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AN EXISTENCE THEOREM FOR PERIODIC BOUNDARY VALUE PROBLEMS FORSYSTEMS OF SECOND ORDER DIFFERENTIAL EQUATIONS

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1. Let I denote the interval $[0, 1]$, let R^n be n-dimensional Euclidean space, let $\Omega \subset R^n$ be a bounded, open, convex set, and assume that $f: I \times R^n \to R^n$ is continuous. Consider the system of second order differential equations

$$x'' = f(t, x) \qquad \left(' = \frac{d}{dt} \right) \tag{1}$$

together with the periodic boundary conditions

$$x(0) = x(1), \qquad x'(0) = x'(1). \tag{2}$$

The result to be established in this paper was motivated by papers by Bebernes and Schmitt [1], Hartman [2], Knobloch [3, 4], Mawhin [6], and Schmitt [7, 8] and to a certain extent unifies the results on periodic solutions presented in the above papers.

In order to avoid notational difficulties and for the sake of brevity we shall only consider equations of the form (1) rather than permitting x' dependence also. By consulting the papers mentioned above it will be clear how to formulate and prove similar results in this more general situation.

The result to be proved is the following.

Theorem. *Assume that, for every $x \in \partial\Omega$ and every outer normal n at x to Ω,*

$$n \,.\, f(t, x) > 0, \qquad 0 \leqq t \leqq 1 \tag{3}$$

Then there exists a solution $x(t)$ of (1), (2) *such that* $x(t) \in \Omega$, $0 \leqq t \leqq 1$.

Remark. Since Ω is convex and bounded, at every $x \in \partial\Omega$ a supporting hyperplane will exist; that is, there exists a vector n and a constant c such that the hyperplane $P = \{y : n \,.\, y = c\}$ has the property that $x \in P$ and $\Omega \subset \{y : n \,.\, y < c\}$. The vectors n in the definition of the hyperplane P are the "outer normals" n in the above theorem.

2. Let k be a positive integer, $g : I \times R^k \to R^k$ be continuous, and let Σ be an open, bounded subset of R^k. Consider the system

$$z' = g(t, z) \tag{4}$$

and assume that every solution of (4) emanating from $\bar{\Sigma}$ is defined on $[0, 1]$. A point

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$z_0 \in \partial\Sigma$ will be called a *nonrecurrence point* for (4) if every solution $z(t)$ of (4) with $z(0) = z_0$ is such that $z(t) \neq z_0$, $0 < t \leqq 1$. Under these assumptions, Krasnosel'skii established the following result which we will use as

Lemma 1. *Let $\partial\Sigma$ consist of nonrecurrence points only with respect to* (4). *Then there exists a solution $z(t)$ of* (4) *with $z(0) = z(1)$, $z(0) \in \Sigma$,* whenever *the topological degree* $\deg(g(0, z), \Sigma, 0) \neq 0$.

The proof of this result requires only elementary properties of the topological degree of a mapping and can be found in ([5], pp. 81–83).

Remark. In proving our theorem, we proceed as follows. First we modify our original differential equation in such a way that the first-order system equivalent to the modification and a properly chosen region Σ satisfy the conditions of lemma 1. This in turn will imply the existence of a solution of the periodic boundary value problem for the modified second-order system. Finally, this solution will be in fact a solution of the original system because of the construction of the modified equation.

3. We now proceed to define a modification $F(t, x)$ of $f(t, x)$.

1. If $x \in \bar{\Omega}$, let

$$F(t, x) = f(t, x), \qquad 0 \leqq t \leqq 1.$$

2. If $x \in \partial\Omega$, then by what has been observed there exists a unit outward normal n at x to Ω. Further, if n_1 and n_2 are unit outward normals at x and α, β are nonnegative, then $\alpha n_1 + \beta n_2$ will be an outward normal at x to Ω, because Ω is convex. Hence, all outward normals to Ω at x will define a "cone" at x, which we shall denote by $C(x)$; i.e., $C(x) = \{x + \alpha n : \alpha \geqq 0, n \text{ is a unit outward normal at } x \text{ to } \Omega\}$. Again by convexity of Ω, if $x, y \in \partial\Omega$, $x \neq y$, then $C(x) \cap C(y) = \varnothing$. Also, one can easily verify that $\bigcup_{x \in \partial\Omega} C(x) = \operatorname{comp} \Omega$. Now define, for $y \in \operatorname{comp} \Omega$,

$$F(t, y) = f(t, x), \qquad 0 \leqq t \leqq 1, \qquad \text{where } y \in C(x)$$

for some $x \in \partial\Omega$.

The function $F : I \times \mathrm{R}^n \to \mathrm{R}^n$ is well-defined, continuous, and bounded.

Next, we define, for $\lambda \geqq 0$, $\Omega_\lambda = \{x : \inf_{z \in \Omega} \| x - z \| < \lambda\}$.

Thus, Ω_λ is a λ-neighborhood of Ω. It is clear that Ω_λ is a bounded convex open set and that $\bigcup_{\lambda > 0} \Omega_\lambda = \mathrm{R}^n$. If $y \in \operatorname{comp} \Omega$, then $y \in \partial\Omega_\lambda$ for some $\lambda > 0$ and hence $y \in C(x)$ for some $x \in \partial\Omega$. This means that $y = x + \alpha n$ where n is an unit outward normal to Ω at x and $\alpha > 0$. We claim that n is also an unit outward normal to Ω_λ at y. Note that $n \cdot y = n \cdot x + \alpha = c + \alpha$ and $\alpha = \lambda$. If $z \in \Omega_\lambda$, then we consider two cases: (a) if $z \in \bar{\Omega}$, then $n \cdot z \leqq c \leqq c + \alpha$; (b) if $z \in \Omega_\lambda - \bar{\Omega}$, then $z = \bar{x} + \beta\bar{n}$ where $\bar{x} \in \partial\Omega$, $\beta > 0$, and $\bar{n}$ is an unit outward normal–space Ω at $\bar{x}$. Thus, $n \cdot z = n \cdot \bar{x} + \beta n \cdot \bar{n} \leqq c + \beta n \cdot \bar{n} \leqq c + \beta \leqq c + \alpha$ since $z \in \Omega_\lambda$ and $\beta < \lambda = \alpha$. This establishes our claim. From these observations, the following lemma is immediate.

Lemma 2. *For every* $\lambda \geqq 0$ $(\Omega_0 = \Omega)$,

$$n \cdot F(t, y) > 0, \qquad 0 \leqq t \leqq 1, \text{ where} \tag{5}$$

$y \in \partial\Omega_\lambda$ and n is any outward normal to Ω_λ at y.

Consider now the differential equation

$$y'' = F(t, y). \tag{6}$$

Lemma 3. *For every t_0, $0 < t_0 \leqq 1$, there exists a constant $N = N(t_0)$ such that if $y(t)$ is any solution of* (6) *with $y(0) \in \Omega$ and $\| y'(0) \| \geqq N$, then $y(t_0) \notin \bar{\Omega}$ and $y(t) \neq y(0)$, $0 < t \leqq t_0$.*

Proof. Using Taylor expansions, we may write

$$y(t) = y(0) + y'(0)\, t + (F_i(\xi_i, y(\xi_i)), \dots, F_n(\xi_n, y(\xi_n)))\, \frac{t^2}{2}, \qquad 0 < \xi_i < t,\ i = 1, \dots, n.$$

Thus, $\| y(t) - y(0) \| \geqq t \left(\| y'(0) \| - \frac{M}{2} \right)$, where M is a bound on $\| F \|$. Thus choose N large enough so that $N > \frac{M}{2}$ and that $t_0 \left(N - \frac{M}{2} \right)$ exceeds the diameter of Ω.

Lemma 4. *Let $y(t)$ be a solution of* (6) *such that $y(t_1) \notin \bar{\Omega}$ and $y(0) \in \Omega$. Then there exists t_0, $0 < t_0 < 1$, such that $y(t) \in \Omega$, $0 \leqq t < t_0$, and $y(t) \notin \bar{\Omega}$, $t_0 < t \leqq 1$.*

Proof. By continuity, there exists $t_0 \in (0, t_1)$ such that $y(t) \in \Omega$, $0 \leqq t \leqq t_0$ and $y(t_0) \in \partial\Omega$. For $t > t_0$, we write

$$y(t) = y(t_0) + y'(t_0)(t - t_0) + (F_i(\xi_i, y(\xi_i)), \dots, F_n(\xi_n, y(\xi_n)))\, \frac{(t - t_0)^2}{2},$$

where $t_0 < \xi < t$. Let n be an outward normal vector to Ω at $y(t_0)$ and let the supporting hyperplane at $y(t_0)$ have the equation $n \,.\, x = c$. Then $n \,.\, y(t) = c + (t - t_0)\, n \,.\, y'(t_0) + \frac{(t - t_0^2)}{2}\, n \,.\, (F_i(\xi_i, y(\xi_i)))$. Clearly, $n \,.\, y'(t_0) \geqq 0$ and by continuity of y and F we have that for $t - t_0$ sufficiently small $n \,.\, (F_i(\xi_i, y(\xi_i))) > 0$. Thus for $t > t_0$, $t - t_0$ sufficiently small $n \,.\, y(t) > c$ meaning that $y(t) \notin \bar{\Omega}$. It follows now from lemma 2 that $y(t)$ cannot reenter Ω.

4. We can now give a proof of our theorem. Let $z = (y, y')$, $g(t, z) = \binom{y'}{F(t, y)}$ and let $t_0 \in (0, 1]$ be given. Let N be a constant of lemma 3 and define $\Sigma = \Omega \times \{y' : \| y' \| < N\}$. Now consider

$$z' = g\,(t, z). \tag{7}$$

Since g is continuous and F is bounded, it follows that every solution of (7) which emanates from $\bar{\Sigma}$ extends to I. That the boundary of Σ consists of nonrecurrence points only follows from lemmas 3 and 4. Since $g(0, z) \neq 0$ on $\partial\Sigma$ it follows that $\deg\big(g(0, z), \Sigma, 0\big)$ is defined. From the basic properties of degree (see [9], Theorem 3.16, pp. 71–72), it follows that $\deg\big(g(0, z), \Sigma, 0\big) \neq 0$ if and only if $\deg\big(F(0, y), \Omega, 0\big) \neq 0$. Fix $x \in \Omega$ and consider

$$(1 - \lambda)(y - x) + \lambda F(0, y), \qquad 0 \leqq \lambda \leqq 1. \tag{8}$$

We claim that this is a homotopy. Let $y \in \partial\Omega$ and let n be an outward normal to Ω at y. Then $n \,.\, [(1 - \lambda)(y - x) + \lambda F(0, y)] = (1 - \lambda)\, c - (1 - \lambda)\, n \,.\, x + \lambda n \,.\, F(0, y) > (1 - \lambda)\, c - (1 - \lambda)\, c = 0$, $0 \leqq \lambda < 1$. Hence, the mapping is nonvanishing for $\lambda \in (0, 1)$ and therefore (8) defines a homotopy. By the homotopy invariance theorem of degree theory, $\deg(F(0, y), \Omega, 0) = \deg(y - x, \Omega, 0) \neq 0$. We therefore conclude that (7) has a solution $z(t)$ with $z(0) = z(1)$ and $z(0) \in \Sigma$. Hence, (6) has a solution $x(t)$ satisfying the periodic boundary conditions (2). Since $x(0) \in \Omega$ and $x(1) \in \Omega$, by lemma 4 it follows that $x(t) \in \Omega$. We now recall the definition of F to conclude that $x(t)$ is a solution of (1).

Remark. Using arguments analogous to those above, one may also prove that, under the conditions of our theorem, (1) has a solution $x(t)$ satisfying the boundary conditions

$$x(0) = A, \qquad x(1) = B \tag{9}$$

for any $A, B \in \Omega$.

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