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*Archivum Mathematicum*, Vol. 8 (1972), No. 3, 137--142

Persistent URL: http://dml.cz/dmlcz/104769

ARCH. MATH. 3, SCRIPTA FAC. SCI. NAT. UJEP BRUNENSIS, VIII: 137-142, 1972

# ON A CHARACTERIZATION OF THE LATTICE OF $\mathfrak{m}$-IDEALS OF AN ORDERED SET

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(Received February 1, 1972)

In [3] O. Frink defined an ideal of an ordered set. In [6] J. Mayer and M. Novotný replaced the notion of an ideal by a more general notion of an $\mathfrak{m}$-ideal. The ideals of Frink are precisely the $\aleph_0$-ideals. The ideals of Frink on lattices (semilattices) are identical with lattice- (semilattice-) ideals. Birkhoff and Frink proved in [2] that a lattice $L$ is isomorphic with the lattice of ideals of some lattice $P$ and only if $L$ is complete, meet-continuous, the set of join-inaccessible elements is a sublattice of $L$ and every element of $L$ is a join of this join-inaccessible elements. Nachbin proved in [5] that a lattice $L$ is isomorphic with the lattice of ideals of some up-semilattice if and only if $L$ is complete and compactly generated. It is known that the concept of complete, meet-continuous lattice every element of which is a join of join-inaccessible elements is equivalent with the concept of complete, compactly generated lattice.

This paper deals with the analogously characterization of the lattice of $\mathfrak{m}$-ideals of an ordered set. This characterization will be deduced from one general theorem given in [7] which characterizes an $\mathfrak{M}$-hull of an ordered set.

## § 1. BASIC CONCEPTS

By an ordered set we mean a partially ordered set. Put $[x) = \{t \in P/t \geqq x\}$, $(x] = \{t \in P/t \leqq x\}$ for every element $x$ of an ordered set $P$. Let $\mathfrak{m}$ be a cardinal number. An ordered set $P$ is called $\mathfrak{m}$-up-directed if for every subset $M$ of $P$ such that card $M < \mathfrak{m}$ there exists $x \in P$ with $x \geqq t$ for every $t \in M$. The system $\mathfrak{A}$ of subsets of a set $P$ is called a closure system if it contains the intersection of every of its subsystem. Especially $P \in \mathfrak{A}$ as $P$ is the intersection of the empty subsystem. Therefore $\mathfrak{A}$ ordered by the set inclusion is a complete lattice. A closure system $\mathfrak{A}$ defines a closure operator for $P$ such that $\bar{X} = \bigcap \{Y \in \mathfrak{A}/X \subseteqq Y\}$. A closure system is called $\mathfrak{m}$-inductive if it contains the union of every of its $\mathfrak{m}$-up-directed subsystem.

## § 2. $\mathfrak{m}$-JOIN-INACCESSIBLE AND $\mathfrak{m}$-COMPACT ELEMENTS

**2.1. Definition:** *Let $P$ be a complete lattice, $\mathfrak{m}$ an infinite cardinal number. An element $x \in P$ is called $\mathfrak{m}$-join-inaccessible if for every non-empty $\mathfrak{m}$-up-directed subset $M$ of $P$ such that $x = \sup M$ it holds $x \in M$.*

The set of $\mathfrak{m}$-join-inaccessible elements of $P$ we shall denote by $\hat{P}(\mathfrak{m})$.

**2.2. Definition:** *Let $P$ be a complete lattice and $\mathfrak{m}$ an infinite cardinal number. An element $x \in P$ is called $\mathfrak{m}$-compact if for every non-empty subset $M$ of $P$ such that $x \leqq$*

$\leqq \sup M$ *there exists a subset* $N$ *of* $M$ *such that* $0 < \text{card}\, N < \mathfrak{m}$ *and* $x \leqq \sup N$.

We define a complete lattice to be $\mathfrak{m}$-compactly generated when every element of $P$ is a join of $\mathfrak{m}$-compact elements of $P$.

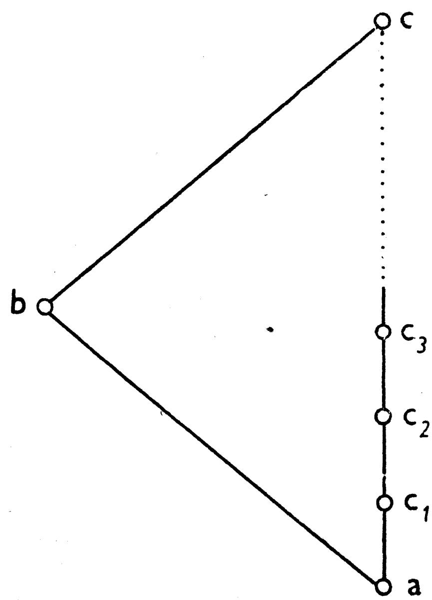
$\aleph_0$-join-inaccessible elements are usual join-inaccessible elements defined in [2]. $\aleph_0$-compact elements are compact elements from [5]. We are going to study relationships between $\mathfrak{m}$-join-inaccessible and $\mathfrak{m}$-compact elements. All results proved in this section are given in [1] for the case $\mathfrak{m} = \aleph_0$.

**2.3. Definition:** *Let* $\mathfrak{m}$ *be a cardinal number. A complete lattice is said to be* $\mathfrak{m}$*-meet-continuous if for every element* $x \in P$ *and every* $\mathfrak{m}$*-up-directed subset* $M \subseteqq P$ $x \wedge \bigvee_{t \in M} t = \bigvee_{t \in M} (x \wedge t)$ *holds.*

The $\aleph_0$-meet-continuity is a usual meet-continuity.

**2.4. Lemma:** *Let* $P$ *be a complete lattice,* $\mathfrak{m}$ *an infinite cardinal number. Then it holds:*
(*i*) *Every* $\mathfrak{m}$*-compact element of* $P$ *is* $\mathfrak{m}$*-join-inaccessible.*
(*ii*) *Let* $P$ *be* $\mathfrak{m}$*-meet-continuous and* $\mathfrak{m}$ *regular. Then every* $\mathfrak{m}$*-join-inaccessible element of* $P$ *is* $\mathfrak{m}$*-compact.*

Proof: (i) Let $x \in P$ be $\mathfrak{m}$-compact, $M$ non-empty $\mathfrak{m}$-up-directed subset of $P$ and $x = \sup M$. By the definition of compactness there exists a subset $N$ of $M$ such that $0 < \text{card}\, N < \mathfrak{m}$ and $x \leqq \sup N$. Since $M$ is $\mathfrak{m}$-up-directed, there exists $y \in M$ with $y \geqq t$ for every $t \in N$. Hence $y \geqq x$, i.e. $x = y \in M$. Thereby $x$ is $\mathfrak{m}$-join-inaccessible.
(ii) Let $x \in P$ be $\mathfrak{m}$-join-inaccessible, $P$ $\mathfrak{m}$-meet-continuous and $\mathfrak{m}$ regular. Let $\emptyset \neq M \subseteqq P$, $x \leqq \sup M$. Let $M' = \{\sup X / X \subseteqq M,\ 0 < \text{card}\, X < \mathfrak{m}\}$. Clearly $\sup M = \sup M'$ and $M'$ is $\mathfrak{m}$-up-directed as $\mathfrak{m}$ is regular. Thus $x = x \wedge \sup M' =$



$= \bigvee_{t \in M'} (x \wedge t)$ and it is easy to see that the set $\{x \wedge t / t \in M'\}$ is $\mathfrak{m}$-up-directed. Then there exists $t \in M'$ such that $x = x \wedge t$, i.e. $x \leqq t$. From the construction of $M'$ it follows that $x$ is $\mathfrak{m}$-compact.

The following example proves that $\mathfrak{m}$-meet-continuity is in the assertion (ii) necessary.

**2.5. Example:** Let $P$ be a complete lattice from the following figure, where $\{c_i / i = 1, 2, \ldots\}$ is a chain of the type $\omega$. $P$ is not meet-continuous for $\bigvee_{i=1}^{\infty} (b \wedge c_i) = a \neq b = b \wedge \bigvee_{i=1}^{\infty} c_i$. The element $b$ is join-inaccessible but it is not compact for $b \leqq \bigvee_{i=1}^{\infty} c_i$ and a finite set $F \subsetneqq \{c_i / i = 1, 2, \ldots\}$, $b \leqq \sup F$ does not exist.

**2.6. Definition:** *Let $P$ be a set, $\mathfrak{A}$ a closure system on $P$. We define an element $X \in \mathfrak{A}$ to be $\mathfrak{m}$-generated if there exists a subset $M$ of $P$ such that card $M < \mathfrak{m}$ and $X = \bar{M}$.*

**2.7. Lemma:** *Let $P$ be a set, $\mathfrak{m}$ an infinite regular cardinal number. Let $\mathfrak{A}$ be an $\mathfrak{m}$-inductive closure system on $P$ ordered by the set inclusion, $X \in \mathfrak{A}$. The following conditions are equivalent:*

*(i) $X$ is $\mathfrak{m}$-generated.*
*(ii) $X$ is $\mathfrak{m}$-join-inaccessible.*
*(iii) $X$ is $\mathfrak{m}$-compact.*

Proof: The $\mathfrak{m}$-inductivity of $\mathfrak{A}$ implies that the complete lattice $\mathfrak{A}$ is $\mathfrak{m}$-meet-continuous. From 2.4. it follows that the conditions (i) and (iii) are equivalent. We shall prove that (i) is equivalent to (ii).

Let (i) hold. Let $\emptyset \neq \mathfrak{B} \subseteqq \mathfrak{A}$ be an $\mathfrak{m}$-up-directed subsystem, $X = \sup \mathfrak{B}$. Hence $X = \bigcup \mathfrak{B}$. There exists $M \subseteqq P$ such that card $M < \mathfrak{m}$ and $X = \bar{M}$. It is easy to see that there exists $B \in \mathfrak{B}$ such that $M \subseteq B$. Hence $X = B$, i.e. (ii) holds.

Let (ii) hold. Let $\mathfrak{B} = \{\bar{B} / B \subseteqq X,\ \text{card}\, B < \mathfrak{m}\}$. Evidently $X = \bar{X} = \bigcup \mathfrak{B}$. We shall prove that $\mathfrak{B}$ is $\mathfrak{m}$-up-directed. Let $\mathfrak{C} \subseteqq \mathfrak{B}$, card $\mathfrak{C} < \mathfrak{m}$. Let us choose $A \subseteqq X$ such that card $A < \mathfrak{m}$ and $Y = \bar{A}$ for every $Y \in \mathfrak{C}$. We obtain a system $\mathfrak{C}'$ of such chosen $A$'s. It is $C = \overline{\bigcup \mathfrak{C}'} \in \mathfrak{B}$ as $\mathfrak{m}$ is regular. Clearly $C$ is an upper bound of $\mathfrak{C}$, i.e. $\mathfrak{B}$ is $\mathfrak{m}$-up-directed. Since $X$ is $\mathfrak{m}$-join-inaccessible, it is $X \in \mathfrak{B}$ and (i) holds.

**2.8. Lemma:** *Let $\mathfrak{m}$ be an infinite cardinal number and $P$ complete, $\mathfrak{m}$-meet-continuous lattice. Then for every subset $M \subseteqq \hat{P}(\mathfrak{m})$, card $M < \mathfrak{m}$ there exists $\sup_{\hat{P}(\mathfrak{m})} M$ and $\sup_{\hat{P}(\mathfrak{m})} M = \sup_P M$.*

Proof: Let $O$ be the least element of $P$. Evidently $O \in \hat{P}(\mathfrak{m})$, i.e. $\sup_{\hat{P}(\mathfrak{m})} \emptyset = O = \sup_P \emptyset$.

Let $M \subseteqq \hat{P}(\mathfrak{m})$, $O <$ card $M < \mathfrak{m}$, $x = \sup_P M$. It suffices to prove that $x \in \hat{P}(\mathfrak{m})$. Let $\emptyset \neq X \subseteqq P$ be an $\mathfrak{m}$-up-directed subset and $x = \sup_P X$. Let $t \in M$. It is $t = t \wedge \sup_P X = \bigvee_{y \in X} (t \wedge y)$. Since $t$ is $\mathfrak{m}$-join-inaccessible, there exists $y_t \in X$ such that $t = t \wedge y_t$, i.e. $t \leqq y_t$. There exists $y \in X$ such that $y \geqq y_t$ for every $t \in M$. Thus $y \geqq t$ for every $t \in M$. Therefore $y \geqq \sup_P M = x$, i.e. $x = y \in X$. We have proved that $x$ is $\mathfrak{m}$-join-inaccessible.

**2.9. Lemma:** *Any complete* $\mathfrak{m}$-*compactly generated lattice is* $\mathfrak{m}$-*meet-continuous.*

Proof: Let $x \in P$, $M \subseteq P$ be an $\mathfrak{m}$-up-directed subset. Clearly $x \wedge \sup M \geq \geq \bigvee_{t \in M} (x \wedge t)$. Let $a \in P$ be $\mathfrak{m}$-compact and $a \leq x \wedge \sup M$. We shall prove $a \leq \leq \bigvee_{t \in M} (x \wedge t)$. From $a \subseteq \sup M$ it follows that there exists a subset $N \subseteq M$ such that $0 < \operatorname{card} N < \mathfrak{m}$ and $a \leq \sup N$. Further, there exists $t_0 \in M$ such that $t_0 \geq t$ for every $t \in N$. Thus $a \leq x$, $a \leq t_0$ and therefore $a \leq \bigvee_{t \in M} (x \wedge t)$.

Since $P$ is $\mathfrak{m}$-compactly generated, there exists a set $A$ such that $x \wedge \sup M = = \bigvee_{\alpha \in A} a_\alpha$, where $a_\alpha \in P$ is $\mathfrak{m}$-compact for every $\alpha \in A$. Hence $\bigvee_{\alpha \in A} a_\alpha \leq \bigvee_{t \in M} (x \wedge t)$ and thereby is the proof accomplished.

## §3. THE LATTICE OF $\mathfrak{m}$-IDEALS

**3.1. Definition:** *Let $P$ be an ordered set, $A \subseteq P$. Put*

$$A^* = \{t \in P / t \geq x \text{ for every } x \in A\},$$

$$A^+ = \{t \in P / t \leq x \text{ for every } x \in A\}.$$

*Denote $A^{*+} = (A^*)^+$.*

It is $\varnothing^* = \varnothing^+ = P$.

**3.2. Definition** (see [6]): *Let $P$ be an ordered set, $\mathfrak{m}$ an infinite cardinal number. A subset $J \subseteq P$ is called an $\mathfrak{m}$-ideal of $P$ if for every subset $M$, $\varnothing \neq M \subseteq J$ with* $\operatorname{card} M < \mathfrak{m}$ *the inclusion $M^{*+} \subseteq J$ holds.*

The system of all $\mathfrak{m}$-ideals of $P$ ordered by the set inclusion is denoted by $\mathfrak{J}_\mathfrak{m}(P)$. It holds $\varnothing \in \mathfrak{J}_\mathfrak{m}(P)$. An $\mathfrak{m}$-ideal $J$ is called principal if $J = (c]$ for some $c \in P$. A subset $I$ of an ordered set $P$ is called a semiideal if $x \in I$, $y \leq x$ implies $y \in I$. Clearly every $\mathfrak{m}$-ideal is a semiideal.

**3.3. Lemma:** *Let $P$ be an ordered set, $\mathfrak{m}$ an infinite cardinal number. Then it holds:*

(*i*) $\mathfrak{J}_\mathfrak{m}(P)$ *is an $\mathfrak{m}$-inductive closure system.*

(*ii*) *Every principal ideal is $\mathfrak{m}$-join-inaccessible in* $\mathfrak{J}_\mathfrak{m}(P)$.

(*iii*) $\mathfrak{J}_\mathfrak{m}(P)$ *is $\mathfrak{m}$-meet-continuous.*

(*iv*) *If $P$ is a complete lattice then $\mathfrak{m}$-ideals of $P$ are precisely $\mathfrak{m}$-up-directed semiideals of $P$.*

Proof: The assertion (i) is evident. Since $(c] = \overline{\{c\}}$, (ii) follows from (i) and the proof of 2.7. The assertion (iii) follows from (i). We shall prove (iv): $\varnothing$ is both on $\mathfrak{m}$-ideal and an $\mathfrak{m}$-up-directed semiideal of $P$. Let $I \neq \varnothing$ be an $\mathfrak{m}$-up-directed semiideal of $P$, $M \subseteq I$, $0 < \operatorname{card} M < \mathfrak{m}$. There exists $x \in I$ with $x \geq t$ for every $t \in M$. Hence $M^{*+} \subseteq (x] \subseteq I$ and $I$ is an $\mathfrak{m}$-ideal. Let $P$ be a complete lattice, $\varnothing \neq J$ an $\mathfrak{m}$-ideal of $P$. Let $\varnothing \neq M \subseteq J$, $\operatorname{card} M < \mathfrak{m}$. It is $J \supseteq M^{*+} = (\sup M]$ and therefore $M$ has an upper bound in $J$. Hence $J$ is an $\mathfrak{m}$-up-directed semiideal.

**3.4. Definition** (see [7]): *Let $P$ be an ordered set, $\mathfrak{N}$ a system of subsets of $P$. An element $x \in P$ is called $\mathfrak{N}$-primitive if $N \in \mathfrak{N}$, $x \leq \sup N$ implies that there exists $y \in N$ with $x \leq y$.*

**3.5. Lemma:** *Let $P$ be a complete, $\mathfrak{m}$-meet-continuous lattice. Then $(\mathfrak{J}_\mathfrak{m}(P) - \{\varnothing\})$-primitive elements of $P$ are precisely $\mathfrak{m}$-join-inaccessible elements of $P$.*

Proof: Let $x \in P$ be $(\mathfrak{F}_\mathfrak{m}(P) - \{\varnothing\})$-primitive. Let $M \neq \varnothing$ be an $\mathfrak{m}$-up-directed subset of $P$, $x = \sup M$. Let $M' = \{t \in P /$ there exists $y \in M$ with $t \leqq y\}$. Clearly $x = \sup M'$ and $M'$ is $\mathfrak{m}$-up-directed. By 3.3. (iv) $M'$ is an $\mathfrak{m}$-ideal. Then there exists $t \in M'$ with $x \leqq t$. Hence $x \in M$ and we have proved that $x$ is $\mathfrak{m}$-join-inaccessible.

Let $x \in P$ be $\mathfrak{m}$-join-inaccessible. Let $\varnothing \neq J \in \mathfrak{F}_\mathfrak{m}(P)$, $x \leqq \sup J$. By 3.3. (iv) $J$ is an $\mathfrak{m}$-up-directed set and therefore $x = x \wedge \sup J = \bigvee_{t \in J} (x \wedge t)$. Since $x$ is $\mathfrak{m}$-join-inaccesible, there exists $t \in J$ with $x \leqq t$. Thus $x$ is $(\mathfrak{F}_\mathfrak{m}(P) - \{\varnothing\})$-primitive.

**3.6. Definition:** (see [7]): *Let $Q \subseteqq P$ be ordered sets, $\mathfrak{M}$ a system of semiideals of $Q$ ordered by the set inclusion. Denote $\omega(x) = \{t \in Q / t \leqq x\}$ for every $x \in P$. The set $P$ is called an $\mathfrak{M}$-hull of $Q$ if $\omega$ is an isomorphismus from $P$ on $\mathfrak{M}$.*

**3.7. Theorem** (see [7]): *Let $Q \subseteqq P$ be ordered sets, $\mathfrak{M}$ a system of semiideals of $Q$. The following conditions are equivalent:*

(*i*) *$P$ is an $\mathfrak{M}$-hull of $Q$.*
(*ii*) *a) $\omega$ is a mapping from $P$ to $\mathfrak{M}$*
*b) Every element of $P$ is a join of elements of $Q$*
*c) There exists* $\sup_P M$ *for every $M \in \mathfrak{M}$*
*d) Every element of $Q$ is $\mathfrak{M}$-primitive in $P$.*

Now, we can prove the main theorem of this paper.

**3.8. Theorem**: *Let $\mathfrak{m}$ be an infinite cardinal number. A lattice $L$ is isomorphic with the lattice of $\mathfrak{m}$-ideals of some ordered set $P$ if and only if $L$ is complete, $\mathfrak{m}$-meet-continuous, and every element of $L$ is a join of $\mathfrak{m}$-join-inaccessible elements.*

Proof: I. Since every $\mathfrak{m}$-ideal $J \neq \varnothing$ is a join of principal ideals generated by its elements, the necessity follows from 3.3.

II. Let $L$ be a complete, $\mathfrak{m}$-meet-continuous lattice and every element of $L$ be a join of $\mathfrak{m}$-join-inaccessible elements. Let $O$ be the least element of $L$. Denote $\hat{L}(\mathfrak{m}) = \hat{L}$. We shall prove that $L \cong \mathfrak{F}_\mathfrak{m}(\hat{L}) - \{\varnothing\}$. According to 3.7. it suffices to verify the condition (ii) of 3.7. for $Q = \hat{L}$, $P = L$, $\mathfrak{M} = \mathfrak{F}_\mathfrak{m}(\hat{L}) - \{\varnothing\}$. Let $x \in L$, $M \subseteqq \omega(x)$, $0 < \operatorname{card} M < \mathfrak{m}$. According to 2.8. there exists $\sup_{\hat{L}} M$. Hence $M^{*+} = (\sup_{\hat{L}} M] \subseteqq \omega(x)$. Thus $\omega(x)$ is an $\mathfrak{m}$-ideal of $\hat{L}$. Since $O \in \omega(x)$, $\omega$ is a mapping from $L$ to $\mathfrak{F}_\mathfrak{m}(\hat{L}) - \{\varnothing\}$. By the supposition the conditions b), c) are fulfilled. Let $x \in \hat{L}$, $J \in \mathfrak{F}_\mathfrak{m}(\hat{L}) - \{\varnothing\}$, $x \leqq \sup J$. Let $J' = \{y \in L /$there exists $t \in J$ with $y \leqq t\}$. Clearly $J' \in \mathfrak{F}_\mathfrak{m}(L) - \{\varnothing\}$. According to 3.5. there exists $y \in J'$ with $x \leqq y$. Further, there exists $z \in J$ such that $y \leqq z$. Thus $x$ is $(\mathfrak{F}_\mathfrak{m}(\hat{L}) - \{\varnothing\})$-primitive.

Since the mapping $J \to J - \{O\}$ defines an isomorphismus $\mathfrak{F}_\mathfrak{m}(\hat{L}) - \{\varnothing\} \cong \mathfrak{F}_\mathfrak{m}(\hat{L} - \{O\})$, the sufficiency is proved.

**3.9. Corollary:** *Let $\mathfrak{m}$ be an infinite regular cardinal number. A lattice $L$ is isomorphic with the lattice of $\mathfrak{m}$-ideals of some ordered set $P$ if and only if $L$ is complete and $\mathfrak{m}$-compactly generated.*

Proof follows from 3.8., 2.4. and 2.9.

In [4] it is stated that a complete lattice $L$ every element of which is a join of a chain of join-inaccessible elements is isomorphic with the lattice of ideals of some ordered set. According to the example 2.5. it is necessary to add to this assertion the supposition that $L$ is meet-continuous.

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