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NOTE ON THE THEORY OF DISPERSIONS OF THE DIFFERENTIAL EQUATION $y'' = q(t)y$

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1.1. Consider a differential equation

$$y'' = q(t)y,\ q \in C^\circ[a, b),\ q(t) < 0,\ t \in [a, b),\ b \leqq \infty, \tag{q}$$

where $C^n[a, b)$ (n being a nonnegative integer) is the set of all continuous functions having continuous derivatives up to and including the order n on $[a, b)$. In all the work we suppose that (q) is an oscillatoric ($t \to b_-$) differential equation, i.e. every non-trivial solution has infinitely many zeros on every interval of the form $[t_0, b)$, $t_0 \in [a, b)$.

Let y_1 (y_2) be a non-trivial solution of (q) such that $y_1(t) = 0$ ($y_2'(t) = 0$), $t \in [a, b)$. If $\varphi(t)$ ($\psi(t)$) is the first zero of y_1 (y_2') lying on the right of t, then $\varphi(\psi)$ is called the basic central dispersion of the 1-st (2-nd) kind (briefly, dispersion of the 1-st (2-nd) kind).

The properties of dispersions can be found in [3]. If δ is the dispersion of the k-th kind, $k = 1, 2$, then

$$\begin{aligned}
&1.\ \delta \in C^3[a, b) && \text{if } k = 1\\
&\delta \in C^1(a, b) && \text{if } k = 2\\
(1)\quad &2.\ \delta'(t) > 0 && \text{on } [a, b)\\
&3.\ \delta(t) > t && \text{on } [a, b)\\
&4.\ \lim_{t \to b_-} \delta(t) = b &&
\end{aligned}$$

hold (see [3] § 13). Let y be an arbitrary non-trivial solution of (q). Then (see [3] § 13.3)

$$\begin{aligned}
\psi'(t) &= \frac{q(t)}{q(\psi(t))} \frac{y'^2(\psi(t))}{y'^2(t)} && \text{if } y'(t) \neq 0,\\
&= \frac{q(t)}{q(\psi(t))} \frac{y^2(t)}{y^2(\psi(t))} && \text{if } y'(t) = 0.
\end{aligned} \tag{2}$$

The dispersion φ of the first kind of (q) fulfils the following non-linear differential equation

$$-\frac{1}{2} \frac{\varphi'''}{\varphi'} + \frac{3}{4} \frac{\varphi''^2}{\varphi'^2} + q(\varphi)\, \varphi'^2 = q(t), \qquad t \in (a, b). \tag{3}$$

1.2. In our later considerations we shall need some results being derived in [1], [4].
(i) Let $\varphi(\psi)$ be the dispersion of the 1-st (2-nd) kind of (q), $q \in C^\circ[a, b]$, $q(t) < 0$

on $[a, b)$, $b \leqq \infty$, (q) oscillatoric on $[a, b)$. Let $t_0 \in (a, b)$. Then

1) $\varphi(t_0) < \psi(t_0)$ iff $\varphi''(t_0) > 0$

2) $\varphi(t_0) = \psi(t_0)$ iff $\varphi''(t_0) = 0$

3) $\varphi(t_0) > \psi(t_0)$ iff $\varphi''(t_0) < 0$

4) $\varphi(t_0) = \psi(t_0)$ iff $\varphi'(t_0)\ \psi'(t_0) = \dfrac{q(t_0)}{q(\psi(t_0))}$

5) $\varphi(t_0) \neq \psi(t_0)$ iff $\varphi'(t_0)\ \psi'(t_0) < \dfrac{q(t_0)}{q(\psi(t_0))}$

(ii) Let (q), $q \in C^\circ\ [a, b)$, $b \leqq \infty$ be oscillatoric on $[a, b)$ and let φ be its dispersion of the 1-st kind.

a) If $\varphi'(t) \leqq 1$ on $[a, b)$, then every solution of (q) is bounded on $[a, b)$.

b) If $\varphi'(t) \leqq$ const < 1 on $[a, b)$, then $b < \infty$ and every solution of (q) tends to zero for $t \to b_-$.

2. In [1] relations between the dispersions of the 1-st and 2-nd kind were examined. The following theorem completes the results derived there.

Theorem 1. Let (q), $q \in C^\circ\ [a, b)$, $q(t) < 0$, $t \in [a, b)$ be an oscillatoric $(t \to b_-)$ differential equation and $\varphi(\psi)$ its dispersion of the 1-st (2-nd) kind. Let $t_0 \in [a, b)$ and

$$f(t) = \frac{q(\psi(t))}{q(t)}\ \psi'(t), \qquad t \in [a, b).$$

Then

a) $\varphi(t_0) < \psi(t_0)$ if, and only if $f'(t_0) < 0$

b) $\varphi[t_0) = \psi(t_0)$ if, and only if $f'(t_0) = 0$

c) $\varphi(t_0) > \psi(t_0)$ if, and only if $f'(t_0) > 0$.

Proof. a) Let y be a solution of (q) such that $y'(t_0) > 0$, $y\ (t_0) = 0$. It follows from (2) that the function f has the derivative and

$$f'(t_0) = \left(\frac{y'^2(\psi(t))}{y'^2(t)}\right)'\Bigg|_{t = t_0} = 2\ \psi'^2_0\ \frac{q^2(\psi_0)}{q(t_0)}\ \frac{y(\psi_0)}{y'(\psi_0)} \tag{4}$$

holds where $\psi_0 = \psi(t_0)$, $\psi_0' = \psi'(t_0)$.

Let $\varphi(t_0) < \psi(t_0)$. Then $y(\psi_0) < 0$, $y'(\psi_0) < 0$ and according (4) we have

$$f'(t_0) < 0. \tag{5}$$

Let (5) ve valid. As $y'(\psi_0) < 0$, it follows from (4) that $y(\psi_0) < 0$ and thus $\varphi(t_0) < \psi(t_0)$

b) c) These cases can be proved in the same way.

The following theorem sums up the results of 1.2. and Theorem 1 concerning the important case $\varphi(t_0) \doteq \psi(t_0)$, $t_0 \in [a, b)$.

Theorem 2. Let $\varphi(\psi)$ be the dispersion of the 1-st (2-nd) kind of an oscillatoric $(t \to b_-)$ differential equation (q), $q \in C^\circ\ [a, b)$, $q(t) < 0$ on $[a, b)$. Then the following assertions are equivalent:

a) $\varphi(t_0) = \psi(t_0)$

b) $\varphi''(t_0) = 0$

c) $\left(\frac{q(\psi(t))}{q(t)}\,\psi'(t)\right)'\Big|_{t=t_0} = 0$

d) $\varphi'(t_0)\,.\,\frac{q(\psi(t_0))}{q(t_0)}\,\psi'(t_0) = 1.$

Remark 1. Theorem 2 indicates that there exists a more profound dependence between the functions φ' and $\frac{q(\psi)}{q(t)}\,.\,\psi'$. The following theorem expresses this dependence more in detail.

Theorem 3. Let (q), $q \in C^\circ\,[a, b)$, $q(t) < 0$ on $[a, b)$ be oscillatoric on $[a, b)$ and φ, ψ be its dispersions of the 1-st and 2-nd kind. Let us put:

$$f(t) = \frac{q(\psi(t))}{q(t)}\,\psi'(t), \qquad t \in [a, b).$$

Then

a) The function φ' has a local maximum (minimum) at $t = t_0$ if, and only if f has a local minimum (maximum) at the point t_0. Moreover,

$$\varphi'(t_0) = \frac{1}{f(t_0)} \tag{6}$$

holds if the point t_0 is an extremant of φ' or f.

b) The function φ' is increasing (decreasing) at $t = t_0$ if, and only if f is decreasing (increasing) at $t = t_0$.

c) If $\varphi'(t) \geqq 1$ $(f(t) \geqq 1)$ holds on $[a, b)$, then $f(t) \leqq 1$ $(\varphi'(t) \leqq 1)$ on $[a, b)$. If $\varphi'(t) \leqq 1$ $(f(t) \leqq 1)$ holds on (a, b), then there exists a number $\bar{t}$, $\bar{t} \in [a, b)$ such that $f(t) \geqq 1$ $(\varphi'(t) \geqq 1)$ on $[\bar{t}, b)$.

Proof. a) b) The relation (6) from the case a) follows from Theorem 2 because if the function $\varphi'(f)$ has a local extreme at the point t_0, then $\varphi''(t_0) = 0$ $(f'(t_0) = 0)$. Further, it follows from Theorem 1 and 1. 2. that $\varphi''(t_0) < 0$, resp. $= 0$, resp. > 0 if, and only if $f'(t_0) > 0$, resp. $= 0$, resp. < 0. Thus if $\varphi''(t_0) \neq 0$ $(f'(t_0) \neq 0)$ holds, then the statement b) is valid. If $\varphi''(t_0) = 0$ $(f'(t_0) = 0)$, then the statements a) b) follows from the following assertions.

1) If $\varphi'(t) \geqq \varphi'(t_0)$ $(f(t) \geqq f(t_0))$, $t \in J$, then $f(t) \leqq f(t_0)$ $(\varphi'(t) \leqq \varphi'(t_0))$, $t \in J$ holds.

2) If $\varphi'(t) \leqq \varphi'(t_0)$ $(f(t) \leqq f(t_0)$, $t \in J$, then $f(t) \geqq f(t_0)$ $(\varphi'(t) \geqq \varphi'(t_0))$, $t \in J_1$ holds, where $J = [t_0, t_0 + \varepsilon)$, resp. $(t_0 - \varepsilon, t_0]$, $\varepsilon > 0$ is an arbitrary number, $\varepsilon \leqq t_0 - a$ and $J_1 = [t_0, t_0 + \varepsilon_1)$, resp. $(t_0 - \varepsilon_1, t_0]$, $\varepsilon_1 \leqq \varepsilon$ is a suitable number and $\varphi''(t_0) = 0$ $(f'(t_0) = 0)$.

The assertion 1) follows directly from 1.2. and Theorem 1. The assertion 2): Let $\varphi'(t) \leqq \varphi'(t_0)$, $t \in J$ and $\bar{\bar{t}} \in J$, $\varphi''(\bar{\bar{t}}) = 0$. Then according to Theorem 2 we have:

$$f(\bar{\bar{t}}) = \frac{1}{\varphi'(\bar{\bar{t}})} \geqq \frac{1}{\varphi'(t_0)} = f(t_0),$$

Let a number t_1, $t_1 \in J$ exist such that $\varphi''(t_1) = 0$, $t_1 \neq t_0$. If $\bar{t} \in J$, $\varphi''(\bar{t}) \neq 0$, $|\bar{t} - t_0| < |t_1 - t_0|$, then φ' is monotone in some neighbourhood of the point $\bar{t}$ and there exist numbers t_2, $t_3 \in J$ such that $\varphi''(t_2) = \varphi''(t_3) = 0$, $\varphi''(t) \neq 0$, $t \in (t_2, t_3)$, $\bar{t} \in (t_2, t_3)$. We have: $f(t_2) \geqq f(t_0)$, $f(t_3) \geqq f(t_0)$. As the function f is monotone on (t_2, t_3), we have

$f(\bar{t}) \geqq f(t_0)$ and the statement is valid in this case. If the above mentioned number t_1 does not exist, then $\varphi''(t) > 0$, resp. < 0 for $t \in J$, $t \neq t_0$ where $J = (t_0 - \varepsilon, t_0]$, resp. $J = [t_0, t_0 + \varepsilon)$. From here it follows (by use of 1.2.) that the function f is increasing, resp. decreasing and in both cases $f(t) \geqq f(t_0)$, $t \in J$ holds. The rest of the statement

$$f(t) \leqq f(t_0), t \in J \Rightarrow \varphi'(t) \geqq \varphi'(t_0), t \in J_1$$

we can proved in the same way.

c) I. Let $\varphi'(t) \geqq 1$ $(f(t) \geqq 1)$, $t \in [a, b)$. Then according to 1.2. we have: $f(t) \leqq 1$ $(\varphi'(t) \leqq 1)$, $t \in [a, b)$ and so the statement is valid in this case.

II. Let $\varphi'(t) \leqq 1$ $(f(t) \leqq 1)$, $t \in [a, b)$ and let M be the set of all numbers $t \in [a, b)$ such that the function $\varphi'(f)$ has a local maximum at $t \in M$. If the infinity is the accumulation point of M, then it follows from a) that $f(\varphi')$ has all local minima at the points $t \in M$ and we have

$$\varphi'(t) \,.\, f(t) = 1, t \in M.$$

From this $f(t) \geqq 1$ $(\varphi'(t) \geqq 1)$, $t \in [t_0, b)$ and the statement is valid in this case.

If the infinity is not the accumulation point of M, then there exists a number $\bar{t} \in [a, b)$ such that the function $\varphi'(f)$ is monotone on $[\bar{t}, b)$.

A. Let $\varphi'(t) \leqq 1$, $t \in [\bar{t}, b)$. Suppose that $\lim\limits_{t \to b} f(t) = c < 1$. Let y be an arbitrary non-trivial solution of (q) and $\{x_k\}_0^\infty$ the sequence of all zeros of y', $x_k \in [\bar{t}, b)$. So $x_k = \psi(x_{k-1})$, $k \geqq 1$ and $|y(x_k)|$ are local maxima of $|y|$. It follows from (2) that

$$0 < \frac{y^2(x_0)}{y^2(x_k)} = \prod_{n=1}^{k} \frac{y^2(x_{n-1})}{y^2(x_n)} = \prod_{n=1}^{k} f(x_{n-1}) \xrightarrow[k \to \infty]{} 0$$

Thus y is unbounded and it is in contradiction with 1.2. (ii). Thus $\lim\limits_{t \to b} f(t) \geqq 1$.

Let φ' be non-decreasing. Then f is non-increasing (see b)) and the statement is valid.

Let φ' be non-increasing. Then $\lim\limits_{t \to b} \varphi'(t) < 1$ and according to 1.2. (ii) we have that an arbitrary solution of (q) converges to zero for $t \to b_-$. Suppose that $\lim\limits_{t \to b} f(t) = 1$. As f is non-decreasing we have $f(t) \leqq 1$, $t \in [\bar{t}, b)$. Let y be an arbitrary non-trivial solution of (q) and $\{x_n\}_0^\infty$ a sequence of the zeros of y', $x_n \in [t_0, b)$. Then y has a local extreme at x_n and by use of (2) we have:

$$\infty \xleftarrow[n \to \infty]{} \frac{y^2(x_0)}{y^2(x_n)} = \prod_{k=1}^{n} \frac{y^2(x_{k-1})}{y^2(x_k)} = \prod_{k=1}^{n} f(x_{k-1}) \leqq 1. \tag{7}$$

But this is the contradiction. So $\lim\limits_{t \to b} f(t) > 1$ and the statement is valid.

B. Let $f(t) \leqq 1$, $t \in [\bar{t}, b)$ and let $\lim\limits_{t \to b} \varphi'(t) = c < 1$. Then (7) is valid and it is the contradiction. Thus $\lim\limits_{t \to b} \varphi'(t) \geqq 1$. If f is non-decreasing, then φ' is non-increasing and the statement is valid. Let f be non-increasing. Then $\lim\limits_{t \to b} f(t) = c < 1$ and φ' is non-decreasing on $(\bar{t}, b)$. In the first part of c) II. A) we proved that the conditions $\varphi'(t) \leqq 1$, $t \in [\bar{t}, b)$, $\lim\limits_{t \to b_-} f(t) < 1$ can not be valid at the same time. From this $\lim\limits_{t \to b} \varphi'(t) > 1$ and the statement of the theorem is proved.

Remark 2. The case c) of Theorem 3 is valid, too if we replace the inequalities $\leqq$, $\geqq$ by $<$, $>$, resp. because if $\varphi'(t_0) = 1$ ($f(t_0) = 1$), $t_0 \in [a, b)$, then according to Theorem 2 we have $f(t_0) = 1$ ($\varphi'(t_0) = 1$).

The results of Theorem 3 gives us a possibility to generalize a theorem from [2] (Theorem 10) concerning the behaviour of solutions of (q).

Theorem 4. *Let (q), $q \in C^\circ[a, b)$, $q(t) < 0$, $t \in [a, b)$, be oscillatoric on $[a, b)$ and let φ, ψ be its dispersions of the 1-st, 2-nd kind, resp. Consider the following assertions on $[a, b)$:*

A) *The sequence of the absolute values of all local extremes (of the derivative) of an arbitraty non-trivial solution of (q) is non-increasing.*

B) The sequence of the absolute values of all local extremes of the derivative of an arbitrary non-trivial solution (of an arbitrary non-trivial solution) of (q) is non-decreasing.

C) $\dfrac{q(\psi)}{q(t)}\,\psi' \geqq 1$ ($\varphi(t) - t$ *is non-decreasing*)

D) $\varphi(t) - t$ *is non-increasing* $\left(\dfrac{q(\psi)}{q(t)}\,\psi' \leqq 1\right)$.

Then $A \Leftrightarrow C \Rightarrow D \Leftrightarrow B$ and there exists a number t_0, $t_0 \in [a, b)$ such that we have $D \Rightarrow C$ on $[t_0, b)$.

Proof. According to Theorem 10 from [2] we must only prove that there exists a number t_0, $t_0 \in [a, b)$ such that $D \Rightarrow C$ on $[t_0, b)$ holds. But this fact follows directly from Theorem 3c).

Remark 3. If we replace „non-increasing", „non-decreasing", $\leqq$, $\geqq$ by „decreasing", „increasing", $<$, $>$, respectively, then Theorem 4 is valid, too.

Theorem 5. *Let (q), $q \in C^\circ[a, b)$, $q(t) < 0$, $t \in [a, b)$ be oscillatoric on $[a, b)$ and let φ, ψ be its dispersions of the 1-st and 2-nd kind. Let $t_0 \in [a, b)$. Then the following assertions are equivalent.*

A. $\varphi(t_0) = \psi(t_0)$, $\varphi'(t_0) = \psi'(t_0)$

B. $\varphi''(t_0) = \varphi'''(t_0) = 0$.

Moreover, if there exists $q'(t_0)$, then the assertion

C) $f'(t_0) = f''(t_0) = 0$ *where* $f(t) = \dfrac{q(\psi(t))}{q(t)}\,\psi'(t)$ *is equivalent with A) and B).*

Proof. $A \Rightarrow B$: According to Theorem 2 we have:

$$\varphi''(t_0) = 0, \qquad \varphi'^2(t_0) = \varphi'(t_0)\ \psi'(t_0) = \frac{q(t_0)}{q(\psi(t_0))} = \frac{q(t_0)}{q(\varphi(t_0))}.$$

From this and from (3)

$$-\frac{1}{2}\,\frac{\varphi'''(t_0)}{\varphi'(t_0)} + \frac{3}{4}\left(\frac{\varphi''(t_0)}{\varphi'(t_0)}\right)^2 = 0$$

holds and thus $\varphi'''(t_0) = 0$.

$B \Rightarrow A$. It follows from Theorem 2 that $\varphi(t_0) = \psi(t_0)$ holds and from (3) we have:

$$q(t_0) = q(\varphi(t_0))\ \varphi'^2(t_0) = q(\psi(t_0))\ .\ \varphi'^2(t_0).$$

From this and from theorem 2 we get: $\varphi'^2(t_0) = \varphi'(t_0)\ .\ \psi'(t_0)$ and thus $\varphi'(t_0) = \psi'(t_0)$

$A \Leftrightarrow C$. Let y be a non-trivial solution of (q) such that $y(t_0) = 0$. Then it follows from (2) that f has the derivative and

$$f'(t) = 2 \,.\, f^2(t)q(t) \frac{y(\psi(t))}{y'(\psi(t))} - 2\,f(t)\,q(t) \,.\, \frac{y(t)}{y'(t)}$$

holds in some neighbourhood of the point t_0. Thus we can see that the function $\frac{f'}{q}$ has the derivative and if $q'(t_0)$ exists, then we have at $t = t_0$:

$$\left(\frac{f'}{q}\right)' = \frac{f''}{q} - \frac{f'q'}{q^2} = \frac{3}{2}\frac{f'^2}{f \,.\, q} + 2f(f \,.\, \psi' - 1). \tag{8}$$

$C \Rightarrow A$: According to (8) we have $f(f\psi' - 1) = 0$ for $t = t_0$ and because $f \neq 0$ we get.

$$f(t_0)\,\psi'(t_0) = 1. \tag{9}$$

Theorem 2 gives us: $\varphi(t_0) = \psi(t_0)$,

$$f(t_0)\,\varphi'(t_0) = 1. \tag{10}$$

Thus $\varphi'(t_0) = \psi'(t_0)$ and the statement is proved.

$A \Rightarrow C$. It follows from the assumptions and Theorem 2 that $f'(t_0) = 0$ and (10) and (9) hold. Then the statement follows from (8).

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