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DISTINGUISHING SUBSETS IN SEMILATTICES

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1. INTRODUCTORY DEFINITIONS AND LEMMAS

1.1 Definition. A semilattice is a set G with an idempotent, commutative, and associative binary operation $\circ$ which assigns to each pair $(x, y) \in G$ a single element $x \circ y \in G$.

1.2 Lemma. *Let G be a join-semilattice (a semilattice under $\cup$). Then G is partially ordered set (poset) where the partial ordering $\leqq$ is defined by the following condition: $x \leqq y$ iff $x \circ y = y$. For all $x, y \in G$, we have $x \cup y = x \circ y$.* (Proof for lattices see [1], Theorem 2.)

1.3 Definition. Let G be a poset, $E \subseteq G$. The set E is called an *end of* G if, for all elements $x \in E$ and $y \in G$, the condition $x \leqq y$ implies $y \in E$.

1.4 Lemma. *Let G be a join-semilattice, $E \subseteq G$ its end. Then E is a join-subsemilattice in G.*

Proof. Let $x, y \in E$. Then $x \circ y \geqq x$ which implies $x \circ y \in E$.

1.5 Definition. Let G be a semigroup, Θ a equivalence relation on G. The relation Θ is called a congruence relation if for all $a, b, c, d \in G$ the conditions $a\Theta b$, $c\Theta d$ imply $a \circ c\Theta b \circ d$.

1.6 Agreement. Let Θ be a congruence relation on a semigroup G. We denote the elements of G/Θ by capital letters $X, Y, \ldots, W$.

1.7 Remark. Let Θ be a congruence relation on a semigroup G. For each $X \in G/\Theta$ and each $Y \in G/\Theta$ there exists such a $Z \in G/\Theta$ that $X \circ Y = \{x \circ y; x \in X, y \in Y\} \subseteq Z$. We put $Y \bullet Y = Z$. (See [4] page 188.)

1.8 Lemma. *Let G be a join-semilattice, Θ a congruence relation on G. The set G/Θ is a join-semilattice.* (See [4] page 189.)

1.9 Lemma. *Let G be a join-semilattice, Θ a congruence relation on G, $X, Y \in G/\Theta$. Let $\leqq$ be an ordering on G/Θ generated by the join-semilattice operation $\bullet$. Then $X \leqq Y$ if for each $x \in X$ there exists $y \in Y$ such that $x \leqq y$.*

Proof. Let $X \leqq Y$. Then $X \bullet Y = Y$ and hence $X \circ Y \subseteq Y$. For arbitrary elements $x \in X$, $y \in Y$, we have $x \circ y \in Y$ and $x \leqq x \circ y$. Now we suppose that for each $x \in X$ there exists an element $y \in Y$ such that $x \leqq y$. Hence $x \circ y = y$ and therefore $X \circ Y \subseteq Y$. The last inclusion is, by 1.7, equivalent to the equation $X \bullet Y = Y$ and $X \leqq Y$.

1.10 Lemma *Let G be a join-semilattice, Θ a congruence relation on G. Then each Θ-class is a join-subsemilattice in G.*

Proof. For lattices see [4] Theorem 75.

1.11 Lemma. *Let G be a join-semilattice, Θ a congruence relation on G. Let $X, Y \in G/\Theta$ be such that $X \leqq Y$. Then $y \circ X \subseteq Y$ holds for each $y \in Y$.*

Proof. Let $x \in X$ be arbitrary. Then there exists an element $z \in Y$ such that $x \leqq z$.

It holds $x \circ z = z \in Y$. Simultaneously $x \circ y \Theta x \circ z$ and we have $x \circ y \in Y$. Hence $y \circ X \subseteq Y$.

1.12 Definition. Let G be either a jon-semilattice or a monoid, $L \subseteq G$ its subset. For $x, y \in G$ we put $(x, y) \in \Xi_{(G, L)}$ if, for each $u, v \in G$, the condition $u \circ x \circ v \in L$ is equivalent to $u \circ y \circ v \in L$.

Some well known results concerning monoids can be formulated for join-semilattices.

1.13 Lemma. *A relation $\Xi_{(G, L)}$ is a congruence relation on the join-semilattice G.*

Proof. See [5] page 386. (The proof is given for monoids).

1.14 Remark. Let G be a join-semilattice, Θ a congruence relation on G. Then Θ is called *principal* if there is a set $L \subseteq G$ such that $\Theta = \Xi_{(G, L)}$. (The definition of principal congruences on semigroups see [6] page 530.)

1.15 Lemma. *Let G be a join-semilattice, $L \subseteq G$ its subset and $X \in G/\Xi_{(G,L)}$.* If $X \cap L \neq \emptyset$, then $X \subseteq L$.

Proof. Let $x \in X$. There exists $y \in X \cap L$. It is $x \Xi_{(G,L)} y$ and $y = y \circ y \in L$ hence $x \circ y \in L$ and also $x = x \circ x \in L$. Thus $X \subseteq L$.

1.16 Corollary. *Let G be a join-semilattice, $L \subseteq G$. Then $L = \bigcup \{X;\ X \in G/\Xi_{(G,L)}$ $X \cap L \neq \emptyset\}$*

1.17 Definition. Let G be a semigroup, $L \subseteq G$ a set, $u \in G$. We say that the elements $x, y \in G$, $x \neq y$, *are distinguished by u with respect to L* if the conditions $u \circ x \in L$, $u \circ y \notin L$ are equivalent. We say that *L distinguishes G* and we write $L \delta G$ if, for each $x, y \in G$, $x \neq y$, there is $u \in G$ such that x, y are distinguished by u with respect to L.

It is easy to prove the following two Theorems. The proofs are similar to the proof of the Theorem 2.6 in [7].

1.18 Theorem. *Let G be a monoid, $L \subseteq G$, Θ a congruence relation on G. Then the following two assertions are equivalent:*

(A) $\Theta = \Xi_{(G,L)}$.

(B) *There exists a subset* **L** *in G/Θ such that $L = \bigcup_{X \in \mathbf{L}} X$ and* **L** *distinguishes G/Θ.*

1.19 Theorem. *Let G be a join-semilattice, $L \subseteq G$, a congruence relation on G. Then the following two assertions are equivalent:*

(A) $\Theta = \Xi_{(G,L)}$.

(B) *There exists a subset* **L** *in G/Θ such that $L = \bigcup_{X \in \mathbf{L}} X$ and* **L** *distinguishes G/Θ.*

1.20 Remark. It is not possible to formulate previous Theorems as one Theorem for semigroups.

1.21 Example. Let B be a semigroup with two elements 0 and a with the following operation: $a \circ a = 0$, $a \circ 0 = 0$, $0 \circ a = 0$, $0 \circ 0 = 0$. Let us put $L = \{a\}$. For all $u, v \in B$ $u \circ a \circ v = 0 \in B - L$, $u \circ 0 \circ v = 0 \in B - L$ and hence $a\ \Xi_{(B, L)}\ 0$. The congruence relation has only one class which is equal to B. Hence the equation $L = \bigcup \{X;\ X \in G/\Xi_{(G,L)},\ X \cap L \neq \emptyset\}$ does not hold.

2. JOIN-SEMILATTICES WITH THE PROPERTY (β)

2.1 Definition. Let G be a join-semilattice. We say that *G has the property* (β) or that *G is of the type* (β) if it has the greatest element i and for each pair $x, y \in G$, $x \neq y$, for which $x \circ y < i$ there exists an element $z \in G$ such that either $x < z$ and simultaneously $z \parallel y$ or $y < z$ and simultaneously $z \parallel x$.

2.2 Lemma. *Let G be a join-semilattice of the type (β) satisfying the maximum condition. Then for each pair $x, y \in G$, $x \neq y$ there exists an element $u \in G$ such that either $x \circ u = i$, $y \circ u \neq i$ or $x \circ u \neq i$, $y \circ u = i$ holds.*

Proof. Let $x, y \in G$.

I. Let $x \circ y = i$. For $x \neq y$, it is $x \neq i$ or $y \neq i$; let us suppose the first case. Then it is sufficient to put $u = x$.

II. Let $x \circ y < i$. Let us denote by the letter a that of the elements x, y to which there exists an element $z_0 \in G$ such that $a < z_0$, and such that it is incomparable with the other of the elements x, y. We denote the other element by b. It is obvious that $z_0 < i$.

α) Let $z_0 \circ b = i$. We put then $u = z_0$ and we get $a \circ u = a \circ z_0 = z_0 < i$, $b \circ u = b \circ z_0 = i$.

β) Let $z_0 \circ b \neq i$. We consider the pair z_0, $b \circ z_0$. To this pair there exists an element $z_1 < i$ for which $z_0 < z_1$, $z_1 \| b \circ z_0$. If $b \circ z_1 = i$, then we put $u = z_1$. In the reverse case we construct an element z_2 by similar way as element z_1 with the property $a < z_0 < z_1 < z_2 < i$, $b \| z_2$. As G satisfies the maximum condition we attain, in a finite number of steps, an element z_n such that $a \circ z_n < i$, $b \circ z_n = i$.

2.3 Corollary. Let G be a join-semilattice with the property (β) satisfying the maximum condition. *Then $\{i\}$ distinguishes G.*

2.4 Lemma. *Let G be a join-semilattice with the greatest element i. Suppose $\{i\}\delta G$. Then G has the property (β).*

Proof. Let us admit that G has not the property (β). Then there exist $x, y, x \neq y$, $x \circ y < i$ such that every $z > x$ is comparable with y and every $z > y$ is comparable with x. There are two possibilities.

I. The elements x, y are comparable, for instance $x < y$. Then for each $z > x$ either $z \leqq y$ or $z > y$ holds. Let $u \in G$ be arbitrary. If $u \circ x = i$, then it is obvious $u \circ y = i$, too. Let $u \circ y = i$, $u \circ x < i$. If $u \circ x = x$, then $u \leqq x$ hence $u \leqq y$ and $u \circ y = y = x \circ y < i$. It is a contradiction. Therefore $x < u \circ x$. Hence $u \circ x \leqq y$ or $u \circ x \geqq y$. In the first case $u \leqq u \circ x \leqq y$ and then $u \circ y = y = x \circ y < i$. It is a contradiction, too. In the second case $i = u \circ y \leqq (u \circ x) \circ y = u \circ x < i$ and it is again a contradiction. We get that $u \circ y = i$ implies $u \circ x = i$.

II. The elements x, y are incomparable. Then the element $z > x$ is comparable with y, it is $z \leqq y$ or $z > y$. The first case implies $x < z \leqq y$ and it is impossible. Therefore $z > x$ implies $z > y$ and conversely. Let $u \in G$ be arbitrary. Let $u \circ x = i$, $u \circ y < i$. Then $u \circ y \geqq y$. If $u \circ y = y$, then it is $u \leqq y$ and hence $i = u \circ x \leqq y \circ x < i$ and this is a contradiction. Therefore $u \circ y > y$ and it implies $u \circ y > x$. Hence we get $i = u \circ x \leqq u \circ (u \circ y) = u \circ y < i$ and it is again a contradiction.

We get that for each $u \in G$ the relation $u \circ x = i$ implies $u \circ y = i$ and conversely $u \circ y = i$ implies $u \circ x = i$. This is a contradiction with the asumption that $\{i\}\delta G$. We have proved that G has the property (β).

2.5 Theorem. *Let G be a join-semilattice satisfying the maximum condition with the greatest element i. Then the following statements are equivalent:*

(A) *$\{i\}\delta G$.*

(B) *G has the property (β).*

2.6 Theorem. *Let G be a dually atomic join-semilattice with the greatest element i. Let M be a set of all dual atoms in G. Suppose $\{i\}\delta G$. Then $M\delta G$.*

Proof. Let $x, y \in G$, $x \neq y$. Since $\{i\}\delta G$, there is an element $u \in G$ such that $u \circ x = i$, $u \circ y \neq i$ or $u \circ x \neq i$, $u \circ y = i$. Let us denote by the letter a that element of

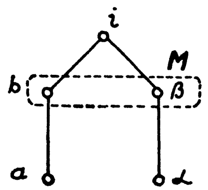
x, y for which the join with the element u is equal i and by the letter b the other element.

Let $u \circ a = i$, $u \circ b \in M$. Then the proof is finished.

Let $u \circ a = i$, $u \circ b \notin M$. G is a dually atomic semillattice and simultaneously $u \circ b < i$. There exists $p \in M$ for which $u \circ b < p$ and hence $(p \circ u) \circ b = p \circ (u \circ b) = p \in M$ and $(p \circ u) \circ a = p \circ (u \circ a) = p \circ i = i \notin M$. We have found $u' \in G$, $u' = u \circ p$ such that $u' \circ a \notin M$ and $u' \circ b \in M$. Thus $M\delta G$.

2.7 Remark. We cannot formulate theorem 2.6 as an equivalence.

2.8 Example. Let G be a join-semilattice with the following diagram:



Then $M = \{b, \beta\}$, $M\delta G$ but $\{i\}$ does not distinguish G.

2.9 Theorem. *Every Boolean algebra has the property* (β).

Proof. In the proof of this theorem we denote the operation $\circ$ by $\cup$.

Let B be Boolean algebra, $x, y \in B$, $x \neq y$. Let us choose the notation in such a way that $y \nleqq x$. If $x \cup y' = i$, then $y = y \cap i = y \cap (x \cup y') = y \cap x$ which implies $y \leqq x$ and we have a contradiction. Therefore $x \cup y' < i$, $y \cup y' = i$ and $\{i\}\delta B$. The statement follows from Lemma 2.4.

3. DISTINGUISHING SUBSETS IN JOIN-SEMILATTICES

3.1 Lemma. *Let G be a join-semilattice, $E \subseteq G$ its end and $M\delta E$. Let $x \in G - E$ and suppose the existence of at least one element $s \in E$ such that, for each $u \in E$, M contains either both elements $u \circ x$, $u \circ s$ or none of them. Then there is precisely one element s with this property.*

Proof. Suppose the existence of $s_1, s_2 \in E$, $s_1 \neq s_2$ with this property. Then, for each $u \in E$, the condition $u \circ s_1 \in M$ implies $u \circ x \in M$ which implies $u \circ s_2 \in M$ and conversely, $u \circ s_2 \in M$ implies $u \circ s_1 \in M$.

It is a contradiction to the hypothesis $M\delta E$.

3.2 Definition. Let G be a join-semilattice, $E \subseteq G$ its end, $M \subseteq E$, $M\delta E$, $x \in G - E$.

We put

$\mathscr{L}(E, M, x) =$
- $\{M, M \cup \{x\}\}$ if, for each $t \in E$, there is $u \in E$ such that M contains precisely one of the elements $u \circ x$, $u \circ t$.
- $\{M\}$ if there is $t \in E$ such that $t \circ x \in M$ and, for each $u \in E$, M contains either both elements $u \circ x$, $u \circ t$ or none of them.
- $\{M \cup \{x\}\}$ if there is $t \in E$ such that $t \circ x \notin M$ and, for each $u \in E$, M contais either both elements $u \circ x$, $u \circ t$ or none of them.

3.3 Lemma. *Let G be a join-semilattice, $E \subseteq G$ its end, $M \subseteq E$, $M\delta E$, $x \in G - E$. Then $\mathscr{L}(E, M, x)$ is the system of all sets L distinguishing $E \cup \{x\}$ such that $L \cap E = M$.*

Proof. We denote by $\mathscr{D}(E, M, x)$ the system of all sets L distinguishing $E \cup \{x\}$ such that $L \cap E = M$.

Clearly, $L \in \mathscr{D}(E, M, x)$ implies either $L = M$ or $L = M \cup \{x\}$.

(i) If $t, z \in E$, $t \neq z$, then there is $u \in E$ such that M contains precisely one of the elements $u \circ t$, $u \circ z$. The following cases can occur:

(1) For each $t \in E$, there is $u \in E$ such that M contains precisely one of the ele.-ents $u \circ x$, $u \circ t$.

We have $\mathscr{L}(E, M, x) = \{M, M \cup \{x\}\} \supseteq \mathscr{D}(E, M, x)$.

We prove $M\delta(E \cup \{x\})$.

Indeed, if $t,z \in E \cup \{x\}$, $x \neq t \neq z$, then we have the following two possibilities: (a) $t \neq x \neq z$ (b) $t \neq x = z$. In the case (a), the condition (i) implies the existence of $u \in E$ such that M contains precisely one of the elements $u \circ t$, $u \circ z$. In the case (b), the condition (1) implies the existence of $u \in E$ such that M contains precisely one of the elements $u \circ z = u \circ x$, $u \circ t$.

We prove $(M \cup \{x\})\ \delta(E \cup \{x\})$.

Indeed, if $t,z \in \{E \cup \{x\}\}$, $x \neq t \neq z$, then we have the following two possibilites: (a) $t \neq x \neq z$ (b) $t \neq x = z$. In the case (a), the condition (i) implies the existence of $u \in E$ such that M contains precisely one of the elements $u \circ t$, $u \circ z$. Since $u \circ t \neq$ $\neq x \neq u \circ z$ the set $M \cup \{x\}$ contains precisely one of the element $u \circ t$, $u \circ z$. In the case (b) the condition (1) implies the existence of $u \in E$ such that M contains precisely one of the elements $u \circ z = u \circ x$, $u \circ t$. Since $u \circ z = u \circ x \neq x \neq u \circ t$ the set $M \cup \{x\}$ contains precisely one of the elements $u \circ t$, $u \circ z$.

We have proved $\mathscr{L}(E, M, x) = \{M, M \cup \{x\}\} \subseteq \mathscr{D}(E, M, x)$.

Thus, $\mathscr{L}(E, M, x) = \mathscr{D}(E, M, x)$.

(2) There is precisely one element $s \in E$ such that $s \circ x \in M$ and, for each $u \in E$, M contains either both elements $u \circ x$, $u \circ s$ or none of them.

We have $\mathscr{L}(E, M, x) = \{M\}$.

We prove $M\delta(E \cup \{x\})$.

Indeed, if $t,z \in E \cup \{x\}$, $x \neq t \neq z$, then we have the following possibilities: (a) $t \neq x \neq z$ (b) $t \neq s$, $z = x$ (c) $t = s$, $z = x$. In the case (a), the condition (i) implies the existence of $u \in E$ such that M contains precisely one of the elements $u \circ t$, $u \circ z$. In the case (b), Lemma 3.1 implies the existence of $u \in E$ such that M contains precisely one of the elements $u \circ t$, $u \circ z = u \circ x$. In the case (c), we have $x \circ t = x \circ s \in M$, $x \circ z = x \circ x = x \notin M$.

We prove that $(M \cup \{x\})\ \delta(E \cup \{x\})$ does not hold. Indeed, $s \neq x$. If $u \in E$, then M contains either both elements $u \circ s$, $u \circ x$ or none of them by (2). Since $u \circ s \neq x \neq$ $\neq u \circ x$ for each $u \in E$ the set $M \cup \{x\}$ contains both elements $u \circ s$, $u \circ x$ or none of them.

Finally, $x \circ s \in M \subseteq M \cup \{x\}$, $x \circ x = x \in M \cup \{x\}$.

We have proved $\mathscr{D}(E, M, x) = \{M\}$.

It follows $\mathscr{L}(E, M, x) = \{M\} = \mathscr{D}(E, M, x)$.

(3) There is precisely one element $s \in E$ such that $s \circ x \notin M$ and, for each $u \in E$, the set M contains either both elements $u \circ x$, $u \circ s$ or none of them.

We have $\mathscr{L}(E, M, x) = \{M \cup \{x\}\}$.

We prove $(M \cup \{x\})\ \delta(E \cup \{x\})$.

Indeed, if $t,z \in E \cup \{x\}$, $x \neq t \neq z$, then we have the following possibilities: (a) $t \neq x \neq z$ (b) $t \neq s$, $z = x$ (c) $t = s$, $z = x$. In the case (a), the condition (i) implies the existence of $u \in E$ such that M contains precisely one of the elements $u \circ t$, $u \circ z$. Since $u \circ t \neq x \neq u \circ z$ the set $M \cup \{x\}$ contains precisely one of the elements $u \circ t$, $u \circ z$. In the case (b), Lemma 3.1 implies the existence of $u \in E$ such that M contains precisely one of the elements $u \circ t$, $u \circ z = u \circ x$. Since $u \circ t \neq x \neq \neq u \circ x = u \circ z$, the set $M \cup \{x\}$ contains precisely one of the elements $u \circ t$, $u \circ z$. In the case (c), we have $x \circ s \notin M$, $x \circ s \neq x$ which implies $x \circ s \notin M \cup \{x\}$, $x \circ x = x \in M \cup \{x\}$.

We prove that $M\delta(E \cup \{x\})$ does not hold.

Indeed, $s \neq x$. If $u \in E$, then M contains either both elements $u \circ s$, $u \circ x$ or none of them.

Finally, $x \circ s \notin M$, $x \circ x = x \notin M$.

We have proved $\mathscr{D}(E, M, x) = \{M \cup \{x\}\}$.

Thus, $\mathscr{L}(E, M, x) = \{M \cup \{x\}\} = \mathscr{D}(E, M, x)$.

The cases (1), (2), (3) represent all possibilities by 3.1. Thus, we have proved $\mathscr{L}(E, M, x) = \mathscr{D}(E, M, x)$ which is the assertion of the Lemma.

3.4 Definition. Let G be a join-semilattice, $L \subseteq G$. Then L is called *hereditary in G* if, for each end E of G, the condition $(E \cap L)\ \delta E$ is satisfied.

3.5 Remark. If G is a join-semilattice, E its end and L is hereditary subset then $E \cap L$ is hereditary in E.

Proof. Indeed, if F is an end of E, then it is an end of G which implies $(F \cap L)\ \delta F$. Since $F \subseteq E$ we have $F \cap (E \cap L) = F \cap L$. Thus $(F \cap (E \cap L))\ \delta F$.

3.6 Lemma. *Let G be a join-semilattice, $E \subseteq G$ its end, L a hereditary subset in E, x a maximal element in $G - E$, $M \subseteq E \cup \{x\}$ a subset such that $M\delta(E \cup \{x\})$, $E \cap M = L$. Then M is hereditary in $E \cup \{x\}$.*

Proof. Let $N \subseteq E \cup \{x\}$ be an end, $t, s \in N$, $t \neq s$. Since $t, s \in E \cup \{x\}$, there is $u \in E \cup \{x\}$ such that M contains precisely one of the elements $u \circ t$, $u \circ s$. It follows, especially, $u \circ t \neq u \circ s$. Clearly, $u \circ t$, $u \circ s \in N$. We can suppose, without loss of generality, that $u \circ s \neq x$.

(a) If $u \circ t \neq x \neq u \circ s$, then $u \circ t$, $u \circ s \in E$ which implies $u \circ t$, $u \circ s \in E \cap N$, the latter set being an end in E. Since L is hereditary in E, we have $(E \cap N \cap L)\ \delta(E \cap N)$. Since $L \subseteq E$, we have $E \cap N \cap L = N \cap L$. Thus $(N \cap L)\ \delta(E \cap N)$. It follows the existence of $v \in E \cap N$ such that $N \cap L$ contains precisely one of the elements $v \circ u \circ t$, $v \circ u \circ s$. Clearly, $v \circ u \circ t \neq x \neq v \circ u \circ s$. Since $N \cap L \subseteq N \cap M \subseteq N \cap (L \cup \{x\})$, the set $N \cap M$ contains precisely one of the elements $v \circ u \circ t$, $v \circ u \circ s$. Clearly $v \circ u \in N$.

(b) If $u \circ t = x \neq u \circ s$, we have $u \leqq x$, $t \leqq x$ which implies $u = t = x$. Thus, $x \neq x \circ s$, x, $x \circ s \in N$ and M contains precisely one of the elements $x = x \circ x$, $x \circ s$. Thus, $x \in N$ and $M \cap N$ contains precisely one of the elements $x \circ t = x \circ x$, $x \circ s$.

We have proved $(N \cap M)\ \delta N$ and M is hereditary in $E \cup \{x\}$.

3.7 Corollary. Let G be a join-semilattice, $E \subseteq G$ its end, L a hereditary subset in E, x a maximal element in $G - E$. Then each $M \in \mathscr{L}(E, L, x)$ is a hereditary subset in $E \cup \{x\}$ such that $M \cap E = L$.

Proof. By 3.3, each $M \in \mathscr{L}(E, L, x)$ distinguishes $E \cup \{x\}$ and $M \cap E = L$. Then M is hereditary in $E \cup \{x\}$ by 3.6.

3.8 Lemma. *Let G be a join-semilattice, $\mathscr{E}$ a chain consisting of ends in G which is ordered by inclusion, $\mathscr{L}$ a chain of subsets in G ordered by inclusion. Let f be a surjection of $\mathscr{E}$ onto $\mathscr{L}$ such, that, for each $E \in \mathscr{E}$, the set $L = f(E)$ is a hereditary subset in E. Suppose that f has the following property:*

(α) *If* $E, E' \in \mathscr{E}$, $E \subseteq E'$, *then* $f(E) = E \cap f(E')$. *Then* $\bigcup_{L \in \mathscr{L}} L$ *is a hereditary subset of* $\bigcup_{E \in \mathscr{E}} E$.

Proof. Let $P \subseteq \bigcup_{E \in \mathscr{E}} E$ be an end in $\bigcup_{E \in \mathscr{E}} E$. Suppose $s, t \in P$, $s \neq t$. Then there is $E_0 \in \mathscr{E}$ such that $s, t \in E_0$. We put $L_0 = f(E_0)$. Then $P \cap E_0$ is an end in E_0; it follows that $(P \cap E_0 \cap L)\, \delta(P \cap E_0)$. Thus, there is an element $u \in P \cap E_0$ such that $P \cap E_0 \cap L_0 = P \cap L_0$ contains precisely one of elements $u \circ s$, $u \circ t$. For instance, we can suppose $u \circ s \in P \cap L_0$, $u \circ t \notin P \cap L_0$. Since $P \cap L_0 \subseteq P \cap (\bigcup_{L \in \mathscr{L}} L)$ we have $u \circ s \in P \cap (\bigcup_{L \in \mathscr{L}} L)$.

Let us admit the existence of $E \in \mathscr{E}$ such that $u \circ t \in f(E) \cap P$. Since $t \in E_0$ we have $u \circ t \geqq t$ and $u \circ t \in E_0$. If $E \subseteq E_0$, then $f(E) = E \cap f(E_0) = E \cap L_0$ and $u \circ t \in f(E) \cap P = E \cap L_0 \cap P \subseteq P \cap L_0$ which is a contradiction. Thus, $E_0 \subseteq E$ which implies $f(E_0) = E_0 \cap f(E)$. It follows $u \circ t \in f(E) \cap P \cap E_0 = f(E_0) \cap P = P \cap L_0$ which is a contradiction.

Thus, $u \circ t \notin f(E) \cap P$ for each $E \in \mathscr{E}$ which implies $u \circ t \notin \bigcup_{E \in \mathscr{E}} (f(E) \cap P) = P \cap (\bigcup_{E \in \mathscr{E}} (E)) = P \cap (\bigcup_{L \in \mathscr{L}} L)$.

We have proved $(P \cap (\bigcup_{L \in \mathscr{L}} L))\, \delta P$ which is by Definition 3.3 the assertion of Lemma.

3.9 Lemma. *Let G be an ordered set satisfying the maximum condition. Then there is a set $\mathscr{E}$ of ends in G having the following properties:*

(i) *$\mathscr{E}$ is well ordered by inclusion; thus, there is an ordinal α such that $\mathscr{E} = \{E_\lambda;\ \lambda < \alpha + 1\}$ and, for $\lambda, \mu < \alpha$, the condition $E_\lambda \subseteq E_\mu$ is equivalent to $\lambda \leqq \mu$.*

(ii) $E_0 = \varnothing$ $E_\alpha = G$

(iii) *for each $\lambda < \alpha$ there is $a_\lambda \in G - E_\lambda$ which is maximal in $G - E_\lambda$ such that $E_{\lambda+1} - E_\lambda = \{a_\lambda\}$.*

(iv) $E_\gamma = \bigcup_{\lambda < \gamma} E_\lambda$ for each limit ordinal $\gamma < \alpha + 1$.

Proof. The assertion is clear if $G = \varnothing$. Thus we can suppose $G \neq \varnothing$. Let $\leqq$ denote the order relation in G. By [4], Theorem 2.3, there is a linear ordering $\preceq$ on G which is an extension of $\leqq$ such that G is well ordered by the dual ordering of $\preceq$. Thus there is an ordinal α and a sequence $(a_\lambda)_{\lambda<\alpha}$ of elements of G such that each element of G appears in this sequence precisely once and that, for $\lambda, \mu < \alpha$ the condition $a_\lambda \preceq a_\mu$ is equivalent to $\lambda \geqq \mu$. We put $E_\lambda = \{a_\varkappa;\ \varkappa < \lambda\}$ for each $\lambda \leqq \alpha$, $\mathscr{E} = \{E_\lambda;\ \lambda < \alpha + 1\}$. Then, for $\lambda, \mu < \alpha + 1$, $E_\lambda \subseteq E_\mu$ is equivalent to the condition $\lambda \leqq \mu$. Thus, $\mathscr{E}$ is isomorph to the set $\{\lambda;\ \lambda < \alpha + 1\}$ which implies that $\mathscr{E}$ is well ordered by set inclusion. If $\lambda < \alpha + 1$, $x \in E_\lambda$, $y \in G$, $x \leqq y$, then there are $\mu, \nu < \alpha$ such that $x = a_\mu$, $y = a_\nu$. Since $x \in E_\lambda$ we have $\mu < \lambda$. The condition $x \leqq y$ implies $x \preceq y$, i.e. $a_\mu \preceq a_\nu$ which implies $\nu \leqq \mu$. Thus, $\nu < \lambda$ and $y = a_\nu \in E_\lambda$. It follows that E_λ is an end with respect to the order relation $\leqq$ for each $\lambda < \alpha + 1$. We have (i). The condition (ii) holds obviously. Clearly, $E_{\lambda+1} - E_\lambda = \{a_\lambda\}$ for each $\lambda < \alpha$; suppose $x \in G - E_\lambda$, $a_\lambda \leqq x$. Then there is $\mu < \alpha + 1$ such that $x = a_\mu$ and $a_\lambda \preceq a_\mu$ which implies $\lambda \geqq \mu$. Clearly, $G - E_\lambda = \{a_\varkappa;\ \varkappa \geqq \lambda\}$. Thus $\mu = \lambda$ and $x = a_\lambda$ is maximal in $G - E_\lambda$. We have (iii). If $\gamma < \alpha + 1$ is a limit ordinal, then $E_\gamma = \{a_\varkappa;\ \varkappa < \gamma\} = \bigcup_{\lambda<\gamma} \{a_\varkappa;\ \varkappa < \lambda\} = \bigcup_{\lambda<\gamma} E_\lambda$ and we have (iv).

3.10 Definition. Let G be an ordered set satisfying the maximum condition. Then each set of ends in G having the properties (i), (ii), (iii), (iv) of Lemma **3.9** is called a *suitable set of ends in G*.

3.11 Definition. Let G be a join-semilattice satisfying the maximum condition, $\mathscr{E} = \{E_\lambda ; \lambda < \alpha + 1\}$ its suitable set of ends.

We put $L_0 = \varnothing$.

Let $0 < \beta < \alpha + 1$ and suppose that we have constructed, for any $\lambda < \beta$, a hereditary subset L_λ of E_λ in such a way that $\lambda < \mu < \beta$ implies $L_\lambda = E_\lambda \cap L_\mu$.

If β is an isolated ordinal, we put $E_\beta - E_{\beta-1} = \{a_{\beta-1}\}$ and we define $L_\beta \in \mathscr{L}(E_{\beta-1}, L_{\beta-1}, a_{\beta-1})$.

If β is a limit ordinal, we put $L_\beta = \bigcup_{\lambda<\beta} L_\lambda$.

By induction, we define L_λ for each $\lambda < \alpha + 1$. Especially, we put $L = L_\alpha$ and we say that L *has been constructed by means the suitable set of ends* $\mathscr{E}$.

3.12 Theorem. *Let G be a join-semilattice satisfying the maximum condition, $L \subseteq G$ a subset. Then the following conditions are equivalent:*

(A) *L is a hereditary subset in G.*

(B) *If $\mathscr{E}$ is an arbitrary suitable set of ends in G, then L has been constructed by means of $\mathscr{E}$.*

Proof. Let (A) hold. Let $\mathscr{E} = \{E_\lambda ; \lambda < \alpha + 1\}$ be an arbitrary suitable set of ends in G. We put $L_\lambda = E_\lambda \cap L$ for each $\lambda < \alpha + 1$.

Then $L_0 = E_0 \cap L = \varnothing$.

Let $0 < \beta < \alpha + 1$. By Remark 3.5, L_λ is a hereditary subset in E_λ for any $\lambda < \beta$ and $\lambda < \mu < \beta$ implies $L_\lambda = L \cap E_\lambda = L \cap E_\lambda \cap E_\mu = E_\lambda \cap L_\mu$.

If β is an isolated ordinal and if $E_\beta - E_{\beta-1} = \{a_{\beta-1}\}$, then L_β is hereditary in $E_\beta = E_{\beta-1} \cup \{a_{\beta-1}\}$ by Remark 3.5 which implies $L_\beta \delta(E_{\beta-1} \cup \{a_{\beta-1}\})$. Further, $L_\beta \cap E_{\beta-1} = L_{\beta-1}$ and $L_{\beta-1}\ \delta E_{\beta-1}$. By Lemma 3.3, we have $L_\beta \in \mathscr{L}(E_{\beta-1}, L_{\beta-1}, a_{\beta-1})$.

If β is a limit ordinal, then

$L_\beta = E_\beta \cap L = (\bigcup_{\lambda<\beta} E_\lambda) \cap L = \bigcup_{\lambda<\beta} (E_\lambda \cap L) = \bigcup_{\lambda<\beta} L_\lambda$.

Finally, $L_\alpha = E_\alpha \cap L = G \cap L = L$.

We have proved that L has been constructed by means of $\mathscr{E}$ which is (B).

Let (B) hold. Then, trivially, L_0 is a hereditary subset in E_0.

Let $0 < \beta < \alpha + 1$ and suppose that L_λ is hereditary in E_λ for each $\lambda < \beta$ and that $\mu < \lambda < \beta$ implies $L_\mu = E_\mu \cap L_\lambda$.

If β is an isolated ordinal, then $L_{\beta-1}$ is hereditary in $E_{\beta-1}$, $E_\beta - E_{\beta-1} = \{a_{\beta-1}\}$, $L_\beta \in \mathscr{L}(E_{\beta-1}, L_{\beta-1}, a_{\beta-1})$, $a_{\beta-1}$ is maximal in $G - E_{\beta-1}$. By Corollary 3.7, L_β is hereditary in $E_{\beta-1} \cup \{_{\beta-1}\} = E_\beta$ and $E_{\beta-1} \cap L_\beta = L_{\beta-1}$. If $\lambda < \beta$, then $\lambda \leqq \beta - 1$ and $E_\lambda \cap L_\beta = E_\lambda \cap E_{\beta-1} \cap L_\beta = E_\lambda \cap L_{\beta-1} = L_\lambda$ by the induction hypothesis.

If β is a limit ordinal, then $L_\beta = \bigcup_{\mu<\beta} L_\mu$ and L_β is hereditary in $\bigcup_{\mu<\beta} E_\mu = E_\beta$ by Lemma 3.8.

If $\lambda < \beta$, then $E_\lambda \cap L_\beta = E_\lambda \cap (\bigcup_{\mu<\beta} L_\mu) = \bigcup_{\mu<\beta} (E_\lambda \cap L_\mu) = \bigcup_{\mu\leqq\lambda} (E_\lambda \cap L_\mu) \cup \bigcup_{\lambda<\mu<\beta} (E_\lambda \cap L_\mu) = \bigcup_{\mu\leqq\lambda} (E_\lambda \cap L_\mu) \cup L_\lambda = L_\lambda$ because $E_\lambda \cap L_\mu \subseteq L_\mu = E_\lambda \cap L_\lambda \subseteq L_\lambda$ for each $\mu \leqq \lambda$.

We have proved that L_β is hereditary in E_β and that $\lambda < \beta$ implies $L_\lambda = E_\lambda \cap L_\beta$.

It follows by transfinite induction that L_λ is hereditary in E_λ for each $\lambda < \alpha + 1$. Especially, $L = L_\alpha$ is hereditary in $E_\alpha = G$, which is (A).

3.13 Corollary. *Let G be a join semilattice satisfying the maximum condition. Then there is a set $L \subseteq G$ such that $(E \cap L)\ \delta E$ for each end E of G.*

3.14 Remark. In [6] following definitions are given: A subset H of a semigroup G is called *indivisible by an equivalence* Θ (by a subset F) if H is contained in some class of Θ ($\Xi_{(G,F)}$). A subset H is called *disjunctive* if the only subsets indivisible by $\Xi_{(G,H)}$ are empty and one-element.

According to these definitions we can formulate the following Corollary:

3.15 Corollary. *Let G be a join-semilattice satisfying the maximum condition. Then there exists a set $L \subseteq G$ such that for each end $E \subseteq G$ the set $L \cap E$ is disjunctive.*

4. SPECIAL CONGRUENCES ON MONOIDS

4.1 Assumption. We shall suppose in the whole fourth paragraph that G is a monoid and Θ a congruence relation on G such that G/Θ is a join-semilattice satisfying the maximum condition. We denote its greatest element by I.

4.2 Definition. Let G/Θ have the property (β). Then we say that the congruence relation Θ *has the property* (β) or that Θ *is of the type* (β).

4.3 Theorem. *Let $L = I \in G/\Theta$. Then the following statements are equivalent:*

(A) $\Theta = \Xi_{(G,I)}$
(B) $\{I\}\ \delta(G/\Theta)$
(C) Θ *has the property* (β).

Proof. The statements (A) and (B) are equivalent according to Theorem 1.18. Simultaneously, by Theorem 2.6 the statements (B) and (C) are equivalent.

4.4 Theorem. *Let Θ be a (β) congruence on G satisfying the assumption* 4.1. *Let M be the set of dual atoms in G/Θ.* (The set of elements which are covered by I). *Then* $\Theta = \Xi_{(G,L)}$, *where* $L = \bigcup_{m \in M} m$.

Proof. From Theorem 4.3 follows that $\{I\}\ \delta(G/\Theta)$. So the conditions of Theorem 2.7 are satisfied and the set $L\delta(G/\Theta)$. By Theorem 1.18 we have $\Theta = \Xi_{(G,L)}$.

4.5 Main Theorem. *Let Θ be a congruence relation on G satisfying the assumption* 4.1. *Then there exists a subset $L \subseteq G$ such that $\Theta = \Xi_{(G,L)}$.*

Proof. According to Corollary 3.13 there exists a subset $\mathbf{L} \subseteq G/\Theta$ $\mathbf{L} = \{X;\ X \in G/\Theta,\ X \subseteq L\}$ in G/Θ which distinguishes G/Θ. Hence by Theorem 1.18 $\Theta = \Xi_{(G,L)}$ holds.

4.6 Corollary. *Let Θ be a congruence relation on G satisfying* 4.1. *Let $\bar{L} \subseteq G/\Theta$ be constructed by* 3.11. *Then $\Xi_{(G/\Theta,\bar{L})} = idG/\Theta$.*

Proof. We have $\bar{L}\delta G/\Theta$ which is equivalent to $\Xi_{(G/\Theta,\bar{L})} = idG/\Theta$ by [3] Theorem 1.7.

4.7 Theorem. *All congruence relations on a join-semilattice S satisfying the maximum condition are principal congruences.*

Proof. Join-semilattice S satisfies the maximum condition. All factor—join-semilattices on S satisfy also the maximum condition and they are join-semilatices. By Corollary 3.13 we obtain a subset $L \subseteq S/\Theta$ for all congruence relations on S which distinguishes S/Θ. Hence by 1.19 $\Theta = \Xi_{(S,L)}$ holds.

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REFERENCES

[1] Birkhoff G., *Generalized Arithmetic*, Duke Jour. 9 (1942), 283—302.
[2] Novák V., *On the Well Dimension of Ordered Sets*, Czechoslovak Mat. Journ. 19 (94), (1969), 1—16.
[3] Novotný M., *On Some Relations defined by Languages*, Prague Studies in Mathematical Linguistics 4, 1972, 157—170.
[4] Szász G., *Einfuhrung in die Verbandsteorie*, Budapest (1962).
[5] Шайн Б. М., *Вмещение полугрупп в обобщенные группы*, Мат. сборник, (Н.С) 55 (1961), 379—400
[6] Schein B. M., *Homomorphism and Subdirect Decompositions of Semigroups* Pac. Tom. of Mat. Vol. 17, No 3, 1966.
[7] Zapletal J., *Distinguishing Subsets of Semigroups and groups*, Arch. Math. Brno (1968), Tomus 4, Fasc. 4, 241—250.

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