II^+) im L^* *is well-ordered* (*in* $\mathscr{W}^*$), *and for each* $z \in Z$ *there exists a set* $Y \subseteq \Gamma z$ *such that sets* $\{f_{j(z)}(z \oplus \mathbf{y});\ y \in Y, \mathbf{y} \in \boldsymbol{\Gamma} y\}$ *and* $\{L^*(z \oplus \mathbf{y});\ y \in Y, \mathbf{y} \in \boldsymbol{\Gamma} y\}$ *are inversely well-ordered* (*in* $\mathscr{V}_{j(z)}$ *and* $\mathscr{W}^*$, *respectively*), *and* $\Gamma z \setminus Y$ *is finite.*
(Namely, $\{L^*(z \oplus \mathbf{y});\ y \in Y, \mathbf{y} \in \boldsymbol{\Gamma} y\}$ is finite if it is inversely-well ordered, as im L^* is well-ordered (cf. § 1.1); the case (II) occurs if the choice $Y = \emptyset$ is admissible for each $z \in Z$.)

3.4. Theorem. *Let*

$$\mathscr{G} = (\Gamma, (P(j); j \in J), (\mathscr{V}_j = (V_j, \leqq_j);\ j \in J), (f_j;\ j \in J))$$

be a game with perfect information.

If $\mathscr{G}$ *is locally finite and, for each* $j \in J$, f_j *satisfies the condition* $D(\bar{1})$ *and* im f_j *is inversely well-ordered* (*in* $\mathscr{V}_j$), *then* $\mathscr{G}$ *has a plain absolute equilibrium point.*

(This follows immediately from §§ 1.21 (cf. (C) $\Rightarrow$ (B)) and 3.2 (cf. case (III); namely, im $\mathrm{L}^\circ \subseteq \{\infty\}$).)

3.5. Remark. Important corollaries of § 3.4 can be obtained by means of § 1.24.1, which presents a general construction of pay-off functions (on locally finite graphs) satisfying $\mathrm{D}(\bar{1})$. In particular, there holds:

The Berge variant of the Zermelo – von Neumann theorem (cf. §§ 0, 3.6). *Any locally finite Bergean game with perfect information and finite-valued evaluation functions* (§§ 2.4.2, 2.4.0) *has a plain absolute equilibrium point.*

[Namely, in the considered case plain absolute equilibrium points can be defined in terms of the natural pay-off functions (cf. §§ 2.10, 2.4.1, 2.4.2), but, evidently, the latter can be expressed in the way mentioned in § 2.4.1 (and, hence, they satisfy $\mathrm{D}(\bar{1})$), and they are finite-valued. Thus, the proposition follows from § 3.4.]

3.6. Remark, definition. Under a ***weak plain absolute equilibrium point*** of a game

$$\mathcal{G} = (\Gamma, (P(j); j \in J), (\mathcal{V}_j = (V_j, \leqq_j); j \in J), (f_j; j \in J))$$

with perfect information we mean $\sigma \in \mathrm{T_F}(\Gamma)$ such that for any $j \in J$ and $\sigma' \in \mathrm{T_F}(\Gamma)$

$$\sigma' \mid (Z' \setminus Z(j)) = \sigma \mid (Z \setminus Z(j)) \Rightarrow f_j(\mathfrak{p}(x, \sigma')) \leqq_j f_j(\mathfrak{p}(x, \sigma)).$$

It is clear that if σ is a plain absolute equilibrium point of $\mathcal{G}$, then σ is a weak plain absolute equilibrium point of $\mathcal{G}$. On the other hand, if Γ contains no *cyclic path* (under a cyclic path of Γ we mean a finite sequence $x_0, \ldots, x_m \in P (m \geqq 0)$ such that $x_j \in \Gamma x_{j-1}$ for $j = 1, \ldots, m$, and $x_0 \in \Gamma x_m$), then any weak plain absolute equilibrium point of $\mathcal{G}$ is a plain absolute equilibrium point of $\mathcal{G}$ [namely, if $\sigma \in \mathrm{T_F}(\Gamma)$, $j \in J$, and if $\mathbf{x} \in \mathbf{X}_\Gamma$ complies with $\sigma \mid (Z \setminus Z(j))$, then there exists $\sigma' \in \mathrm{T_F}(\Gamma)$ such that $\mathbf{x} = \mathfrak{p}(x, \sigma')$ (etc., cf. § 2.9)]. Clearly, if Γ is locally finite, then it contains no cyclic path.

The notion of weak plain absolute equilibrium point expresses, in terms of this paper (and for the games considered here), the original Berge's notion of "(absolute) equilibrium point" (see [2], Ch. I, §§ 7 and 3). The preceding remarks show that the formulation of the Berge theorem used in § 3.5 is equivalent to the original Berge's formulation (in [2], Ch. I, § 7), although the latter concerns weak plain absolute equilibrium points.

3.7. Theorem. *Let*

$$\mathcal{G} = (\Gamma, (P(j); j \in J), (\mathcal{V}_j = (V_j, \leqq_j); j \in J), (f_j; j \in J))$$

be a two-player game with perfect information, let $J = \{j_1, j_2\}$.

Let there exist real numbers λ_k, $c_k(k = 1, 2)$ and a real-valued pay-off function f (on Γ) such that the conditions (α), (β), (γ) (§ 1.22) are satisfied, and

$$f_{j_k} = c_k + \lambda_k f \qquad (k = 1, 2),$$
$$\lambda_1 . \lambda_2 < 0.$$

Let there occur at least one of the following two cases:

(I') *P is finite.*
(II') *$\{|f(\mathbf{x})|;\ \mathbf{x} \in \mathbf{X}\}$ is inversely well-ordered, and Γx is finite for each $x \in P$.*

Then $\mathscr{G}$ has a plain absolute saddle point.

Proof. Let $\mathscr{W}^*$, f^*, L^* be the same as in § 1.22, let $f^*_{j_k} = f^* .\ \operatorname{sgn} \lambda_k (k = 1, 2)$. By means of §§ 1.22, 1.25.2, 1.26 (especially, cf. part c)) we conclude that L^* with $\mathscr{W}^*$ is a pseudolength (on Γ) and, for each $j \in J$, f^*_j is an L^*-qualitative pay-off function such that f_j is an L^*-quasiqualitative pay-off function complying with f^*_j. Evidently, if $\{|f(\mathbf{x})|;\ \mathbf{x} \in \mathbf{X}\}$ is inversely well-ordered, then (cf. § 1.22) im L^* is well-ordered. Consequently, if (K') holds for $K = \mathrm{I}$ or $K = \mathrm{II}$, then we have the case (K) from § 3.2; therefore (see §§ 1.26, 2.11) $\mathscr{G}$ has a plain absolute saddle point.

3.8. Remark. Of course, it would be possible to derive a certain corollary of the main theorem for the antagonistic case (by means of a suitable re-formulation of the "technical conditions" (A), (B) (§ 3.0) to pay-off functions constructed by means of §§ 1.22 and 1.26, etc.), but we take interest in (immediately applicable) important particular cases (cf. § 3.1); thus, we have based the antagonistic-case result (§ 3.7) directly on § 3.2. (Note that the case (III) in § 3.2 is not useful here, as it would lead to

(III') $\qquad \{|f(\mathbf{x})|;\ \mathbf{x} \in \mathbf{X}\}$ *and* $\{\mathrm{L}(\mathbf{x});\ \mathbf{x} \in \mathbf{X}\}$ *are finite,*

but this gives only a very special case of the situation considered in § 3.4.)

3.9. Remark. Important corollaries of § 3.7 can be obtained by means of § 1.23, which presents a general construction of (some) functions f (§ 1.22), **and by means of the special cases considered in §§ 1.24.2–3.** Especially, the result obtained by means of §§ 3.7 and 1.24.3 [it is easy to see how the direct expression of the notion of plain absolute saddle point is to be formulated (cf. [6], § 3.16, too); note that it is sufficient to take the case

(II'') *Γx is finite for each $x \in P$,*
cf. §§ 1.24.3, 1.22] involves the original result of Zermelo (proved for chess in [14]); cf. § 0.

(To be continued)

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