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QUADRATIC PHASES OF DIFFERENTIAL EQUATIONS $y'' = q(t)y$

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1. INTRODUCTION

The phases form a significant pillar of the transformation theory of O. BORŮVKA, see [1]. This theory deals with the differential equations

$$(q) \qquad y'' = q(t)y,$$

where $q(t)$ are real continuous functions on open intervals $]a, b[$ of the real axis $\mathbf{R}$. Sometimes we call q the carrier. Any ordered pair of linearly independent solutions of (q) on domain Dom $q =]a, b[$ of q is called a basis of (q).

A phase $\alpha(t)$ of (q) is every continous function in $]a, b[$ fulfilling everywhere, except the roots of the denominator, the relation

$$\langle 1\rangle \qquad \operatorname{tg} \alpha(t) = \frac{y(t)}{z(t)},$$

where $\langle y, z\rangle$ is an arbitrary basis of (q).

It is known that for every basis $\langle y, z\rangle$ of (q) the phases exist in the whole interval $]a, b[$ and if $\alpha(t)$ is one of them, then all are $\alpha(t) + \nu\pi$, where ν ranges over the set of all integers $\mathbf{Z}$. Moreover, all the phases α of (q) satisfy in $]a, b[$ the differential equation

$$(-1, q) \qquad -\{\alpha, t\} - \alpha'^2 = q(t),$$

where $\{\alpha, t\} = \frac{1}{2}\left(\frac{\alpha''}{\alpha'}\right)' - \frac{1}{4}\left(\frac{\alpha''}{\alpha'}\right)^2$ is the Schwarz' derivative. Note that by solutions of differential equations we always mean the largest solutions. It is an important fact that every solution α of $(-1, q)$ exists in $]a, b[$ and is a phase of a suitable basis $\langle y, z\rangle$ of (q).

The phases arose from the transformation of a basis $\langle y, z\rangle$ of (q) to the polar form

$$\langle 2\rangle \qquad y = \pm r \sin \alpha, \quad z = \pm r \cos \alpha, \ r > 0.$$

The important position of phases in the classical theory [1] issues from their existence in the whole interval $]a, b[$ for any equation (q) and from their object meaning $\langle 1\rangle$, $\langle 2\rangle$.

For two given equations $(i = 1, 2)$

$$(q_i) \qquad y_i'' = q_i(t_i)y_i,$$

where Dom $q_i =]a_i, b_i[$, it is a matter of transformations of the form

$$\langle 3\rangle \qquad y_2 = \frac{y_1(\gamma)}{\sqrt{|\gamma'|}}$$

of solutions y_2 of (q_2) to solutions y_1 of (q_1). It appears that in a suitable subinterval of $]a_2, b_2[$ the function γ satisfies the equation

$$(q_1, q_2) \qquad -\{\gamma, t\} + q_1(\gamma)\,\gamma'^2 = q_2(t)$$

and on the contrary every solution γ of (q_1, q_2) in some interval transforms solutions y_2 of (q_2) to solutions y_1 of (q_1) according to the formula $\langle 3\rangle$. Generally only parts of y_2 are transformed to parts of y_1, i.e. γ transforms only $y_2/\mathrm{dom}\,\gamma$ to $y_1/\mathrm{im}\,\gamma$.

If the equation (q_1, q_2) has a solution γ which maps $]a_2, b_2[$ onto $]a_1, b_1[$, then each such γ is called a complete solution of (q_1, q_2) or a complete transformation of (q_2) to (q_1). Then by the formula $\langle 3\rangle$ an arbitrary solution y_2 of (q_2) in $]a_2, b_2[$ is transformed to the solution y_1 of (q_1) in $]a_1, b_1[$. Evidently the complete transformations have the main significance in comparison with the others.

For three given equations $(i = 1, 2, 3)$

$$(q_i) \qquad y_i'' = q_i(t_i)\, y_i$$

where $\mathrm{Dom}\, q_i =]a_i, b_i[$, the solutions γ_{ij} of the equations (q_i, q_j) $(i, j = 1, 2, 3)$ have the following arithmetic:

1° if γ_{ij} is a solution of (q_i, q_j), then the inverse function γ_{ij}^{-1} is a solution of (q_j, q_i),

2° if γ_{ij} is a solution of (q_i, q_j) and $\tilde{\gamma}_{jk}$ is a solution of (q_j, q_k) $(i, j, k = 1, 2, 3)$, then the composed function $\gamma_{ij} \circ \gamma_{jk}$ (if it exists in some open interval) is a solution of (q_i, q_k).

Let us remind how the character, see [1], of an equation (q) can be univocally determined by means of the phases of the equation (q). The equation (q) is called nonoscillatory of the type $m \geqq 1$ if it has solutions with m roots but has no solution with $(m + 1)$ roots. Just then every phase α of (q) maps the interval $]a, b[= \mathrm{Dom}\, q$ onto an interval of the length $d(\alpha) \in](m - 1)\pi, m\pi]$, where for all α, there is $d(\alpha) < m\pi$ or for all α there is $d(\alpha) = m\pi$. According to that the equation (q) is called either general or special.

If just one of the ends of $]a, b[$ is an accumulation point of roots of any solution of (q), then (q) is called one-sided oscillatory. Just then every phase α of (q) maps the interval $]a, b[$ onto an interval of the form either $]-\infty, c[$ or $]c, +\infty[$.

If both ends of $]a, b[$ are accumulation points of roots of any solution of (q), then (q) is called both-sided oscillatory. Just then every phase α of (q) maps the interval $]a, b[$ on $\mathbf{R} =]-\infty, +\infty[$.

By the character of an equation (q) there is meant any one of the possibilities:

1° (q) is general and of the type $m \geqq 1$ in $]a, b[$,
2° (q) is special and of the type $m \geqq 1$ in $]a, b[$,
3° (q) is one-sided oscillatory,
4° (q) is both-sided oscillatory.

There exist countably many characters of differential equations (q). Each (q) has a quite definite character which can be determined univocally by means of its phases.

Let us have two equations (q_i), $i = 1, 2$. The phases α_i of (q_i) are called similar if their ranges are identical, i.e. $\mathrm{Im}\, \alpha_1 = \mathrm{Im}\, \alpha_2$.

There holds, see [1]:

1° iff (q_1) and (q_2) have the same character, there exist similar phases α_i of (q_i),

2° iff the phases α_i of (q_i) are similar, the transformation $\gamma = \alpha_1^{-1}\alpha_2$ is a complete one of (q_2) to (q_1).

Consequently (q_1) and (q_2) have the same character if and only if there exist complete solutions of the equation (q_1, q_2).

From these important theorems of the theory of transformations of O. BORŮVKA the significance of the phases for the whole theory is evident. Note that for arbitrary values $t_0 \in]a, b[$, $\alpha_0 \in \mathbf{R}$, $0 \neq \alpha_0' \in \mathbf{R}$, $\alpha_0'' \in \mathbf{R}$ there exists uniquely the phase α of (q) fulfilling the initial conditions $\alpha(t_0) = \alpha_0$, $\alpha'(t_0) = \alpha_0'$, $\alpha''(t_0) = \alpha_0''$. Consequently it is shown that for the equation (q_1, q_2) the existence and uniqueness of initial value problem hold.

Besides the mentioned classical phases there was introduced another sort of phases, see [5], [6], [7]. Analogically to $\langle 2 \rangle$ a basis $\langle y, z \rangle$ of (q) was transformed to the form

$$\langle 4 \rangle \qquad y = \pm r \operatorname{sh} \vartheta, \quad z = \pm r \operatorname{ch} \vartheta, \; r > 0,$$

so that in suitable intervals the function ϑ, called hyperbolic phase, fulfilled the relation

$$\langle 5 \rangle \qquad \operatorname{th} \vartheta = \frac{y}{z}.$$

The analogy of hyperbolic and trigonometric functions made possible get some formulae analogous to that of the classical theory.

Hyperbolic phases ϑ of (q) represent transformations of the equation (q) to the equation

$$(1) \qquad y'' = y.$$

An equation (q) can be completely transformed to (1) if and only if (q) is general and of the type 1. Note that in this case (q) can be also transformed to the equation

$$(-1) \qquad y'' = -y,$$

restricted to any open interval of length less than π.

2. MECHANISM OF DEFINITION OF PHASES

Let (a, b) denote any one of the intervals $[a, b]$, $[a, b[$, $]a, b]$, $]a, b[\subseteq \mathbf{R}$ for $-\infty \leqq a < b \leqq +\infty$, i.e. $[a, b]$ closed, $]a, b[$ open etc.

Let us fix an arbitrary equation

$$(Q) \qquad Y'' = Q(t)\, Y$$

with Q real and continuous in $\operatorname{Dom} Q = (A, B)$ and let us fix some of its bases $\langle Y, Z \rangle$.

For an arbitrary equation (q) with q real and continuous in $\operatorname{Dom} q = (a, b)$ and for any one of its bases $\langle y, z \rangle$ let us define a function Θ in some interval $\subseteq (a, b)$ by the requirement of *continuity* and by the relation

$$\langle 6 \rangle \qquad \frac{Y(\Theta)}{Z(\Theta)} = \frac{y}{z}$$

everywhere in that interval except the roots of the denominator.

2.1. Lemma. *If Θ exists in some interval $\subseteq (a, b)$, then there holds in that interval*

$$\langle 7\rangle \qquad y = \pm \frac{Y(\Theta)}{\sqrt{\frac{W\Theta'}{w}}}, \quad z = \pm \frac{Z(\Theta)}{\sqrt{\frac{W\Theta'}{w}}}$$

where W and w, resp., are the Wronskians of the basis $\langle Y, Z\rangle$ or $\langle y, z\rangle$, resp. Moreover, Θ fulfils the differential equation (Q, q) in that interval.

On the contrary, every solution Θ of the equation (Q, q) in some interval $\subseteq (a, b)$ fulfils there the relation $\langle 6\rangle$, e.g. for the basis $\langle y, z\rangle$ of (q) given by $\langle 7\rangle$.

Proof. I. Let Θ exist for some basis $\langle y, z\rangle$ in some interval. Then there exists a continuous function k fulfilling

$$\langle 8\rangle \qquad y = kY(\Theta), \quad z = kZ(\Theta)$$

as $k = \dfrac{y}{Y(\Theta)} = \dfrac{z}{Z(\Theta)}$ is continuously defined by these relations because $Y(\Theta)$, $Z(\Theta)$ do not vanish simultaneously.

Differentiating the relation $\langle 6\rangle$ we get

$$\langle 9\rangle \qquad \Theta' = \frac{w}{W}\frac{1}{k^2} \quad \text{or} \quad k = \pm \frac{1}{\sqrt{\frac{W\Theta'}{w}}},$$

so that there holds $\langle 7\rangle$. As it is evident that Θ has a continuous derivative of the 3rd order we get by the differentiation of $\langle 7\rangle$ the equation

$$\langle 10\rangle \qquad \sqrt{\frac{W\Theta'}{w}}\left(\frac{1}{\sqrt{\frac{W\Theta'}{w}}}\right)'' + Q(\Theta)\,\Theta'^2 = q(t) \text{ or } (Q, q).$$

II. On the contrary, every solution Θ of the equation (Q, q) in some interval fulfils there $\langle 6\rangle$ for the basis $\langle y, z\rangle$ defined by $\langle 7\rangle$.

2.2. Remark. The function Θ fulfilling the equation (Q, q) in some interval $\subseteq (a, b)$ represents there a transformation of (q) to (Q). If we wished to express the function k occurring in $\langle 8\rangle$ otherwise than by means of $\langle 9\rangle$, namely in the terms of $\langle y, z\rangle$ only (which we are able to do for $\langle Y, Z\rangle = \langle \sin t, \cos t\rangle$ and for $\langle Y, Z\rangle = \langle \operatorname{sh} t, \operatorname{ch} t\rangle$ where $k = \pm\sqrt{y^2 + z^2}$ or $k = \pm\sqrt{z^2 - y^2}$), we should have to know some corresponding quadratic relations for the basis $\langle Y, Z\rangle$.

2.3. Definition. *If for some basis $\langle y, z\rangle$ of (q) the relation $\langle 6\rangle$ together with the requirement of continuity determine a function Θ in the whole interval $]a, b[$, then we call Θ a Q-phase (with regard to the basis $\langle Y, Z\rangle$) of the basis $\langle y, z\rangle$ of (q).*

Then, by a Q-phase of (q) any Q-phase of a suitable basis $\langle y, z\rangle$ of (q) is meant.

For the choice of a convenient sort of phases a requirement of universality evidently plays the main role, e.g. that the equation (Q) in $]A, B[$ may be both-sided oscillatory so that all the characters can be represented by suitable restrictions of (Q).

Another requirement, that of simplicity of the function Q, is not irrelevant, e.g. such that some quadratic relations may exist among the solutions of (Q).

O. BORŮVKA proves in [2] that for a linear system

$$\langle 11\rangle \qquad y' = My,$$

where M is a square $n \times n$ matrix of continuous functions in some interval $]A, B[$, there exists a quadratic relation ${}^t yNz = \text{const.}$ for any two solutions y, z of $\langle 11\rangle$ (y, z being $n \times 1$ matrices of functions with the first continuous derivative in $]A, B[$, N being some convenient constant symmetric matrix, ${}^t y$ denoting the transported matrix) if and only if there holds

$$\langle 12\rangle \qquad {}^t MN + NM = 0.$$

This applied to the equation (Q) in $]A, B[$, interpreted as the system

$$\langle 13\rangle \qquad \begin{bmatrix} Y \\ Y' \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ Q & 0 \end{bmatrix} \begin{bmatrix} Y \\ Y' \end{bmatrix},$$

we get that for $N = [n_{ij}]$, $i, j = 1, 2$, there holds $\langle 12\rangle$ if and only if Q is a constant, $n_{12} = n_{21} = 0$, $n_{11} = -Qn_{22}$.

We can see that just for $Q = \text{const.}$ there holds the quadratic relation

$$\langle 14\rangle \qquad -QYZ + Y'Z' = \text{const.}$$

for any two solutions Y, Z of (Q).

Among all equations (Q) with $Q = \text{const.}$ it suffices to choose three representatives for $Q = -1$, $Q = 0$, $Q = 1$ on the real axis $\mathbf{R}$. The corresponding equations (Q) have then the characters: both-sided oscillatory, special and of the type 1, general and of the type 1, respectively.

Note that all the characters can be realized with $Q = -1$ by suitable restrictions of the domain.

Between the solutions Y, Z of (Q) and their first derivatives we get the quadratic relations

$$\langle 14^*\rangle \qquad YZ + Y'Z' = \text{const. for } Q = -1,$$

$$\langle 14^{**}\rangle \qquad Y'Z' = \text{const. for } Q = 0,$$

$$\langle 14^{***}\rangle \qquad -YZ + Y'Z' = \text{const. for } Q = 1,$$

which give specially the classical formulae $\sin^2 t + \cos^2 t = 1$, $\operatorname{ch}^2 t - \operatorname{sh}^2 t = 1$.

From the point of view of simplicity we have to choose any one of the carriers $Q = -1$, $Q = 0$, $Q = 1$. From the point of view of universality we get to the unique carrier $Q = -1$, since any other universal carrier Q misses the simplicity and any other simple carrier Q misses the universality.

Accordingly as $Q = -1, 0, 1$ we call Q-phases elliptic, parabolic, hyperbolic. The classical phases are elliptic phases with regard to the basis $\langle \sin t, \cos t\rangle$. Further we shall deal with parabolic phases ζ only with regard to the basis $\langle t, 1\rangle$ so that they fulfil for any basis $\langle y, z\rangle$ of (q) the relation

$$\langle 15\rangle \qquad \zeta = \frac{y}{z},$$

and we call them *paraphases* of (q). Finally we shall deal with hyperbolic phases ϑ only with regard to the basis $\langle \operatorname{sh} t, \operatorname{ch} t\rangle$ so that they fulfil for any basis $\langle y, z\rangle$ of (q) the relation $\langle 5\rangle$ and we call them *hyperphases* of (q).

2.4. Remark. Note that the requirement of continuity is a part of definition o- paraphases ⟨15⟩. By this they differ from the so-called semiphases of (q), whicff are quotients $\frac{y}{z}$ of linearly independent solutions of (q) in $]a, b[$ $=$ Dom q without regard to the continuity. See [4].

Certainly, elliptic, parabolic and hyperbolic phases can be called quadratic phases with regard to the exclusiveness of the existence of quadratic relations among solutions of (Q) for $Q = -1, 0, 1$.

3. APPLICATION OF PRINCIPAL SOLUTIONS

Let (q) be an equation with q real and continuous in Dom $q = (a, b)$, nonoscillatory at a or b, respectively, see [3]. A solution u or v, resp. is called principal at a or b, resp. if $u \neq 0$ or $v \neq 0$, resp. in some deleted neighbourhood $0^{\bullet}_{a^+}$ or $0^{\bullet}_{b^-}$, resp., and for $t \in 0^{\bullet}_{a^+}$ or $t \in 0^{\bullet}_{b^-}$, resp., there holds

$$\langle 16\rangle \qquad \int_a^t \frac{ds}{u^2(s)} = +\infty \text{ or } \int_t^b \frac{ds}{v^2(s)} = +\infty, \text{ resp.}$$

Everywhere in further the letter u or v, resp., denotes the principal solution at a or b, resp.

The principal solution u or v, resp., is also characterized by the fact that for any linearly independent solution z there holds $\lim\limits_{t \to a^+} \frac{u}{z} = 0$ or $\lim\limits_{t \to b^-} \frac{v}{z} = 0$, resp.

In comparison with [3] we shall call an equation (q) in (a, b) disconjugate i each of its solutions has at most one zero in $]a, b[$. Specially (q) in $[a, b]$ having a solution $y \neq 0$ in $]a, b[$ such that $y(a) = y(b) = 0$ is disconjugate according to our definition, whereas according to [3] it is not. Only in this case both definitions differ. If (q) is not disconjugate, we call it conjugate so that exactly a conjugate equation (q) has a fundamental central dispersion $\varphi(t)$ defined in some non-trivial interval.

3.1. Generalized separation theorem. *Let (q) in (a, b) be a disconjugate equation for which $z \neq 0$ in $]a, b[$ is a principal solution simultaneously at a and b. Then every solution y linearly independent on z has exactly one zero in $]a, b[$.*

Proof. The function $\frac{y}{z}$ is strictly monotonic in $]a, b[$ and maps this interval onto $\mathbf{R}$ so that it has exactly one zero in $]a, b[$.

3.2. Lemma. *For an equation (q) in $(a, b]$ the principal solution at b is that one which has zero at the point b. Similarly for an equation (q) in $[a, b)$ the principal solution at a is that one which has zero at the point a.*

Proof. Let the solution y of (q) have zero b. For any linearly independent solution z in a convenient deleted neighbourhood $0^{\bullet}_{b^-}$ there holds $\left(\frac{z}{y}\right)' = \frac{k}{y^2}$, where $k \neq 0$ is a constant. Hence $\lim\limits_{t \to b^-} \frac{z}{y} = \pm\infty$. For $t \in 0^{\bullet}_{b^-}$ we have $\pm\infty = \frac{z(t)}{y(t)} + \int_t^b k \frac{ds}{y^2(s)}$.

Hence $\int_t^b \frac{ds}{y^2(s)} = +\infty$ so that y is a principal solution at b. Similarly for the second part.

Let (q) be a conjugate equation in (a, b) which is nonoscillatory at the point a. Let R denote the set of all $t \in (a, b)$ such that $\varphi^{-1}(t) \in (a, b)$ exists. Put $r = \inf R$. In [1] O. BORŮVKA calls a solution y of (q) with the zero r a left fundamental solution. Similarly a right fundamental solution z of a conjugate equation in (a, b) which is nonoscillatory at the point b is a solution having the zero $s = \sup S$ where S is the set of all $t \in (a, b)$ such that $\varphi(t) \in (a, b)$.

3.3. Theorem. *For a conjugate equation (q) the left fundanental solution is identical (until the linear dependence) with the principal solution at the point a. Similarly the right fundamental solution is identical (until the linear dependence) with the principal solution at the point b.*

Proof. Let y be the left fundamental solution of (q) and u be the principal solution of (q) at the point a. If y, u are linearly independent, then u has a zero in $]a, r[\cup]r, b[$ since the equation (q) would be otherwise disconjugate. If u has a zero in $]r, b[$, then u also has a zero in $]a, r[$ according to the significance of $r = \inf R$. Then according to 3.1. and 3.2, y has a zero in $]a, r[$, which contradicts the definition of r. Thus y, u are linearly dependent. For the second part the proof is analogous.

3.4. Remark. The priority of the concept of principal solutions with respect to that of left or right fundamental solutions consists in their existence for (q) nonoscillatory at a or b, resp., regardless of the fact (q) being conjugate or disconjugate.

3.5. Lemma. *The phases of the basis $\langle y, z\rangle$ acquire the values $\nu\pi$, $\nu \in \mathbf{Z}$ exactly in the zeros of y, whereas they acquire the values $(\nu + 1/2)\pi$, $\nu \in \mathbf{Z}$ exactly in the zeros of z.*

3.6. Lemma. *For an equation (q) in (a, b) let z be an arbitrary solution linearly independent on the principal solution u or v, respectively.*

For the phases α of the basis $\langle u, z\rangle$ there holds $\alpha(a^+) = \lim_{t\to a^+} \alpha(t) = \nu\pi$, $\nu \in \mathbf{Z}$, whereas for the phases α of the basis $\langle v, z\rangle$ there holds $\alpha(b^-) = \lim_{t\to b^-} \alpha(t) = \nu\pi$, $\nu \in \mathbf{Z}$.

For the phases α of the basis $\langle z, u\rangle$ there holds $\alpha(a^+) = \lim_{t\to a^+} \alpha(t) = (\nu + {}^1/_2)\pi$, $\nu \in \mathbf{Z}$, whereas for the phases α of the basis $\langle z, v\rangle$ there holds $\alpha(b^-) = \lim_{t\to b^-} \alpha(t) = (\nu + {}^1/_2)\pi$, $\nu \in \mathbf{Z}$.

The proof follows from the fact that $\lim_{t\to a^+} \frac{u}{z} = 0$, $\lim_{t\to b^-} \frac{v}{z} = 0$.

3.7. Theorem. *A nonoscillatory equation (q) in (a, b) is general or special according to the fact whether u, v are linearly independent or dependent.*

Proof. If u, v are linearly independent, then for the phase α of the basis $\langle u, v\rangle$ there holds $\alpha(a^+) = \mu\pi$, $\alpha(b^-) = (\nu + 1/2)\pi$ for some μ, $\nu \in \mathbf{Z}$ and thus the length of $\operatorname{Im} \alpha$ is not of the form $m\pi$, $m \in \mathbf{Z}$ so that (q) is general.

If u, v are linearly dependent, then for an arbitrary solution z which is linearly independent on u, the phase α of the basis $\langle u, z\rangle$ fulfils $\alpha(a^+) = \mu\pi$, $\alpha(b^-) = \nu\pi$ for suitable μ, $\nu \in \mathbf{Z}$ so that the length of $\operatorname{Im} \alpha$ is a multiple of π and thus (q) is special.

3.8. Remark. Recall that an equation (q) in (a, b) is of the type 1, i.e. each of its solutions has at most one zero in (a, b) if and only if it is disconjugate except the case of special disconjugate equation (q) in $[a, b]$ (when there are solutions with zeros a, b).

3.9. Lemma. *An equation (q) in (a, b) is disconjugate if and only if there exists a solution $z \neq 0$ in $]a, b[$, e.g. u, v.*

Proof. If a solution $z \neq 0$ in $]a, b[$ exists, then every solution y of (q) has in $]a, b[$ at most one zero according to the separation theorem. Thus (q) is disconjugate. If (q) is disconjugate, then in $]a, b[$ the principal solutions u, v have no zero according to 3.2 and 3.1.

3.10. Lemma. *An equation is general and disconjugate in (a, b) if and only if there exist linearly independent solutions y, z both having no zero in $]a, b[$.*

Proof. If (q) is general and disconjugate, then linearly independent solutions u, v have no zero in $]a, b[$ according to 3.9. If (q) is special and disconjugate, then every solution y linearly independent on u, v has one zero according to 3.1. in $]a, b[$ and thus there do not exist linearly independent solutions y, z having no zero in $]a, b[$.

4. PARAPHASES

In this paragraph we shall deal with parabolic phases of differential equation (q) in (a, b) with regard to the basis $\langle t, 1\rangle$, i.e. with so-called paraphases.

4.1. Lemma. *Paraphases of a basis $\langle y, z\rangle$ of (q) exist in $]a, b[$ if and only if $z \neq 0$ in $]a, b[$.*

Proof. Iff $z \neq 0$ in $]a, b[$, then $\zeta = \dfrac{y}{z}$ is continuous in $]a, b[$.

4.2. Corollary. *Paraphases of an equation (q) in (a, b) exist if and only if (q) is disconjugate.*

4.3. Theorem. *Let (q) in (a, b) be disconjugate and general. Let u, v be principal solutions fulfilling the inequalities $u > 0$, $v > 0$ in $]a, b[$. Then all the bases $\langle y, z\rangle$ of (q), for which $\zeta = \dfrac{y}{z}$ is a paraphase, are given by the formula*

$$\begin{bmatrix} y \\ z \end{bmatrix} = [a_{ij}] \begin{bmatrix} u \\ v \end{bmatrix}, \det [a_{ij}] \neq 0, \; a_{21}a_{22} \geqq 0,$$

where a_{21}, a_{22} are not simultaneously zero.

Proof. For $z = a_{21}u + a_{22}v$, where $a_{21}a_{22} \geqq 0$ and a_{21}, a_{22} are not simultaneously zero, there is evidently $z \neq 0$ in $]a, b[$. On the contrary if, e.g., $z = a_{21}u + a_{22}v > 0$ in $]a, b[$, then for $t \to a^+$ we get $0 < \dfrac{z}{v} = a_{21}\dfrac{u}{v} + a_{22} \to a_{22} \geqq 0$ and similarly $a_{21} \geqq 0$ so that $a_{21}a_{22} \geqq 0$ and a_{21}, a_{22} are not simultaneously zero.

4.4. Theorem. *Let (q) in (a, b) be disconjugate and special. Then all the bases $\langle y, z\rangle$ of (q), for which $\zeta = \dfrac{y}{z}$ is a paraphase, have z as the common principal solution and y as an arbitrary linearly independent solution.*

Proof. Follows from 3.1.

4.5. Theorem. *Let (q) in (a, b) be disconjugate and special. Then every paraphase ζ is strictly monotonic and maps $]a, b[$ onto $\mathbf{R}$.*

Proof. For an arbitrary paraphase $\zeta = \dfrac{y}{z}$ we have, according to 4.4, $\lim\limits_{t \to a^+} \zeta = -\infty \operatorname{sgn} \zeta'$, $\lim\limits_{t \to b^-} \zeta = +\infty \operatorname{sgn} \zeta'$.

4.6. Theorem. *Let (q) in (a, b) be disconjugate and general. Let $k \neq l$ be arbitrary values in $\tilde{\mathbf{R}} = [-\infty, +\infty]$, one of them, at least, being finite. Then there exist ∞^1 (i.e. a oneparametric continuum) paraphases ζ of (q) fulfilling $\zeta(a^+) = \lim_{t \to a^+} \zeta(t) = k$, $\zeta(b^-) = \lim_{t \to b^-} \zeta(t) = l$.*

Proof. According to 4.3. any paraphase is of the form $\zeta = \dfrac{a_{11}u + a_{12}v}{a_{21}u + a_{22}v}$, where $a_{21} \geqq 0$, $a_{22} \geqq 0$ can be supposed without loss of generality, of course a_{21}, a_{22} being not simultaneously zero.

I. If $a_{21} > 0$, $a_{22} > 0$, then $\zeta(a^+) = \dfrac{a_{12}}{a_{22}}$, $\zeta(b^-) = \dfrac{a_{11}}{a_{21}}$ so that for $k \neq l \in \mathbf{R}$ we have $a_{11} = la_{21}$, $a_{12} = ka_{22}$. Since it is the matter of homography it is possible to eliminate one of the parameters $a_{21} > 0$, $a_{22} > 0$ and thus to each pair $k \neq l \in \mathbf{R}$ there are ∞^1 paraphases ζ with the boundary values $\zeta(a^+) = k$, $\zeta(b^-) = l$.

II. For $a_{21} = 0$, $a_{22} = 1$ we have $\zeta = a_{11}\dfrac{u}{v} + a_{12}$, where $a_{11} \neq 0$, a_{12} are arbitrary numbers. Then $\zeta(a^+) = a_{12}$, $\zeta(b^-) = \pm\infty$ according to the fact if $a_{11} \gtrless 0$. Hence for the pair $k = a_{12}$, $l = \pm\infty$ there exist ∞^1 paraphases ζ having boundary values $\zeta(a^+) = k \in \mathbf{R}$, $\zeta(b^-) = \pm\infty$, the parameter being $a_{11} \gtrless 0$ according to the fact if $l = \pm\infty$.

III. For $a_{21} = 0$, $a_{22} = 0$ we have $\zeta = a_{11} + a_{12}\dfrac{v}{u}$, where $a_{12} \neq 0$, a_{11} are arbitrary numbers. Then $\zeta(a^+) = \pm\infty$ according to the fact if $a_{12} \lessgtr 0$, $\zeta(b^-) = a_{11}$. Hence for the pair $k = \pm\infty$, $l = a_{11} \in \mathbf{R}$ there exist ∞^1 of paraphases ζ with boundary values $\zeta(a^+) = \mp\infty$, $\zeta(b^-) = l \in \mathbf{R}$, the parameter being $a_{12} \lessgtr 0$ according to the fact if $k = \mp\infty$.

4.7. Theorem. *Let (q) in (a, b) be disconjugate so that paraphases exist. Then all the (real) homographies*

$$\langle 17 \rangle \qquad \zeta_2 = \frac{b_{11}\zeta_1 + b_{12}}{b_{21}\zeta_1 + b_{22}},$$

which transform the paraphase ζ_1 to ζ_2 with the boundary values $\zeta_i(a^+) = k_i \neq l_i = \zeta_i(b^-)$, $i = 1, 2$, are given by the formulae

$$\langle 18 \rangle \qquad b_{11} = \frac{k_1k_2 - l_1l_2}{k_1 - l_1} b_{21} + \frac{k_2 - l_2}{k_1 - l_1} b_{22},$$

$$b_{12} = \frac{k_1l_1(l_2 - k_2)}{k_1 - l_1} b_{21} + \frac{k_1l_2 - k_2l_1}{k_1 - l_1} b_{22},$$

where b_{21}, b_{22} are arbitrary parameters fulfilling the condition

$$\langle 19 \rangle \qquad (b_{22} + k_1b_{21})(b_{22} + l_1b_{21}) > 0.$$

The determinant of the matrix $[b_{ij}]$ is then

$$\langle 20 \rangle \qquad |b_{ij}| = \frac{k_2 - l_2}{k_1 - l_1}(b_{22} + k_1b_{21})(b_{22} + l_1b_{21}).$$

Proof. Any two semiphases ζ_1, ζ_2 of (q), see remark 2.4., are connected together with the formula $\langle 17\rangle$. If both semiphases ζ_1, ζ_2 are continuous in $]a, b[$, then $\zeta_i(a^+) = k_i \lesseqgtr l_i = \zeta_i(b^-)$ according to the fact whether ζ_i is strictly increasing or decreasing, mapping the interval $]a, b[$ onto a suitable interval $]r_i, s_i[$. Then $\langle 17\rangle$ maps $]r_1, s_1[$ onto $]r_2, s_2[$ by means of one of both branches of the corresponding hyperbola, namely by the increasing or decreasing one according to the fact wheather $|b_{ij}| \gtrless 0$.

Without regard to the sign of the determinant $|b_{ij}|$ there hold the equations

$$\langle 21\rangle \qquad k_2 = \frac{b_{11}k_1 + b_{12}}{b_{21}k_1 + b_{22}}, \qquad l_2 = \frac{b_{11}l_1 + b_{12}}{b_{21}l_1 + b_{22}},$$

being $k_i = r_i, l_i = s_i$ or $k_i = s_i, l_i = r_i$ according to that if ζ_i is increasing or decreasing. From $\langle 21\rangle$ we get $\langle 18\rangle$.

There are four possibilities.

I. If $k_1 < l_1, k_2 < l_2$, then $\langle 17\rangle$ is realized by an increasing branch and $|b_{ij}| > 0$.
II. If $k_1 < l_1, k_2 > l_2$, then $\langle 17\rangle$ is realized by a decreasing branch and $|b_{ij}| < 0$.
III. If $k_1 > l_1, k_2 < l_2$, then $\langle 17\rangle$ is realized by a decreasing branch and $|b_{ij}| < 0$.
IV. If $k_1 > l_1, k_2 > l_2$, then $\langle 17\rangle$ is realized by an increasing branch and $|b_{ij}| > 0$.

Considering these possibilities together with $\langle 20\rangle$ we get the condition $\langle 19\rangle$, which eliminates the homographies transforming the numbers k_1, l_1 to k_2, l_2 according to $\langle 21\rangle$ in such a way that the point (k_1, k_2) lies on one branch whereas the point (l_1, l_2) lies on the other branch of the hyperbola.

4.8. Lemma. *Let us have equations (q_i) in (a_i, b_i), $i = 1, 2, 3$. Let us consider equations (q_i, q_j) in open intervals $]a_j, b_j[\times]a_i, b_i[$ only.*

If γ_{ij} is a complete solution of (q_i, q_j), then γ_{ij}^{-1} is a complete solution of (q_j, q_i).

If γ_{ij} or $\tilde{\gamma}_{jk}$, resp., is a complete solution of (q_i, q_j) or (q_j, q_k), resp., then the composed function $\gamma_{ij} \circ \tilde{\gamma}_{jk}$ is a complete solution of (q_i, q_k).

Proof. We denote as usually the domain of a function f by $\operatorname{Dom} f$ and the range of f as $\operatorname{Im} f$. There is $\operatorname{Dom} \gamma_{ij} \circ \tilde{\gamma}_{jk} = \tilde{\gamma}_{jk}^{-1} (\operatorname{Im} \tilde{\gamma}_{jk} \cap \operatorname{Dom} \gamma_{ij}) = \tilde{\gamma}_{jk}^{-1} (\operatorname{Im} \tilde{\gamma}_{jk}) = \operatorname{Dom} \tilde{\gamma}_{jk} =]a_k, b_k[$, $\operatorname{Im} \gamma_{ij} \circ \tilde{\gamma}_{jk} = \gamma_{ij}(\operatorname{Im} \tilde{\gamma}_{jk} \cap \operatorname{Dom} \gamma_{ij}) = \gamma_{ij}(\operatorname{Dom} \gamma_{ij}) = \operatorname{Im} \gamma_{ij} =]a_i, b_i[$ so that $\gamma_{ij} \circ \tilde{\gamma}_{jk}$ is a complete solution of (q_i, q_k). There is $\operatorname{Dom} \gamma_{ij}^{-1} = \operatorname{Im} \gamma_{ij} =]a_i, b_i[$, $\operatorname{Im} \gamma_{ij}^{-1} = \operatorname{Dom} \gamma_{ij} =]a_j, b_j[$ so that γ_{ij}^{-1} is a complete solution of (q_j, q_i).

4.9. Definition. *Let the equation (q_i) in (a_i, b_i), $i = 1, 2$, be disconjugate. Let ζ_i be a paraphase of (q_i). The paraphases ζ_1, ζ_2 are called similar if* $\operatorname{Im} \zeta_1 = \operatorname{Im} \zeta_2$.

4.10. Corollary. *Disconjugate equations (q_i) in (a_i, b_i), $i = 1, 2$ admit similar paraphases if and only if they are both general or both special.*

Proof follows from 4.5 and 4.6.

4.11. Theorem. *Let (q_i) in (a_i, b_i), $i = 1, 2$ be disconjugate equations both general or both special. Let γ be some transformation of (q_1) to (q_2), i.e. a solution of the equation (q_2, q_1). Let ζ_i be some paraphase of (q_i). Then γ is a complete transformation if and only if $\gamma = \zeta_2^{-1}\zeta_1$ where ζ_1, ζ_2 are similar.*

Proof. I. Let ζ_1, ζ_2 be similar. Put $\gamma = \zeta_2^{-1}\zeta_1$. Then γ is a solution of (q_2, q_1) since ζ_i is a solution of $(0, q_i)$. There is $\operatorname{Dom} \gamma = \zeta_1^{-1}(\operatorname{Im} \zeta_1 \cap \operatorname{Im} \zeta_2) = \operatorname{Dom} \zeta_1 =]a_1, b_1[$, $\operatorname{Im} \gamma = \zeta_2^{-1}(\operatorname{Im} \zeta_1 \cap \operatorname{Im} \zeta_2) = \operatorname{Dom} \zeta_2 =]a_2, b_2[$ so that γ is a complete solution of (q_2, q_1).

II. Let γ be a complete solution of (q_2, q_1), ζ_2 be an arbitrary paraphase of (q_2). Put $\zeta_1 = \zeta_2 \circ \gamma$. Then ζ_1 is a solution of the equation $(0, q_1)$. Since $\operatorname{Dom} \zeta_1 =$

$= \gamma^{-1}(\mathrm{Im}\,\gamma \cap \mathrm{Dom}\,\zeta_2) = \gamma^{-1}(\mathrm{Im}\,\gamma) = \mathrm{Dom}\,\gamma =]a_1, b_1[$, ζ_1 is a paraphase of (q_1). Since $\mathrm{Im}\,\zeta_1 = \zeta_2(\mathrm{Im}\,\gamma \cap \mathrm{Dom}\,\zeta_2) = \zeta_2(\mathrm{Dom}\,\zeta_2) = \mathrm{Im}\,\zeta_2$, ζ_1 is similar to ζ_2.

4.12. Remark. For two disconjugate and special equations (q_1), (q_2) all paraphases ζ_1, ζ_2 are similar and they are complete solutions of $(0, q_1)$, $(0, q_2)$, respectively, see 4.5. According to 4.8. all complete solutions γ of (q_1, q_2) are of the form $\gamma = \zeta_1^{-1}\zeta_2$, which follows from 4.11., too.

4.13. Theorem. *Let the equations (q_i) in (a_i, b_i), $i = 1, 2$, be disconjugate and either both general or both special. Among the (classical) phases α_1 and α_2, resp., corresponding to paraphase ζ_1 and ζ_2, resp., of (q_1) and (q_2), resp., there exist pairs of similar phases α_1, α_2 if and only if ζ_1, ζ_2 are similar.*

Proof. I. Let ζ_1, ζ_2 be similar. For α_i we have $\mathrm{tg}\,\alpha_i = \zeta_i$. Since $\mathrm{Im}\,\zeta_i = \mathrm{tg}\,(\mathrm{Im}\,\alpha_i)$ and $\mathrm{Im}\,\alpha_1$, $\mathrm{Im}\,\alpha_2$ are intervals of length less than π, there exists $\nu \in \mathbf{Z}$ such that $\alpha_2, \alpha_1 + \nu\pi$ are similar.

II. If $\mathrm{Im}\,\alpha_1 = \mathrm{Im}\,\alpha_2$, then $\zeta_i = \mathrm{tg}\,\alpha_i$ gives $\mathrm{Im}\,\zeta_1 = \mathrm{Im}\,\zeta_2$.

4.14. Corollary. For two disconjugate equations (q_i), $i = 1, 2$, of the same character in $]a_i, b_i[$ resp. all the complete transformations $\gamma \in (q_1, q_2)$ are given by the formula $\alpha_1^{-1}\alpha_2$ where α_1, α_2 range over all pairs of similar (classical) phases of (q_1), (q_2) resp., *but they are also given by the formula $\gamma = \zeta_1^{-1}\zeta_2$ where ζ_1, ζ_2 range over all the pairs of similar paraphases of* (q_1), (q_2), resp.

To the end of this paragraph we generalize to semiphases some results concerning the paraphases.

4.15. Lemma. *Let (q) in (a, b) be disconjugate. Then the boundary values $\zeta(a^+) = k$, $\zeta(b^-) = l$ of an arbitrary semiphase ζ of (q) in $]a, b[$ consisting of* (i) *increasing* (ii) *decreasing branches, resp., fulfil the conditions*

$$\langle 22\rangle \qquad \text{(i)} \quad -\infty \leqq k < l \leqq +\infty \quad \text{or} \quad -\infty < l \leqq k < +\infty,$$

$$\text{(ii)} \quad -\infty \leqq l < k \leqq +\infty \quad \text{or} \quad -\infty < k \leqq l < +\infty, \text{ resp.},$$

according to that if $\mathrm{Im}\,\zeta$ *is an interval or a union of two (disjoint and non bounded) intervals where in $\langle 22\rangle$ in any case at least one of the signs of equality falls off for a general equation whereas for a special one all signs of equality take place in any case.*

Proof. Any semiphase ζ of (q) is of the form $\zeta = \mathrm{tg}\,\alpha$ where α is some (classical) phase of (q) in $]a, b[$. Denote by $\alpha(a^+) = c$, $\alpha(b^-) = d$, $\zeta(a^+) = k$, $\zeta(b^-) = l$ the boundary values of α and ζ. Put $r = c$, $s = d$ or $r = d$, $s = c$ according to that if α increases or decreases, i.e. if $c < d$ or $d < c$, the phase α mapping $]a, b[$ onto $]r, s[$ of the length $0 < s - r < \pi$ or $s - r = \pi$ in case of (q) being general or special.

Then the function tg maps $]r, s[$ onto the interval $]\mathrm{tg}\,r^+, \mathrm{tg}\,s^-[$ or onto the union of two disjoint intervals $]-\infty, \mathrm{tg}\,s^-[\cup]\mathrm{tg}\,r^+, +\infty[$ according to that if in $]r, s[$ a number of the form $(\nu + 1/2)\pi$, $\nu \in \mathbf{Z}$ does not lie or lies. According to that, for a general equation (q), there is $]\mathrm{tg}\,r^+, \mathrm{tg}\,s^-[\subset \mathbf{R}$ (a proper subset) or $\mathrm{tg}\,s^- < \mathrm{tg}\,r^+$, whereas for a special one there is $]\mathrm{tg}\,r^+, \mathrm{tg}\,s^-[= \mathbf{R}$ or $\mathrm{tg}\,s^- = \mathrm{tg}\,r^+$.

According to that if α increases or decreases we have $k = \zeta(a^+) = \lim\limits_{t \to a^+} \zeta(t) = \lim\limits_{t \to a^+} \mathrm{tg}\,\alpha(t) = \lim\limits_{\alpha \to c\pm} \mathrm{tg}\,\alpha \overset{\text{den.}}{=} \mathrm{tg}\,c\pm$, $l = \zeta(b^-) = \lim\limits_{t \to b^-} \zeta(t) = \lim\limits_{t \to b^-} \mathrm{tg}\,\alpha(t) = \lim\limits_{\alpha \to d\mp} \mathrm{tg}\,\alpha \overset{\text{den.}}{=} \mathrm{tg}\,d^\mp$, and thus $k = \mathrm{tg}\,r^+$ or $k = \mathrm{tg}\,s^-$ simultaneously with $l = \mathrm{tg}\,s^-$ or $l = \mathrm{tg}\,r^+$. Hence the assertion.

4.16. Remark. The conditions $\langle 22\rangle$ (i) and (ii), resp., define the **admissible** boundary values k, l for semiphases ζ of a disconjugate equation (q) in (a, b) consisting of (i) increasing or (ii) decreasing branches.

4.17. Theorem. *Let (q) in (a, b) be disconjugate. Then for arbitrary admissible boundary values k, l for the semiphases ζ of (q) consisting of* (i) *increasing or* (ii) *decreasing branches there exists the unique interval $]r, s[(\bmod \pi)$ of the length $\leqq \pi$ such that* (i) $\operatorname{tg} r^+ = k$, $\operatorname{tg} s^- = l$ *or* (ii) $\operatorname{tg} r^+ = l$, $\operatorname{tg} s^- = k$. *Put in case* (i) $c = r$, $d = s$ *and in case* (ii) $c = s$, $d = r$. *If α ranges over all the* (i) *increasing or* (ii) *decreasing* (*classical*) *phases of (q) with the boundary values $\alpha(a^+) = c$, $\alpha(b^-) = d$, then $\zeta = \operatorname{tg} \alpha$ ranges over all the semiphases of (q) consisting of* (i) *increasing or* (ii) *decreasing branches with the boundary values $\zeta(a^+) = k$, $\zeta(b^-) = l$.*

Proof. The theorem is formulated separately for the semiphases ζ consisting of increasing branches and the decreasing ones. We shall prove it for semiphases ζ consisting of decreasing branches. Be k, l admissible boundary values for ζ. Then $\operatorname{Im} \zeta$ is an interval or a union of two (disjoint and non bounded) intervals according to that $l < k$ or $k \leqq l$. For the appointment of the interval $]r, s[$ we have the conditions $0 < s - r \leqq \pi$, $\operatorname{tg} r^+ = l$, $\operatorname{tg} s^- = k$. The existence and uniqueness of the interval $]r, s[$ $(\bmod \pi)$ is evident from the behaviour of the function tg.

For the semiphases ζ consisting of decreasing branches the corresponding phases are decreasing, too, and thus for the boundary values of the phases there is $\alpha(a^+) = c = s$, $\alpha(b^-) = d = r$.

It is known that there exist phases α with boundary values c, d. Then $\zeta = \operatorname{tg} \alpha$ have the boundary values k, l.

Passing on to the semiphases $\zeta = \operatorname{tg} \alpha$ the influence of translations mod π for boundary values of phases α is eliminated so that all semiphases ζ with the boundary values k, l are obtained as soon as α ranges over all the phases with the boundary values c, d.

5. HYPERPHASES.

Let $\langle y, z\rangle$ be a basis of the equation (q) in an interval (a, b). We shall deal with the hyperbolic phases of $\langle y, z\rangle$ with respect to the basis $\langle \operatorname{sh} t, \operatorname{ch} t\rangle$, i.e. with continuous functions ϑ in $]a, b[$ fulfilling $\langle 5\rangle$. Since $\operatorname{th} \vartheta$ is bounded, it must be $z \neq 0$ in $]a, b[$. That is why there exists a function $k = \dfrac{y}{\operatorname{sh} \vartheta} = \dfrac{z}{\operatorname{ch} \vartheta}$ continuous and having no zero in $]a, b[$. Then the relations $y = k \operatorname{sh} \vartheta$, $z = \operatorname{ch} \vartheta$ give $z^2 - y^2 = k^2 > 0$ in $]a, b[$ and it is the matter of expression of the basis $\langle y, z\rangle$ in the form $y = \pm r \operatorname{sh} \vartheta$, $z = \pm r \operatorname{ch} \vartheta$, where $r > 0$ and ϑ are continuous functions in $]a, b[$.

5.1. Lemma. *For a basis $\langle y, z\rangle$ of an equation (q) in (a, b) hyperphases ϑ exist if and only if there holds $|y| < |z|$ in $]a, b[$.*

Proof. I. If $y = \pm r \operatorname{sh} \vartheta$, $z = \pm r \operatorname{ch} \vartheta$, $r > 0$ and ϑ being continuous in $]a, b[$, then $\dfrac{y}{z} = \operatorname{th} \vartheta$, $z^2 - y^2 = r^2 > 0$ and thus $|y| < |z|$ in $]a, b[$.

II. If $|y| < |z|$ in $]a, b[$, then $|z| > 0$ and the equation $\dfrac{y}{z} = \operatorname{th} \vartheta$ has a unique and continuous solution $\vartheta(t)$ in $]a, b[$.

5.2. Lemma. *An equation (q) in (a, b) has a basis $\langle y, z\rangle$ fulfilling $|y| < |z|$ in $]a, b[$ if and only if (q) is disconjugate and general.*

Proof. I. Let (q) be disconjugate and general. We can suppose that the principal solutions u, v in $]a, b[$ fulfil $u > 0$, $v > 0$. Put $y = u - v$, $z = u + v$. Then there holds $|y| = \pm(u - v) < u + v = |z|$ in $]a, b[$, the solutions y, z being linearly independent.

II. Let $\langle y, z\rangle$ be a basis of (q) fulfilling $|y| < |z|$ in $]a, b[$. Then (q) is disconjugate in (a, b) since $z \neq 0$ in $]a, b[$. Moreover. $\frac{y}{z}$ is strictly monotonic in $]a, b[$ and since $\frac{y}{z} < 1$, the solution z is linearly independent on u (and on v, too) because there holds $\lim_{t\to a^+} \frac{y}{z} \neq \pm\infty$ and $\lim_{t\to b^-} \frac{y}{z} \neq \pm\infty$. According to 3.10. (q) is disconjugate and general since z, u are linearly independent solutions without zero in $]a, b[$.

5.3. Corollary. *Hyperphases exist exactly for disconjugate and general equations* (q).

5.4. Remark. If (q) in (a, b) has a solution $z \neq 0$ in $]a, b[$, which is not a principal one at the point a, then z, u are linearly independent so that (q) is disconjugate and general according to 3.9. and 3.10. On the contrary, for an equation (q) in (a, b), which is disconjugate and general, the principal solution v at the point b has no zero in $]a, b[$ and is not a principle one at the point a.

5.5. Theorem. *In* (a, b) *let* (q) *be disconjugate and general. Let the principle solutions* u, v *fulfil* $u > 0$, $v > 0$ *in* $]a, b[$. *Then all the bases* $\langle y, z\rangle$ *of* (q), *for which there is* $|y| < |z|$ *in* $]a, b[$. *are given by the formula*

$$\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} a_{11}a_{12} \\ a_{21}a_{22} \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} \tag{23}$$

where $[a_{ij}]$ *ranges over all the matrices fulfilling one of the conditions*

$$-1 \leqq \frac{a_{12}}{a_{22}} < \frac{a_{11}}{a_{21}} \leqq 1, \quad a_{21}a_{22} > 0, \tag{24}$$

$$-1 \leqq \frac{a_{11}}{a_{21}} < \frac{a_{12}}{a_{22}} \leqq 1, \quad a_{21}a_{22} > 0. \tag{25}$$

Proof. In $]a, b[$ the condition $|y| < |z|$ is equivalent to the condition $|a_{11}u + a_{12}v| < |a_{21}u + a_{22}v|$ where $[a_{ij}]$ is a real matrix with the determinant $\gtrless 0$.

In $]a, b[$ put $\zeta = \frac{u}{v}$ so that the semiphase ζ, which is also a paraphase, is positive. By differentiation we get $\zeta' = -\frac{W(u, v)}{v^2}$ where $W(u, v)$ is the Wronskian of the basis $\langle u, v\rangle$.

If the curves given by the functions u, v did not intersect each other in $]a, b[$ it would be always $u < v$ or $u > v$, which is a contradiction. Hence there exists some $t_0 \in]a, b[$ such that $u(t_0) = v(t_0)$. Then it is $W(u, v) = u(t_0)\, v'(t_0) - u'(t_0)\, v(t_0) = u(t_0)[v'(t_0) - u'(t_0)] < 0$ according to the properties of principal solutions, see [3].

Hence $\zeta' > 0$ and thus ζ increases in $]a, b[$ from zero to $+\infty$. We are to find all non-singular real matrices $[a_{ij}]$ such that $|h(\zeta)| < 1$ for $\zeta \in]0, \infty[$ where $h(\zeta) = \frac{a_{11}\zeta + a_{12}}{a_{21}\zeta + a_{22}}$.

The condition $|h(\zeta)| < 1$ makes $a_{21} \neq 0$ necessary so that we have a hyperbola with the asymptotes $\zeta = -\frac{a_{22}}{a_{21}}$ and $h = \frac{a_{11}}{a_{21}}$. The hyperbola has increasing or decreasing branches according to that $\det [a_{ij}] \gtrless 0$.

For $\zeta \to +\infty$ it can be $|h(\zeta)| < 1$ only if $\left|\frac{a_{11}}{a_{21}}\right| \leqq 1$, which really takes place if (i) $\frac{a_{11}}{a_{21}} = 1$ and $\det [a_{ij}] > 0$, or if (ii) $\frac{a_{11}}{a_{21}} = -1$ and $\det [a_{ij}] < 0$ or if (iii) $\left|\frac{a_{11}}{a_{21}}\right| < 1$.

Since $|h(\zeta)| < 1$ is to be for all $\zeta \in]0, \infty[$, we have to complete the cases (i), (ii), (iii) by the conditions $-\frac{a_{22}}{a_{21}} < 0$, $\left|\frac{a_{12}}{a_{22}}\right| = |h(0)| \leqq 1$. Hence the theorem.

5.6. Remark. The case of $\det [a_{ij}] < 0$ follows from that of $\det [a_{ij}] > 0$, and vice versa, by the substitution of the numbers a_{11}, a_{12} by the numbers $-a_{11}, -a_{12}$.

5.7. Theorem. *In (a, b) let (q) be general and disconjugate. Let $c \neq d$ be arbitrary values in $\tilde{\mathbf{R}} = [-\infty, +\infty]$. Then there exists a one-parametric continuum (briefly ∞^1) of hyperphases ϑ of (q) fulfilling $\lim_{t\to a^+} \vartheta(t) = c$ and $\lim_{t\to b^-} \vartheta(t) = d$.*

Proof. A hyperphase ϑ of a basis $\langle y, z\rangle$ of (q) exists iff $|y| < |z|$ in $]a, b[$. Since $\zeta = \frac{y}{z}$ is strictly monotonic in $]a, b[$,the limits $\zeta(a^+) \overset{\text{den.}}{=} \lim_{t\to a^+} \zeta(t)$ and $\zeta(b^-) \overset{\text{den.}}{=} \lim_{t\to b^-} \zeta(t)$ exist and lie in the interval $[-1, 1]$. Since the function th t increases in $\mathbf{R}$ from -1 to $+1$, the existence of all hyperphases ϑ with the boundary values $\vartheta(a^+) = c \neq d = \vartheta(b^-)$ in $\tilde{\mathbf{R}}$ is equivalent to the existence of all paraphases $\zeta = \frac{y}{z}$. i.e. continuous semiphases, with the property $|y| < |z|$ in $]a, b[$ and the boundary values $\zeta(a^+) = k \neq l = \zeta(b^-)$ laying in $[-1, 1]$, the values th $c = k$ and th $d = l$ being arbitrarily given.

Take the principal solutions u, v of (q) both positive in $]a, b[$ so that their Wronskian $W(u, v)$ is negative. Then every admissible basis $\langle y, z\rangle$, i.e. fulfilling $|y| < |z|$ in $]a, b[$, is given by $\langle 23\rangle$, $\langle 24\rangle$, $\langle 25\rangle$. For $\det [a_{ij}] \gtrless 0$ we have $W(y, z) \lessgtr 0$ so that $\zeta = \frac{y}{z}$ and thus ϑ increases or decreases.

For an arbitrary non-singular matrix $[a_{ij}]$ and the basis given by $\langle 23\rangle$ the semiphase $\zeta = \frac{y}{z}$ fulfils $\zeta(a^+) = \frac{a_{12}}{a_{22}}$, $\zeta(b^-) = \frac{a_{11}}{a_{21}}$. The quotients in $\langle 24\rangle$ and $\langle 25\rangle$ thus mean the boundary values of ζ. Hence ϑ increases or decreases according to that if $\det [a_{ij}] \gtrless 0$.

If the values $\zeta(a^+) = k \neq l = \zeta(b^-)$ are arbitrarily given in $[-1, 1]$, then ζ increases or decreases in $]a, b[$ according to that if $k \lessgtr l$. Then the corresponding matrices $[a_{ij}]$ are determined by the conditions $\frac{a_{11}}{a_{21}} = l$, $\frac{a_{12}}{a_{22}} = k$. We can see that the set of all these matrices depends on two parameters a_{21}, a_{22} having the same sign.

Since it is a matter of homographies, one of these parameters can be cancelled and thus we have always ∞^1 semiphases $\zeta = \frac{y}{z}$ with the given boundary values $\zeta(a^+) = k$ and $\zeta(b^-) = l$, $k \neq l$.

5.8. Corollary. *For every general and disconjugate equation* (q) *in* $]a, b[$ *there exist* ∞^1 *increasing and* ∞^1 *decreasing hyperphases* ϑ *with the boundary values* $\pm\infty$. *That equation is of the same character as* $Y'' = Y$ *in* $\mathbf{R}$ *and teh mentioned hyperphases* ϑ *represent all the complete transformations of* (1) *to* (q). *For each* ϑ *the set of all solutions* y *of* (q) *in* $]a, b[$ *corresponds to the set of all solutions* Y *of* (1) *in* $\mathbf{R}$ *by the formula* $y = \dfrac{Y(\vartheta)}{\sqrt{|\vartheta'|}}$. *See* [5], [6], [7].

5.9. Remark. For a general and disconjugate equation (q) in (a, b) there exist continuous semiphases ζ_i, $i = 1, 2$, in $]a, b[$ fulfilling $|\zeta_i| < 1$ and thus there exist hyperphases ϑ_i determined by the continuity and the relations $\operatorname{th}\vartheta_i = \zeta_i$. For the boundary values $\vartheta_i(a^+) = c_i \mp d_i = \vartheta_i(b^-)$ and $\zeta_i(a^+) = k_i \mp l_i = \zeta_i(b^-)$ we have $k_i = \operatorname{th} c_i$, $l_i = \operatorname{th} d_i$, $|k_i| \leqq 1$, $|l_i| \leqq 1$.

For $|k_i| = 1$, $|l_1| = 1$ the hyperphases ϑ_i have the boundary values $\pm\infty$. In these four cases all the transformations $\langle 17\rangle$ are $\zeta_2 = \pm \dfrac{b_{22}\zeta_1 + b_{21}}{b_{21}\zeta_1 + b_{22}}$ where b_{21}, b_{22} fulfil only the condition $b_{22}^2 - b_{21}^2 > 0$. Put $b_{21} = \pm r_0 \operatorname{sh}\vartheta_0$, $b_{22} = \pm r_0 \operatorname{ch}\vartheta_0$. Then $\operatorname{th}\vartheta_2 = \zeta_2 = \pm \dfrac{\zeta_1 + \operatorname{th}\vartheta_0}{1 + \zeta_1 \operatorname{th}\vartheta_0} = \pm\operatorname{th}(\vartheta_1 + \vartheta_0)$. Hence $\vartheta_2 = \pm(\vartheta_1 + \vartheta_2)$, ϑ_0 *being a (real) parameter*. See [5], [6], [7].

By the formula $\vartheta_2 = \pm(\vartheta_1 + \vartheta_0)$ *there are given both one-paramatric systems of hyperphases with the boundary values* $\pm\infty$. As a consequence we can see that there exist general and disconjugate equations (q) in a finite interval $]a, b[$ such that for any one of its hyperphases ϑ with the boundary values $\pm\infty$ the limits $\lim\limits_{t\to a^+}\vartheta'(t)$, $\lim\limits_{t\to b^-}\vartheta'(t)$ do not exist.

5.10. Remark. For $i = 1, 2$ let (q_i) be a general and disconjugate equation in (a_i, b_i). Let ϑ_i be a hyperphase of (q_i), γ a transformation of (q_1) to (q_2), i.e. a solution of (q_1, q_2). Then γ *is a complete transformation, iff* $\gamma = \vartheta_1^{-1}\vartheta_2$ *for similar* ϑ_1 *and* ϑ_2, i.e. fulfilling $\operatorname{Im}\vartheta_1 = \operatorname{Im}\vartheta_2$.

Moreover, there exist all types of phases, i.e. the classical ones α_i, the parabolical ones ζ_i and the hyperbolical ones ϑ_i. *If* $\operatorname{tg}\alpha_i = \zeta_i = \operatorname{th}\vartheta_i$, *then* ζ_1 *and* ζ_2 *are similar iff* ϑ_1 *and* ϑ_2 *are similar and this is equivalent to the existence of suitable pairs of similar phases* α_1 *and* α_2. *Each of the complete transformations* $\gamma \in (q_1, q_2)$ *is expressable as* $\gamma = \alpha_1^{-1}\alpha_2 = \zeta_1^{-1}\zeta_2 = \vartheta_1^{-1}\vartheta_2$.

6. UNIVERSAL PHASES.

In this final paragraph we will deal with the concept of "universal Q-phases".

6.1. Definition. *An equation* (Q) *in an interval* $]A, B[$ *will be called universal if for every equation* (q) *in* $]a, b[$ *and for each (largest) solution* α *of* (Q, q) *there holds* $\operatorname{Dom}\alpha =]a, b[$. *The solutions* α *are called universal* Q*-phases.*

6.2. Lemma. *Let* (Q) *in* $]A, B[$ *be a universal equation. Let* (q_i) *in* $]a_i, b_i[$, *for* $i = 1, 2$, *be arbitrary equations. Then*

(i) *for every pair* $(i = 1, 2)$ *of similar solutions* α_i *of* (Q, q_i), *i.e. such that* $\operatorname{Im}\alpha_1 = \operatorname{Im}\alpha_2$, *the function* $\gamma = \alpha_1^{-1}\alpha_2$ *is a complete solution of* (q_1, q_2),

(ii) *for each complete solution* γ *of* (q_1, q_2) *and arbitrary solution* α_1 *of* (Q, q_1) *the function* $\alpha_2 = \alpha_1\gamma$ *is a solution of* (Q, q_2), *which is similar to* α_1.

6.3. Lemma. If for each equation (q) in $]a, b[$ and for each solution α of (Q, q) there holds $\mathrm{Dom}\ \alpha^{-1}\alpha =]a, b[$, then (Q) is a universal equation.

6.4. Corollary. If $\gamma = \alpha_1^{-1}\alpha_2$ is not a complete solution of (q_1, q_2) for some solutions α_i of (Q, q_i), $i = 1, 2$, then (Q) cannot be a universal equation.

6.5. Lemma. Let (Q) in $]A, B[$ be a universal equation. Then

(i) for every (q) in $]a, b[$ and each solution α of (Q, q) it is $\mathrm{Im}\ \alpha^{-1} =]a, b[$, in other verbs for every (q) and each solution β of (q, Q) it is $\mathrm{Im}\ \beta =]a, b[$.

(ii) For every solution Φ of (Q, Q) it is $\mathrm{Dom}\ \Phi = \mathrm{Im}\ \Phi =]A, B[$, in other verbs all the dispersions Φ of (Q, Q) are complete solutions of (Q, Q).

6.6. Theorem. *Let (Q) in $]A, B[$ be such that each solution Φ of (Q,Q) is complete. Then (Q) is a universal equation.*

Proof. This theorem goes a little more profoundly to the properties of classical phases, namely in the fact, that each solution Φ of (Q, Q) is complete iff the values of each phase A of (Q) range over the whole real line $\mathbf{R}$.

On the other hand, for each phase A of (Q), (Q) being arbitrary, and for each phase α of another arbitrary (q) in $]a, b[$ the domain of $A^{-1}\alpha$ is $]a, b[$ iff $\mathrm{Im}\ \alpha \subseteq \mathrm{Im}\ A$. Under the supposition of the theorem 6.6. it is $\mathrm{Im}\ A = \mathbf{R}$ and thus each solution of (Q, q) exists in $]a, b[$. I.E. (Q) is universal.

Since the values of each phase A of (Q) range over the whole real line $\mathbf{R}$ iff the equation (Q) is both-sided oscillatory, we have

6.7. Corollary. (Q) *is a universal equation iff it is both-sided oscillatory.*

Now it is clear that the carrier -1 is the unique one with both required properties, that of simplicity and that of universality.

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