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ON STRONG HOMOMORPHISMS OF FINITELY SEMIGENERATED LANGUAGES

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0. INTRODUCTION

M. Novotný [3] proved that finitely characterizable languages are preserved under strong homomorphisms. Finitely characterizable languages have been defined by the use of configurations. Taking semiconfigurations instead of configurations we can define similarly finitely semigenerated languages. The problem is, whether these languages are preserved under strong homomorphisms. In this paper the positive answer to the question mentioned above is given.

1. SOME DEFINITIONS

Let V be a set. We denote V^* the free monoid over V, i.e. the set of all finite sequences of elements of the set V including the empty sequence Λ. We identify one-member sequences with elements of the set V; it follows $V \subseteq V^*$. If $x \in V^*$, $x = x_1 \dots x_n$, where $x_i \in V (i = 1, \dots, n)$ and n is a natural number, we put $|x| = n$; further, we define $|\Lambda| = 0$.

Let V, U be sets, f a surjection of V onto U. Then there exists the only homomorphism of V^* onto U^*. This homomorphism (denoted as f_*) is defined as follows: for every $x \in V^*$, $x = x_1x_2 \dots x_n$, where n is a natural number and $x_i \in V (i = 1, 2, \dots, n)$, we put $f_*(x) = f(x_1) f(x_2) \dots f(x_n)$; further, we define $f_*(\Lambda) = \Lambda$.

Let us assume that $x \in V^*$, $f_*(x) = y_1y_2 \dots y_m$, where m is a natural number and $y_i \in U^* (i = 1, 2, \dots, m)$. It follows that there exist elements $x_1, x_2, \dots, x_m$ of the set V^* such that $x = x_1x_2 \dots x_m$ and $f_*(x_i) = y_i$ for $i = 1, 2, \dots, m$. It is obvious that $|x| = |f_*(x)|$ for every $x \in V^*$.

2. LANGUAGES AND GENERALIZED GRAMMARS

2.1. Definition. Let V be a set, L a subset of the set V^*. The ordered pair (V, L) is called a *language*. The elements of the set V are called *word-forms*, the elements of the set V^* are called *strings*, the elements of the set L are called *marked strings*. The set V is called a *vocabulary of the language* (V, L). The ordered pair (V, L) is called the *language over the vocabulary* V.

2.2. Definition. Let V, R be sets with the property $R \subseteq V^* \times V^*$. For $x, y \in V^*$ we put $x \to y(R)$ if $(x, y) \in R$. Further, for $x, y \in V^*$ we put $x \Rightarrow y(R)$ if there exist elements $u, v, t, z \in V^*$ such that $x = utv$, $y = uzv$, $t \to z(R)$. For $x, y \in V^*$ we put

$x \overset{*}{\Rightarrow} y(R)$ if there exists an integer number $p \geqq 0$ and elements $t_0, t_1, \ldots, t_p \in V^*$ such that $x = t_0$, $t_p = y$ and $t_{i-1} \Rightarrow t_i(R)$ for $i = 1, 2, \ldots, p$. The sequence of strings $(t_i)_{i=0}^p$ is called an *x-derivation of y of the length p in R. An x-derivation of length* 0 is called a *trivial derivation of x*.

2.3. Definition. Let V, V_T, S, R be sets such that $V_T \subseteq V$, $S \subseteq V^*$, $R \subseteq V^* \times V^*$. Then the quadruple $G = \langle V, V_T, S, R \rangle$ is called a *generalized grammar*.

2.4. Definition. Let $G = \langle V, V_T, S, R \rangle$ be a generalized grammar. We put $\mathscr{L}(G) = \{x;\ x \in V_T^*$ and there exists some $s \in S$ such that $s \overset{*}{\Rightarrow} x(R)\}$. The language $(V_T, \mathscr{L}(G))$ is called the *language generated by the generalized grammar G*.

2.5. Definition. A generalized grammar $G = \langle V, V_T, S, R \rangle$ is called a generalized special grammar if $V = V_T$. In this case we write $\langle V, S, R \rangle$ instead of $\langle V, V, S, R \rangle$.

2.6. Definition, A generalized grammar $G = \langle V, V_T, S, R \rangle$ is called a *grammar* if the sets V, S, R are finite.

2.7. Definition. A grammar $G = \langle V, V_T, S, R \rangle$ is called a *special grammar* if $V = V_T$, in this case we write $\langle V, S, R \rangle$ instead of $\langle V, V, S, R \rangle$.

3. SEMICONFIGURATIONS AND STRONG HOMOMORPHISMS

3.1. Definition. Let (V, L), (U, M) be languages, f be a surjection of V onto U. The surjection f is called a *weak homomorphism* of the language (V, L) onto (U, M) if $f_*(L) = M$. The surjection f is called a *strong homomorphism* of the language (V, L) onto (U, M) if $f_*^{-1}(M) = L$.

3.2. Remark. *It is obvious that each strong homomorphism is at the same time a weak homomorphism and a bijective weak homomorphism is strong.*

3.3. Definition. A bijective strong homomorphism of (V, L) onto (U, M) is called an *isomorphism*.

3.4. Definition. Let $G = \langle V, V_T, S, R \rangle$, $H = \langle U, U_T, P, Q \rangle$ be generalized grammars, f a surjection of V onto U. The surjection f is called a strong homomorphism of G onto H if following conditions are satisfied:

(A) For every $x \in V$, the condition $x \in V_T$ is equivalent to $f(x) \in U_T$.

(B) For every $x \in V^*$, the condition $x \in S$ is equivalent to $f_*(x) \in P$.

(C) For every $x, y \in V^*$, the condition $(x, y) \in R$ is equivalent to $(f_*(x), f_*(y)) \in Q$.

3.5. Definition. A bijective strong homomorphism of G onto H is called an *isomorphism*.

3.6. Theorem. *Let $G = \langle V, V_T, S, R \rangle$, $H = \langle U, U_T, P, Q \rangle$ be generalized grammars, f a strong homomorphism of G onto H. Then the following assertions hold true:*

(I) *For $t' \in U^*$, $s \in V^*$, $(x', y') \in Q$ the following conditions are equivalent:*

(A) $t' \Rightarrow f_*(s)\ (\{(x', y')\})$.

(B) *there exist $t \in f_*^{-1}(t')$, $x \in f_*^{-1}(x')$, $y \in f_*^{-1}(y')$ such that $t \Rightarrow s(\{(x, y)\})$.*

(II) *For $t' \in U^*$, $s \in V^*$ the condition $t' \Rightarrow f_*(s)\ (Q)$ is equivalent to the existence of $t \in f_*^{-1}(t')$ such that $t \Rightarrow s(R)$.*

(III) *For $t' \in U^*$, $s \in V^*$ the condition $t' \overset{*}{\Rightarrow} f_*(s)\ (Q)$ is equivalent to the existence of $t \in f_*^{-1}(t')$ such that $t \Rightarrow s(R)$.*

(IV) *For $x \in V^*$ the condition $x \in \mathscr{L}(G)$ is equivalent to $f_*(x) \in \mathscr{L}(H)$.*

(V) *f/V_T is the strong homomorphism $(V_T, \mathscr{L}(G))$ onto $(V_T, \mathscr{L}(H))$.*

Proof. $t' \Rightarrow f_*(s)$ $(\{(x', y')\})$ means, in other words, that there exist elements $u', v' \in U^*$ such that $t' = u'x'v'$, $f_*(s) = u'y'v'$. It means that there exist elements $u \in f_*^{-1}(u')$, $x \in f_*^{-1}(x')$, $v \in f_*^{-1}(v')$ such that $s = uyv$, $uxv \Rightarrow s(\{(x, y)\})$. Thus, assertion (I) is proved.

It is obvious that assertion (II) follows from (I). The assertion $t' \overset{*}{\Rightarrow} f_*(s)$ (Q) is equivalent with the existence of an integer $p \geqq 0$ and strings $t' = t_0', t_1', \ldots, t_p' = f_*(s)$ in U^* such that $t_{i-1}' \Rightarrow t_i'$ (Q) for $i = 1, 2, \ldots, p$. According to (II) there is possible to prove by induction that the existence of such a t_i' is equivalent with the existence of elemensts $t_i \in V^* (i = 0, 1, \ldots, p-1)$ such that $t_i \in f_*^{-1}(t_i')$, $t_{i-1} \Rightarrow t_i(R)$ for $i = 1, 2, \ldots, p-1$, $t_{p-1} \Rightarrow s(R)$. This is equivalent with the existence of $t \in f_*^{-1}(t')$ such that $t \overset{*}{\Rightarrow} s(R)$. So the assertion (III) is proved.

Now, the following conditions are equivalent for $x \in V^*$: (1) $f_*(x) \in U_T^*$ and there exists $s' \in P$ such that $s' \overset{*}{\Rightarrow} f_*(x)$ (Q); (2) $x \in V_T^*$ and there exists $s \in f_*^{-1}(s') \subseteq S$ such that $s \overset{*}{\Rightarrow} x(R)$. From it follows the proof of the assertion (IV).

Assertion (V) follows obviously from (IV) and from 3.1.

3.7. Definition. Let (V, L) be a language. The element $x \in V^*$ is called *necessary* in the language (V, L) (which we symbolize by $x\, \nu\, (V, L)$) if there exist elements $u, v \in V^*$ such that $uxv \in L$.

3.8. Definition. Let (V, L) be a language. For elements $x, y \in V^*$ we put $x > y(V, L)$ (x can be substituted by y in the language (V, L)), if $u, v \in V^*$, $uxv \in L$ imply $uyv \in L$.

3.9. Lemma. *Let (V, L), (U, M) be languages, f a strong homomorphism of the language (V, L) onto (U, M). Then following assertions hold:*

(A) *For every $x \in V^*$ the condition $x\, \nu\, (V, L)$ is equivalent to $f_*(x)\, \nu\, (U, M)$.*

(B) *For every $x, y \in V^*$ the condition $x > y(V, L)$ is equivalent to $f_*(x) > f_*(y)$ (U, M).*

This lemma can be found in [3] as the lemma 1.

3.10. Definition. Let (V, L) be a language, $x, y \in V^*$. We say that x is a *semiconfiguration* in the language (V, L) with the resultant y if following conditions are satisfied:

(1) $y\, \nu\, (V, L)$

(2) $y > x(V, L)$, $y \neq x$, $|y| \leqq |x|$.

We denote by $E(V, L)$ the set of all pairs (y, x), where x is a semiconfiguration of the language (V, L) with the resultant y.

3.11. Definition. We put $R(V, L) = \{(y, x); y\, \nu\, (V, L),\ y > x(V, L), |y| \leqq |x|\}$.

3.12. Remark. *From the definitions of the sets $E(V, L)$ and $R(V, L)$ it follows that $E(V, L) \subseteq R(V, L)$ and also, for every $s, t \in V^*$, the condition $s \Rightarrow t(E(V, L))$ implies $s \Rightarrow t(R(V, L))$ and for every $s, t \in V^*$, $s \neq t$, the condition $s \Rightarrow t(R(V, L))$ implies $s \Rightarrow t(E(V, L))$.*

3.13. Lemma. *Let be $s, t \in V^*$. Then $s \overset{*}{\Rightarrow} t(E(V, L))$ iff $s \overset{*}{\Rightarrow} t(R(V, L))$.*

Proof. I. Let us have $s, t \in V^*$, $s \overset{*}{\Rightarrow} t(E(V, L))$. Then, by 3.12, $s \overset{*}{\Rightarrow} t(R(V, L))$ holds.

II. If $s, t \in V^*$, $s \overset{*}{\Rightarrow} t(R(V, L))$, then there exist elements $t_0, t_1, \ldots, t_p \in V^*$ such that $s = t_0$, $t_p = t$ and $t_{i-1} \Rightarrow t_i(R(V, L))$ for $i = 1, 2, \ldots, p$. If $t_{i-1} \neq t_i$ is valid for $i = 1, 2, \ldots, p$, then a by 3.12, the condition $t_{i-1} \Rightarrow t_i(E(V, L))$ is satisfied for $i = 1, 2, \ldots, p$ and so $s \overset{*}{\Rightarrow} t(E(V, L))$.

III. Suppose $s, t \in V^*$. Let $(t_i)_{i=0}^p$ be an s-derivation of t of length p in $R(V, L)$ and k an integer $k \in \{1, 2, \ldots, p\}$ such that $t_{k-1} = t_k$. If we cancel the string t_k in $(t_i)_{i=0}^p$ we obtain an s-derivation of t of the length $p - 1$ in $R(V, L)$. Repeating this procedure we obtain an s-derivation of t of length $l \leq p$ in $R(V, L)$ such that $t_{i-1} \neq t_i$ for $i = 1, 2, \ldots, l$. By II, this s-derivation is an s-derivation of t in $E(V, L)$, and so $s \overset{*}{\Rightarrow} t(E(V, L))$.

3.14. Definition. Let (V, L) be a language. For $x \in L$ we put $x \in B_E(V, L)$ iff, for every $t \in L$, the condition $t \overset{*}{\Rightarrow} x(E(V, L))$ implies $|t| = |x|$.

For $x \in L$ we put $x \in B_R(V, L)$ iff for every $t \in L$, the condition $t \overset{*}{\Rightarrow} x(R(V, L))$ implies $|t| = |x|$.

3.15. Theorem. Let (V, L) be a language. Then $B_E(V, L) = B_R(V, L)$.

Proof. If $x \in L$, $x \in B_E(V, L)$, $t \in L$, $t \overset{*}{\Rightarrow} x(R(V, L))$, then, by 3.13, it holds true that $t \overset{*}{\Rightarrow} x(E(V, L))$ and according to the definition of $B_E(V, L)$, we have $|t| = |x|$. Thus, $x \in B_R(V, L)$.

If $x \in L$, $x \in B_R(V, L)$, $t \in L$, $t \overset{*}{\Rightarrow} x(E(V, L))$, then, by 3.13, it holds true that $t \overset{*}{\Rightarrow} x(R(V, L))$ and so $|t| = |x|$. It follows that $x \in B_E(V, L)$.

3.16. Lemma. *Let (V, L) be a language. Then the following assertions hold:*

(A) *For every $x \in L$ there exists $s \in B_E(V, L)$ such that $s \overset{*}{\Rightarrow} x(E(V, L))$.*

(B) *For every $x \in L$ there exists $s \in B_R(V, L)$ such that $s \overset{*}{\Rightarrow} x(R(V, L))$.*

Proof. There exists at least one element $s \in L$ such that $s \overset{*}{\Rightarrow} x(E(V, L))$. One can consider for example the trivial s-derivation in $E(V, L)$. If the element of minimum length from those mentioned above is chosen, there is evident that this element belongs to $B_E(V, L)$.

That is the proof of assertion (A).

Assertion (B) follows from 3.15, 3.13 and (A).

3.17. Definition. Two definitions are condensed in 3.17; the first is obtained when reading the conditions denoted by 1° the second is obtained when reading the conditions denoted by 2°. 3.23 must be interpreted similarly.

Let (V, L) be a language. If $s, t \in V^*$ are the strings such that 1° $s \Rightarrow t(E(V, L))$, 2° $s \Rightarrow t(R(V, L))$, we put 1° $|(s, t)|_E = \min\{|q|;\ (p, q) \in E(V, L),\ s \Rightarrow t(\{(p, q)\})\}$, 2° $|(s, t)|_R = \min\{|q|;\ (p, q) \in R(V, L),\ s \Rightarrow t(\{(p, q)\})\}$.

If $s, t \in V^*$ are strings and $(t_i)_{i=0}^p$ is an s-derivation of t in 1° $E(V, L)$, 2° $R(V, L)$, $p > 0$, then we put 1° $\|(t_i)_{i=0}^p\|_E = \max\{|(t_{i-1}, t_i)|_E;\ i = 1, 2, \ldots, p\}$, 2° $\|(t_i)_{i=0}^p\|_R = \max\{|(t_{i-1}, t_i)|_R;\ i = 1, 2, \ldots, p\}$. The number 1° $\|(t_i)_{i=0}^p\|_E$, 2° $\|(t_i)_{i=0}^p\|_R$ is called the norm of the s-derivation $(t_i)_{i=0}^p$ of t in 1° $E(V, L)$, 2° $R(V, L)$. The norm of a trivial s-derivation in 1° $E(V, L)$, 2° $R(V, L)$ is defined to be zero.

If $s, t \in V^*$ are the strings such that 1° $s \overset{*}{\Rightarrow} t(E(V, L))$, 2° $s \overset{*}{\Rightarrow} t(R(V, L))$, we define the norm 1° $\|(s, t)\|_E$, 2° $\|(s, t)\|_R$ of the ordered pair (s, t) to be the minimum of norms of all s-derivations of t in 1° $E(V, L)$, 2° $R(V, L)$. If $t \in L$, we put 1° $\|t\|_E = \min\{\|(s, t)\|_E;\ s \in B_E(V, L),\ s \overset{*}{\Rightarrow} t(E(V, L))\}$, 2° $\|t\|_R = \min\{\|(s, t)\|_R;\ s \in B_R(V, L),\ s \overset{*}{\Rightarrow} t(R(V, L))\}$. The number 1° $\|t\|_E$, 2° $\|t\|_R$ is called the norm of t in 1° $E(V, L)$, 2° $R(V, L)$.

3.18. Lemma. *Suppose $s, t \in V^*$. If $|(s, t)|_E$, exists, then $|(s, t)|_R$ exists and it holds that $|(s, t)|_E = |(s, t)|_R$. Further, for $t \in V^*$, $t \Rightarrow t(R(V, L))$ it holds evidently that $|(t, t)|_R = 0$.*

Proof. The proof follows from 3.12, and 3.17.

3.19. Lemma. *Suppose $s, t \in V^*$, let $(t_i)_{i=0}^p$ be an s-derivation of t in $E(V, L)$. Then $(t_i)_{i=0}^p$ is also an s-derivation of t in $R(V, L)$ and the equation $\|(t_i)_{i=0}^p\|_E = \|(t_i)_{i=0}^p\|_R$ holds. On the contrary, if $s, t \in V^*$, and if $(t_i)_{i=0}^p$ is an s-derivation of t in $R(V, L)$ such that $t_{i-1} \neq t_i$ for $i = 1, 2, \ldots, p$, then $(t_i)_{i=0}^p$ is an s-derivation in $E(V, L)$ and the equation $\|(t_i)_{i=0}^p\|_E = \|(t_i)_{i=0}^p\|_R$ holds.*

Proof. The proof follows from 3.18 and 3.17.

3.20. Remark. *Suppose* $s, t \in V^*$. *If* $(t_i)_{i=0}^p$ *is an s-derivation of t in* $R(V, L)$, *then it is obvious from* 3.18 *and* 3.17 *that the elements* t_i *of the s-derivation of t, such that* $t_{i-1} = t_i$ *have no influence on the value of* $\| (t_i)_{i=0}^p \|_R$.

3.21. Lemma. *If* $s, t \in V^*$, $s \overset{*}{\Rightarrow} t(E(V, L))$, *then* $\| (s, t) \|_E = \| (s, t) \|_R$.

Proof. The proof follows from 3.17, 3.19 and 3.20.

3.22. Theorem. *If* $t \in L$, *then* $\| t \|_E = \| t \|_R$.

Proof. Assume $t \in L$. Then, by 3.15 and 3.13, for every $s \in B_E(V, L)$, the condition $s \overset{*}{\Rightarrow} t(E(V, L))$ implies $s \in B_R(V, L)$, $s \overset{*}{\Rightarrow} t(R(V, L))$. Further, by 3.21 it holds that $\| (s, t) \|_E = \| (s, t) \|_R$, thus, according to the definition of $\| t \|_E$ and $\| t \|_R$ it holds that $\| t \|_E \geqq \| t \|_R$. Similarly, it is possible to prove that $\| t \|_E \leqq \| t \|_R$; thus $\| t \|_E = \| t \|_R$.

3.23. Lemma. *Let* (V, L) *be a language. Then, for every* $t \in L$, *there exists a string* 1° $s \in B_E(V, L)$, 2° $s \in B_R(V, L)$ *and an s-derivation of t in* 1° $E(V, L)$, 2° $R(V, L)$ *such that the norm of this s-derivation is equal to* 1° $\| t \|_E$, 2° $\| t \|_R$.

Proof. According to 3.17, there exists an element $s \in B_E(V, L)$ such that $\| (s, t) \|_E = \| t \|_E$. It means that there exists such an s-derivation of t in $E(V, L)$ that its norm is equal to $\| t \|_E$.

Similar proof takes place in the case of $R(V, L)$.

3.24. Definition. Let (V, L) be a language. Then we put $X_E(V, L) = \{(y, x); (y, x) \in E(V, L), | x | > \| t \|_E$ for every $t \in L\}$, $X_R(V, L) = \{(y, x); (y, x) \in R(V, L), | x | > \| t \|_R$ for every $t \in L\}$, $Z_E(V, L) = E(V, L) - X_E(V, L)$, $Z_R(V, L) = R(V, L) - X_R(V, L)$:

3.25. Lemma. *It holds that* $X_E \subseteq X_R$, $Z_E \subseteqq Z_R$.

Proof. The proof follows from 3.11 and 3.24.

3.26. Lemma. *Let be* $s, t \in V^*$. *Then* $s \overset{*}{\Rightarrow} t(Z_E(V, L))$ *iff* $s \overset{*}{\Rightarrow} t(Z_R(V, L))$.

Proof. I. Assume $s, t \in V^*$, $s \overset{*}{\Rightarrow} t(Z_E(V, L))$. Then it follows, by 3. 25, that $s \overset{*}{\Rightarrow} t(Z_R(V, L))$.

II. Suppose $s, t \in V^*$, $s \overset{*}{\Rightarrow} t(Z_R(V, L))$. Then there exists an s-derivation $(t_i)_{i=0}^p$ of t in $Z_R(V, L)$. If $t_{i-1} \neq t_i (i = 1, 2, \ldots, p)$, then this is an s-derivation of t in $Z_E(V, L)$. If $t_{i-1} = t_i$ for some $i \in \{1, 2, \ldots, p\}$, then it is possible to omit these t_i and to obtain again an s-derivation of t in $Z_R(V, L)$ which is at the same time the s-derivation of t in $Z_E(V, L)$.

3.27. Theorem. *Let* (V, L) *be a language. Then the following assertions hold:*

(A) *For every* $t \in L$ *there exists at least one element* $s \in B_E(V, L)$ *such that* $s \overset{*}{\Rightarrow} t (Z_E(V, L))$.

(B) *For every* $t \in L$ *there exists at least one element* $s \in B_R(V, L)$ *such that* $s \overset{*}{\Rightarrow} t (Z_R(V, L))$.

Proof. I. According to 3.23 for every $t \in L$ there exists a string $s \in B_E(V, L)$ and an s-derivation $(t_i)_{i=0}^p$ of t in $E(V, L)$ such that $\| (t_i)_{i=0}^p \|_E = \| t \|_E$. It follows that $| (t_{i-1}, t_i) |_E \leqq \| t \|_E$ for $i = 1, 2, \ldots, p$ by 3.17. Further, for every $i = 1, 2, \ldots, p$ there exists an element $(p_i, q_i) \in E(V, L)$ such that $t_{i-1} \Rightarrow t_i (\{(p_i, q_i)\})$ and $| q_i | = | (t_{i-1}, t_i) |_E \leqq \| t \|_E$. Thus, $(p_i, q_i) \in Z_E(V, L)$ for $i = 1, 2, \ldots, p$ and consequently $s \overset{*}{\Rightarrow} t(Z_E(V, L))$.

That is the proof of assertion (A).

The proof of assertion (B) follows from (A), 3.15 and 3.26.

3.28. Definition. Let (V, L) be a language. We put $K_E(V, L) = \langle V, B_E(V, L), Z_E(V, L) \rangle$, $K_R(V, L) = \langle V, B_R(V, L), Z_R(V, L) \rangle$. $K_E(V, L)$, respectively $K_R(V, L)$ is a generalized special grammar called further a *generalized semiconfigurational* respectively *R-semiconfigurational* grammar.

3.29. Theorem. *Let* (V, L) *be a language. Then* $\mathscr{L}(K_E(V, L)) = \mathscr{L}(K_R(V, L)) = L$.

Proof. I. According to 3.27 we have $L \subseteq \mathscr{L}(K_E(V, L))$.

II. Let $V(n)$ be the following assertion: if $t \in \mathscr{L}(K_E(V, L))$ and if there exists an element $s \in B_E(V, L)$ and an s-derivation of t of length n in $Z_E(V, L)$, then $t \in L$.

If $t \in \mathscr{L}(K_E(V, L))$ and if there exists an element $s \in B_E(V, L)$ and a trivial s-derivation of t in $Z_E(V, L)$, then $t = s \in B_E(V, L) \subseteq L$. Thus, $V(0)$ is valid.

Let be now $m \geqq 0$ and assume that $V(m)$ holds. Suppose further that $t \in \mathscr{L}(K_E(V, L))$, $s \in B_E(V, L)$ and that $(t_i)_{i=0}^{m+1}$ is an s-derivation of length $m + 1$ in $Z_E(V, L)$. Then, according to $V(m)$, it holds that $t_m \in L$. Further, $t_m \Rightarrow t\,(Z_E(V, L))$. It means that there exist elements $x, y, u, v \in V^*$ such that $t_m = uxv$, $t = uyv$, $(x, y) \in Z_E(V, L) \subseteq E(V, L)$. It follows that $x > y(V, L)$ which implies $t \in L$. Thus $V(m + 1)$ holds. Hence $V(m)$ holds for $m = 0, 1, \ldots$. It means that $\mathscr{L}(K_E(V, L)) \subseteq L$.

The assertion $\mathscr{L}(K_E(V, L)) = L$ has been proved.

III. Suppose $t \in \mathscr{L}(K_E(V, L))$ and $s \in B_E(V, L)$, $s \overset{*}{\Rightarrow} t(Z_E(V, L))$. Then, by 3.15, it holds that $s \in B_R(V, L)$ and, by 3.26, it holds that $s \overset{*}{\Rightarrow} t(Z_R(V, L))$. Thus, $t \in \mathscr{L}(K_R(V, L))$ and we have $\mathscr{L}(K_E(V, L)) \subseteq \mathscr{L}(K_R(V, L))$.

Assume $t \in \mathscr{L}(K_R(V, L))$ and $s \in B_R(V, L)$, $s \overset{*}{\Rightarrow} t(Z_R(V, L))$. Then, by 3.15, it holds that $s \in B_E(V, L)$ and by 3.26, it holds that $s \overset{*}{\Rightarrow} t(Z_E(V, L))$. It means that $t \in \mathscr{L}(K_E(V, L))$ and we have $\mathscr{L}(K_R(V, L)) \subseteq \mathscr{L}(K_E(V, L))$.

Thus, the assertion $\mathscr{L}(K_E(V, L)) = \mathscr{L}(K_R(V, L))$ has been proved.

3.30 Lemma. *Let* (V, L), (U, M) *be languages,* f *a strong homomorphism of* (V, L) *onto* (U, M). *Let us have* $t' \in U^*$, $s \in V^*$. *Then the following assertions hold:*

(A) *If* $t' \Rightarrow f_*(s)\ (R(U, M))$, *then there exists* $t \in f_*^{-1}(t')$ *such that* $t \Rightarrow s(R(V, L))$ *and* $|(t, s)|_R \leqq |(t', f_*(s))|_R$.

(A′) *If there exists* $t \in f_*^{-1}(t')$ *such that* $t \Rightarrow s(R(V, L))$, *then* $t' \Rightarrow f_*(s)\ (R(U, M))$ *and* $|(t', f_*(s))|_R \leqq |(t, s)|_R$.

(B) *If* $(t_i')_{i=0}^{p}$ *is a* t'*-derivation of the string* $f_*(s)$ *in* $R(U, M)$, *then there exist* $t_i \in f_*^{-1}(t_i')$ *for* $i = 0, 1, \ldots, p$, $t_p = s$ *such that* $(t_i)_{i=0}^{p}$ *is* t_0*-derivation of the string* s *in* $R(V, L)$ *such that* $\|(t_i)_{i=0}^{p}\|_R \leqq \|(t_i')_{i=0}^{p}\|_R$.

(B′) *If* $t \in f_*^{-1}(t')$ *and if* $(t_i)_{i=0}^{p}$ *is a* t*-derivation of the string* s *in* $R(V, L)$, *then* $(f_*(t_i))_{i=0}^{p}$ *is a* t'*-derivation of the string* $f_*(s)$ *in* $R(U, M)$ *such that* $\|(f_*(t_i))_{i=0}^{p}\|_R \leqq \|(t_i)_{i=0}^{p}\|_R$.

(C) *If* $t' \Rightarrow f_*(s)\ (R(U, M))$, *then there exists* $t \in f_*^{-1}(t')$ *such that* $t \overset{*}{\Rightarrow} s(R(V, L))$ *and* $\|(t, s)\|_R \leqq \|(t', f_*(s))\|_R$.

(C′) *If* $t \in f_*^{-1}(t')$ *and* $t \overset{*}{\Rightarrow} s(R(V, L))$, *then* $t' \overset{*}{\Rightarrow} f_*(s)\ (R(U, M))$ *and* $\|(t', f_*(s))\|_R \leqq \|(t, s)\|_R$.

Proof. 0. If f is a strong homomorphism of the language (V, L) onto (U, M), then it is also a strong homomorphism of the generalized grammar $\langle V, V, L, R(V, L)\rangle$ onto $\langle U, U, M, R(U, M)\rangle$. It follows from 3.30.

1. Assume $(x', y') \in R(U, M)$, $t' \Rightarrow f_*(s)\ (\{(x', y')\})$ and $|(t', f_*(s))|_R = |y'|$. According to 3.6 there exist $t \in f_*^{-1}(t')$, $x \in f_*^{-1}(x')$, $y \in f_*^{-1}(y')$ that $t \Rightarrow s(\{(x, y)\})$. It follows $|(t, s)|_R \leqq |y| = |y'| = |(t', f_*(s))|_R$ and (A) holds.

1′. Assume $(x, y) \in R(V, L)$, $t \Rightarrow s(\{(x, y)\})$ and $|(t, s)|_R = |y|$. According to 3.6 there is $t' \Rightarrow f_*(s)\ (\{(f_*(x), f_*(y))\})$ and, further, $|(t', f_*(s))|_R \leqq |f_*(y)| = |y| = |(t, s)|_R$ and (A′) holds.

2. We put $t_p = s$. Then $\|(t_i)_{i=p}^{p}\|_R = 0 = \|(t_i')_{i=p}^{p}\|_R$. Suppose $0 < k \leqq p$ and assume that we have such $t_i \in f_*^{-1}(t_i')$ for $i = k, k + 1, \ldots, p$ that $(t_i)_{i=k}^{p}$ is a t_k-derivation of the string s in $R(V, L)$ with the property $\|(t_i)_{i=k}^{p}\|_R \leqq \|(t_i')_{i=k}^{p}\|_R$. Then $t_{k-1}' \Rightarrow f_*(t_k)\ (R(U, M))$. According to (A) there exists $t_{k-1} \in f_*^{-1}(t_{k-1}')$ such that

$t_{k-1} \Rightarrow t_k(R(V, L))$ and $|(t_{k-1}, t_k)|_R \leqq |(t'_{k-1}, t'_k)|_R$. It follows $\|(t_i)_{i=k-1}^p\|_R = \max\{|(t_{k-1}, t_k)|_R, \|(t_i)_{i=k}^p\|_R\} \leqq \max\{|(t'_{k-1}, t'_k)|_R, \|(t'_i)_{i=k}^p\|_R\} = \|(t'_i)_{i=k-1}^p\|_R$. Assertion (B) could be proved by induction.

2'. There is $\|(f_*(t_i)_{i=0}^0\|_R = 0 = \|(t_i)_{i=0}^0\|_R$. Suppose $0 \leqq k < p$ and assume $\|(f_*(t_i))_{i=0}^k\|_R \leqq \|(t_i)_{i=0}^k\|_R$. There exists the string $t_k \in f_*^{-1}(t'_k)$ such that $t_k \Rightarrow t_{k+1}(R(V, L))$. Then $f_*(t_k) \Rightarrow f_*(t_{k+1})$ $(R(U, M))$ and $|(f_*(t_k), f_*(t_{k+1}))|_R \leqq |(t_k, t_{k+1})|_R$ according to (A'). It follows $\|(f_*(t_i))_{i=0}^{k+1}\|_R = \max\{\|(f_*(t_i)_{i=0}^k\|_R, |(f_*(t_k), f_*(t_{k+1}))|_R \leqq \max\{\|(t_i)_{i=0}^k\|_R, |(t_k, t_{k+1})|_R\} = \|(t_i)_{i=0}^{k+1}\|_R$. Assertion (B') could be proved by induction.

3. Let be $(t'_i)_{i=0}^p$ a t'-derivation of the string $f_*(s)$ in $R(U, M)$ such that $\|(t', f_*(s))\|_R = \|(t'_i)_{i=0}^p\|_R$. According to (B) there exist $t_i \in f_*^{-1}(t'_i)$ for $i = 0, 1, \dots, p$, $t_p = s$ such that $(t_i)_{i=0}^p$ is a t_0-derivation of the string s in $R(V, L)$ with the property $\|(t_i)_{i=0}^p\|_R \leqq \|(t'_i)_{i=0}^p\|_R$. For $t = t_0$, it follows that $\|(t, s)\|_R \leqq \|(t_i)_{i=0}^p\|_R$ and this is the proof of the assertion (C).

3'. Let $(t_i)_{i=0}^p$ be a t-derivation of the string s in $R(V, L)$ such that $\|(t, s)\|_R = \|(t_i)_{i=0}^p\|_R$. According to (B'), $(f_*(t_i))_{i=0}^p$ is a t'-derivation of the string $f_*(s)$ in $R(U, M)$ such that $\|(f_*(t_i))_{i=0}^p\|_R \leqq \|(t_i)_{i=0}^p\|_R$. It follows that $\|(t', f_*(s))\|_R \leqq \|(f_*(t_i))_{i=0}^p\|_R \leqq \|(t_i)_{i=0}^p\|_R = \|(t, s)\|_R$ and this is the proof of the assertion (C').

3.32 Lemma. *Let* (V, L), (U, M) *be languages, f a strong homomorphism of* (V, L) *onto* (U, M). *Then* $B_R(V, L) = f_*^{-1}(B_R(U, M))$ *and* $\|z\|_{R(V,L)} = \|f_*(z)\|_{R(U,M)}$ *for every* $z \in L$.

Proof. 1. It holds that $B_R(V, L) \subseteq f_*^{-1}(B_R(U, M))$. Indeed, suppose $z \in B_R(V, L)$. Then we have $z \in L$ and consequently $f_*(z) \in M$. Suppose $s' \in M$ and $s' \overset{*}{\Rightarrow} f_*(z)$ $(R(U, M))$. According to 3.31 (C) there exists $s \in f_*^{-1}(s')$ such that $s \overset{*}{\Rightarrow} z(R(V, L))$. We have $s \in L$ and it follows $|s| = |z|$. Further, it implies $|s'| = |f_*(s)| = |s| = |z| = |f_*(z)|$. It follows $f_*(z) \in B_R(U, M)$ and, thus, $z \in f_*^{-1}(B_R(U, M))$. This proves immediatly the assertion.

2. It holds that $f_*^{-1}(B_R(U, M)) \subseteq B_R(V, L)$. Indeed, suppose $z \in f_*^{-1}(B_R(U, M))$. Then we have $z \in f_*^{-1}(M) = L$. Suppose $s \in L$, $s \overset{*}{\Rightarrow} z(R(V, L))$. According to (C'), it holds that $f_*(s) \overset{*}{\Rightarrow} f_*(z)$ $(R(U, M))$ and $f_*(s) \in M$, $f_*(z) \in B_R(U, M)$. Thus, $|f_*(s)| = |f_*(z)|$ and it follows $|s| = |f_*(s)| = |f_*(z)| = |z|$. Therefore $z \in B_R(V, L)$.

3. Assertions 1 and 2 imply that $B_R(V, L) = f_*^{-1}(B_R(U, M))$.

4. For every $z \in L$, the condition $\|z\|_{R(V,L)} \leqq \|f_*(z)\|_{R(U,M)}$ holds. Indeed if $z \in L$ then $f_*(z) \in M$ and there exists $s' \in B_R(U, M)$ such that $s' \overset{*}{\Rightarrow} f_*(z)$ $(R(U, M))$ and $\|(s', f_*(z))\|_{R(U,M)} = \|f_*(z)\|_{R(U,M)}$. According to 3.31 (C) there exists $s \in f_*^{-1}(s')$ such that $s \overset{*}{\Rightarrow} z(R(V, L))$ and $\|(s, z)\|_{R(V,L)} \leqq \|(s', f_*(z))\|_{R(U,M)} = \|f_*(z)\|_{R(U,M)}$. Now, we have $s \in f_*^{-1}(B_R(U, M)) = B_R(V, L)$ according to 3 and therefore $\|z\|_{R(V,L)} \leqq \|(s, z)\|_{R(V,L)} \leqq \|f_*(z)\|_{R(U,M)}$.

5. For every $z \in L$ the condition $\|f_*(z)\|_{R(U,M)} \leqq \|z\|_{R(V,L)}$ holds. Indeed there exists $s \in B_R(V, L)$ such that $s \overset{*}{\Rightarrow} z(R(V, L))$ and $\|(s, z)\|_{R(V,L)} = \|z\|_{R(V,L)}$. According to 3.31 (C'), we have $f_*(s) \overset{*}{\Rightarrow} f_*(z)$ $(R(U, M))$ and $\|(f_*(s), f_*(z))\|_{R(U,M)} \leqq \|(s, z)\|_{R(V,L)} = \|z\|_{R(V,L)}$. Now, we have $f_*(s) \in f_*(B_R(V, L)) = f_*(f_*^{-1}(B_R(U, M))) = B_R(U, M)$ according to 3 and therefore $\|f_*(z)\|_{R(U,M)} \leqq \|(f_*(s), f_*(z))\|_{R(U,M)} \leqq \|z\|_{R(V,L)}$.

6. It follows from 4 and 5 that $\|f_*(z)\|_{R(U,M)} = \|z\|_{R(V,L)}$ for every $z \in L$.

3.33. Lemma. *Let* (V, L), (U, M) *be languages, f a strong homomorphism* (V, L) *onto* (U, M). *Then, for every* $x, y \in V^*$, *the following assertions hold:*

(A) $(y, x) \in X_R(V, L)$ *iff* $(f_*(y), f_*(x)) \in X_R(U, M)$.
(B) $(y, x) \in Z_R(V, L)$ *iff* $(f_*(y), f_*(x)) \in Z_R(U, M)$.

Proof. Suppose $x, y \in V^*$, $(y, x) \in X_R(V, L)$. Then, by 3.24, we have $(y, x) \in \in R(V, L)$ and $t \in L$ implies $| x | > \| t \|_{R(V,L)}$. By 3.30, the condition $(f_*(y), f_*(x)) \in \in R(U, M)$ holds. Now suppose $z \in M$. It follows from the definition of a strong homomorphism that there exists a string $z' \in L$ such that $f_*(z') = z$. For this string, the condition $| x | > \| z' \|_{R(V,L)}$ holds. By 3.32, we have $\| z' \|_{R(V,L)} = \| f_*(z') \|_{R(U,M)} = \| z \|_{R(U,M)}$. Thus, it holds that $| f_*(x) | > \| z \|_{R(U,M)}$ for every $z \in M$. It means that $(f_*(y), f_*(x)) \in X_R(U, M)$. Now, suppose $x, y \in V^*$, $(f_*(y), f_*(x)) \in X_R(U, M)$. Then $(f_*(y), f_*(x)) \in R(U, M)$ and, by 3.30, it follows $(y, x) \in R(V, L)$. If $t \in L$ then $f_*(t) \in M$ and $| f_*(x) | > \| f_*(t) \|_{R(U,M)}$. By 3.32, $\| f_*(t) \|_{R(U,M)} = \| t \|_{R(V,L)}$ and therefore $| x | > \| t \|_{R(V,L)}$ for every $t \in L$. It means that $(y, x) \in X_R(V, L)$.
This is the proof of the assertion (A).
The assertion (B) follows from (A) and 3.30.

3.34. Theorem. *Let (V, L), (U, M) be languages, f a strong homomorphism (V, L) onto (U, M). Then f is a strong homomorphism $K_R(V, L)$ onto $K_R(U, M)$.*
Proof. The proof follows from 3.32 and 3.33.

3.35. Theorem. *Let (V, L), (U, M) be languages.*

(A) *If f is a strong homomorphism of $K_E(V, L)$ onto $K_E(U, M)$, then f is also the strong homomorphism of the language (V, L) onto (U, M).*
(B) *If f is a strong homomorphism of $K_R(V, L)$ onto $K_R(U, M)$, then f is also the strong homomorphism of the language (V, L) onto (U, M).*
Proof. The proof follows from 3.6 and 3.29.

3.36. Theorem. *Let (V, L), (U, M) be languages, f a surjection V onto U. Then f is a strong homomorphism of the language (V, L) onto (U, M) iff f is the strong homomorphism of $K_R(V, L)$ onto $K_R(U, M)$.*
Proof. The proof follows immediately by 3.34 and 3.35.

4. FINITELY SEMIGENERATED LANGUAGES

4.1. Definition. A language (V, L) is called finitely semigenerated if the sets V, $B_E(V, L)$, $Z_E(V, L)$ are finite.

A language (V, L) is called *finitely R-semigenerated* if the sets V, $B_R(V, L)$, $Z_R(V, L)$ are finite.

4.2. Theorem. *Let (V, L) be a language. Then*
(A) *(V, L) is finitely semigenerated iff the following two conditions are satisfied:*
  (a) *The sets V, $B_E(V, L)$ are finite.*
  (b) *There exists a number N such that $\| z \|_E \leqq N$ for every $z \in L$.*
(B) *(V, L) is finitely R-semigenerated iff the following two conditions are satisfied:*
  (a) *The sets V, $B_R(V, L)$ are finite.*
  (b) *There exists a number N such that $\| z \|_R \leqq N$ for every $z \in L$.*

Proof. 1. If the language (V, L) is finitely semigenerated, the sets V, $B_E(V, L)$, $Z_E(V, L)$ are finite. We put $N = max\{| q |; (p, q) \in Z_E(V, L)\}$. Let us have an arbitrary $z \in L$. By 3.27, there exists $s \in B_E(V, L)$ such that $s \stackrel{*}{\Rightarrow} z(Z_E(V, L))$. Let $(s_i)_{i=0}^n$ be an s-derivation of z in $Z_E(V, L)$.

Then $|(s_{i-1}, s_i)|_E \leqq N$ for $i = 1, 2, \ldots, n$. If follows $||(s_i)_{i=0}^n||_E \leqq N$ and that implies $||(s, z)||_E \leqq N$ and finally $||z||_E \leqq N$.

2. Let V, $B_E(V, L)$ be finite and suppose the existence of a number N such that $||z||_E \leqq N$ for every $z \in L$. Let us have an arbitrary $(p, q) \in Z_E(V, L)$. Then there exists $z \in L$ such that $|p| \leqq |q| \leqq ||z||_E \leqq N$. It follows that the set $Z_E(V, L)$ is finite.

That is the proof of the assertion (A).

The assertion (B) could be proved in a similar way.

4.3. Theorem. *Let (V, L) be a language. Then (V, L) is finitely semigenerated iff it is finitely R-semigenerated.*

Proof. The proof follows from 3.15, 3.22 and 4.2.

4.4. Theorem. *Let (V, L), (U, M) be languages, f a strong homomorphism of (V, L) onto (U, M).*

(A) *If (V, L) is finitely R-semigenerated, then (U, M) is also finitely R-semigenerated.*

(B) *If V is a finite set and (U, M) a finitely R-semigenerated language, then (V, L) is also finitely R-semigenerated.*

Proof. The proof follows from 3.32 and 4.2.

4.5. Corollary. *Let (V, L), (U, M) be languages, f a strong homomorphism of (V, L) onto (U, M).*

(A) *If (V, L) is finitely semigenerated, then (U, M) is also finitely semigenerated.*

(B) *If V is a finite set and (U, M) a finitely semigenerated language, then (V, L) is also finitely semigenerated.*

It follows by 4.3 and 4.4.

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